A STUDY OF MIDDLE SCHOOL AND COLLEGE STUDENTS' MENTAL MATHEMATICS ABILITIES IN REAL-WORLD CONTEXTS

By

Elizabeth Madell Brion

A Master's Thesis
Submitted in Partial Fulfillment
of the Requirements for the Degree of
Master of Science in Education
Mathematics Education (7-12)
Department of Mathematical Sciences
State University of New York at Fredonia
Fredonia, New York

June 2016

State University of New York at Fredonia
Department of Mathematical Sciences

CERTIFICATION OF PROJECT WORK

We, the undersigned, certify that this project entitled A study of middle school and college students' mental mathematics abilities in real-world contexts by Elizabeth Madell Brion, Candidate for the degree of Master of Science in Education, Mathematics Education (7-12), is acceptable in form and content and demonstrates a satisfactory knowledge of the field covered by this project.

[redacted]

Keary Howard, Ph.D.
Master's Thesis Advisor
Department of Mathematical Sciences

[redacted]______

H. Joseph Straight, Ph.D.
Department Chair
Department of Mathematical Sciences

_[redacted]
J[redacted]
Associate Provost for Graduate Studies,
Sponsored Research, and Faculty Development

Abstract

This study examines the thought processes employed by middle school and undergraduate college students to mentally solve situational mathematics problems. It was hypothesized that non-major undergraduate students would use more efficient mental strategies than middle school students to solve in-context arithmetic problems, though both undergraduate students and middle school students would have equal accuracy. The study also compared age and strategy choice, as well as indicated perception of mathematical abilities and accuracy. It was further hypothesized that both undergraduate students and middle school students would lack confidence in their mental computation abilities, possibly affecting their accuracy. The results of this study were in partial support of the hypothesis; it was indicated that college students were more efficient than middle school students (p-value = 0.019), but there was no statistical significance in accuracy between middle school and college students when solving in-context mathematics problems mentally. In-context problems consisted of calculating change, percent tip, percent discount, and gas mileage. Furthermore, there was no statistical significance between confidence in mathematical abilities and accuracy on the assessment (p-value = 0.298). Overall confidence in mathematics skills (on a Likert scale of 1-5) for middle school students ($\bar{X}_{MS} = 4.3$) and college students ($\bar{X}_{CS} = 4.0$) was not statistically significant (p-value = 0.306). Additional results indicate that the use of efficient strategies had statistical significance on accuracy when solving these problems.

Table of Contents

Introduction

This study examines the thought processes employed by middle school and undergraduate college students to mentally solve situational mathematics problems. As technology develops and its role increases in society, the necessity for quick mental calculations decreases. Calculators readily available on cellphones and built into cash registers have had an immense effect on the ability to accurately determine change, tips, or discounts when shopping. Students are given the opportunity to use calculators to complete work in their mathematics classes and on state tests. This technology has benefitted society in many ways; however, some may feel that the quick availability of technology may have caused some deficiencies in the capability to correctly solve arithmetic problems mentally. This study was designed to uncover information on how technology affects the generations currently in middle school and undergraduate school.

Interest in this topic was developed in my elementary school years. Whenever I went shopping with my father we played a game; I would collect the coins from the change if I was able to correctly calculate the change prior to the cash register displaying it. Just this simple challenge with addition and subtraction fostered number sense, arithmetic ability, and a love of numbers. Teachers loved this game and peers were astounded: this was in the early 2000s. The advancement and availability of technology has skyrocketed over the past decade, and specific interest lies in whether or not mental arithmetic abilities for simple situations like calculating change still exist.

Further, are mental strategies introduced and fostered in educational programs? All too often students at any level reach for their calculators for the smallest of calculations. Some can compute the answer mentally but do not elicit confidence in their abilities and therefore feel the need to check answers with a calculator. If students do not believe it is necessary to calculate mentally, or that they cannot calculate mentally, their accuracy may be compromised. These underlying problems define the hypothesis statement for this research:

> *It is hypothesized that non-major undergraduate students will use more efficient mental strategies than middle school students to solve in-context arithmetic problems, though both undergraduate students and middle school students will have equal accuracy. It is further hypothesized that both undergraduate students and middle school students will lack confidence in their mental computation abilities, which may affect their accuracy.*

Two key terms of the study are introduced and utilized in the hypothesis, namely "efficient strategies" and "in-context problems." The working definitions for these terms based on their role in this specific study are described in the figure below.

Key Terms	Working Definitions
Efficient Strategies	Mental strategies used to solve a given problem accurately and in a short amount of time.
In-Context Problems	Problems that have been written in the context of real-world situations, such as calculating change, gas mileage, tips, or discounts.

Figure 1. Definitions of key terms of the study.

This hypothesis was tested using an interview-style formal assessment containing four in-context problems and a survey administered individually to middle school students and non-major undergraduate students enrolled in a pre-calculus class. Students' responses were recorded using an Apple iPad for later coding. Two versions of demographic surveys were developed, one for middle school students and another for undergraduate students. These surveys were administered to participants after the interview to gain information such as opinions about mathematics and mental computation ability, as well as impressions of the problems presented during the interview. In this study we simulated real-world mental arithmetic problems in an attempt to understand the thought processes used to solve the problems. The literature review was completed prior to the study and explores the mental processes that occur during mental computation, strategies used to solve mathematics problems mentally, instruction of mental strategies, and real-world applications, which is then followed by the experimental design of the study. Results of the study were collected and analyzed, and implications for instruction based on the results are then discussed.

Literature Review

The purpose of this literature review is to analyze existing research focused on the processes and components of mental mathematics and its connection to real-world situations in order to discover a context for further research. This literature review contains five sections: Psychological Processes Utilized in Mental Mathematics, Flexible and Inflexible Mental Computation, Strategies Used in Mental Computation, Mental Mathematics Instruction, and Everyday Mental Mathematics, respectively. The first section examines research that has pinpointed the areas of the brain where each mental process takes place. Understanding of these mental processes is expanded in the flexible and inflexible section and applied in the strategies section. Next, research that pertains to the instruction of mental mathematics strategies is analyzed, including the ongoing debate of whether or not these strategies should be taught to students. Finally, in the Everyday Mental Mathematics section, research that places the previously presented information about mental mathematics processes and strategies in real-world context is described.

Psychological Processes Utilized in Mental Mathematics

Mental Mathematics consists of calculations performed without using any sort of technology. Carrying out mental calculations utilizes one's working memory and information processing structures, which are located in the prefrontal cortex and the parietal lobe (Rickard et al., 2000; Zarnhofer et al., 2012). Different areas of these brain structures are activated depending on the complexity of the problem and the operations that must be solved (Childers, Houston, & Heckler, 1985; Fürst & Hitch, 2000; Simon, 1979). The speed and accuracy of mentally solving problems is affected by the level of brain activation as well as by other psychological aspects, including math anxiety (Miller & Bischel, 2004).

According to Rickard et al. (2000) and Zarnhofer et al. (2012), the processes involved in mental calculations primarily take place in the prefrontal cortex and parietal lobe. The prefrontal cortex is responsible for brain functions regarding complex cognition, such as mental mathematics. Sensory information is processed in the parietal lobe. Situations involving mental calculations often occur from visual or auditory information; this information is processed in the parietal lobe and then analyzed in the prefrontal cortex. Different areas of the prefrontal cortex are activated depending on which operations are identified in the problem. Operations concerning retrieval of known facts such as single-digit addition and multiplication do not utilize the prefrontal cortex because they are stored as long-term memories in the hippocampus. Operations requiring the analysis of these known facts, such as multi-digit addition and multiplication, division, and subtraction, do take place in the prefrontal cortex (Rickard et al., 2000).

Results from the work of Zarnhofer et al. (2012) indicate that subtraction problems activate a wider range of areas in the prefrontal cortex and parietal lobe than multiplication problems, and that more inferior frontal brain areas are utilized when problems are presented in number word form compared to Arabic numeral form. A greater number of language-processing regions, including Broca's area, are activated when problems are given in numerical word form. Verbal and visual processors were compared in this study to determine if different brain regions are utilized based on processing type. No significant correlations were found between processing type and figural, numerical, or verbal intelligence, nor between processing type and brain activation (Zarnhofer et al., 2012). Thus processing method does not affect brain activation, while operation and complexity of problem does.

Complex problems involving the manipulation of known facts involve a number of executive processes. Problems that require recall of known facts do not involve as many executive processes, but still utilize some. Fürst and Hitch (2000) discovered this in their study designed to separate the phonological and executive processes of mental calculations. Their results indicate that mental calculations involving carrying operations require the utilization of more executive than phonological processes. However, when verbal information was presented and removed, more reliance was placed on the phonological loop to store that information and use it to solve the problem. Less reliance on the phonological loop, and more on executive processes, occurred when the problem was presented visually then removed prior to solving. Working memory is involved in both situations and plays a key role in any mental calculation.

Working memory contains two subsystems: visual working memory and verbal working memory. Both are directly involved in mental calculations, and essentially equally according to Miller and Bischel (2004). The presentation of problems affects the level of reliance on each subsystem, and variation in either subsystem affects mathematics performance. A number of factors can cause that variation, but mathematics anxiety is the "most significant factor predicting variance in mathematics performance" (Miller & Bischel, 2004, p. 603). Visual working memory may be more affected by math anxiety than verbal working memory due to the decreased ability to focus on and process visual information, which therefore impairs mental calculations. Despite these effects, however, among peers that suffer from math anxiety those with higher working memory ability, specifically higher visual working memory, outperform those with lower working memory ability. While those who do not suffer from mathematics anxiety have higher accuracy, students who do are able to still be accurate if their general working memory ability is high. Differences in verbal working memory do not seem to affect mathematics performance (Miller & Bischel, 2004, pp. 603-4).

Understanding of mathematical concepts expands as related problems are processed and solved. While brain structure is genetic, brain functions such as conceptual understanding can be practiced and improved. One's level of conceptual understanding affects strategy choice in mental mathematics situations and ultimately the efficiency of those calculations.

Flexible and Inflexible Mental Computation

Conceptual understanding can be characterized as either flexible or inflexible and is comprised of number sense, domain-specific knowledge, and metacognitive strategies. It is facilitated by the knowledge of different strategies, which are often developed through connections to real-world contexts introduced in educational settings (Blöte, Klein, & Beishuizen, 2000; Heirdsfield, 2001; Nys & Content, 2010). "Mental flexibility" is defined as the ability to manipulate numbers in problems to use different strategies to solve those problems. "Mental inflexibility" occurs when one strategy is applied over and over again to problems without variation. A deeper conceptual understanding supports flexibility, but may not necessarily affect accuracy (Blöte, Klein, & Beishuizen, 2000; Heirdsfield & Cooper, 2004;

Threlfall, 2002). The following figure provides the definitions of mental flexibility and mental inflexibility as defined by Heirdsfield (2001).

Mental Flexibility	The ability to solve problems utilizing multiple types of strategies.
Mental Inflexibiity	Consistent utilization of one strategy to solve any given problem.

Figure 2. Definitions of mental flexibility and inflexibility.

Students who are accurate mental computers “possess extensive and connected knowledge bases” which support efficient strategies (Heirdsfield, 2001, p. 279). These students recall their number facts quickly and accurately and are able to use number fact strategies when they cannot recall a fact. They have a developed number sense and are confident in their mathematical abilities. Using this number sense, they are able to immediately notice the characteristics of numbers in problems and choose the most efficient strategy they can think of to solve the problem. Mentally flexible students analyze problems prior to attempting to solve them and use a variety of methods rather than one method repeatedly, which often results in greater efficiency (Heirdsfield, 2001, pp. 276-9).

Mentally inflexible students tend to rely on one strategy for solving problems of a similar nature. This strategy is often a visualization of the paper-and-pencil algorithm of whichever operation is being solved. While this may not be the most efficient method, familiarity supports accuracy and these students are able to be as accurate as their mentally flexible peers (Heirdsfield, 2001, pp. 276-9). Past mathematical instruction has focused on rote practice of algorithms, which supports inflexibility. Current standards and instruction, however, encourage the introduction and practice of multiple strategies for solving problems. This type of instruction is meant to expand students’ knowledge bases and allow for stronger mental flexibility.

While some find such instructional methods to be beneficial and consider them the best way to teach these concepts, there is some speculation about their effectiveness (Heirdsfield & Cooper, 2004; Threlfall, 2002). Heirdsfield and Cooper (2004) mention that “flexibility and number sense is neither sufficient nor necessary in mental computation” (p. 443). Students who are familiar with a single strategy and believe in the accuracy of that strategy as well as their ability to apply it are able to be accurate in their calculations. These students, classified as accurate and inflexible, do not have well-developed number sense, but they are comfortable with their chosen method and are therefore accurate. They do not need developed number sense in order to be successful in their calculations.

Some researchers believe that this ability is enough, that teaching too many strategies makes choosing which strategy to apply more difficult for the student. It requires students to fully analyze the problem and rule out which strategies do not make sense to use and slows down calculation. John Threlfall (2002) declares “the idea that criteria can be taught to children for deciding in advance which strategy to use does not seem feasible” (p. 38) in his article “Flexible Mental Calculations.” According to Threlfall, teaching different strategies actually *limits* the development of flexibility in mental calculations. When given the option of which strategies to introduce to students, teachers often choose the strategies they know best or they feel their students will understand the best. Since students are not discovering new ways to find solutions to problems in this instructional format, the expansion of mental flexibility is undermined. When students are given the opportunity to analyze a problem and must use whatever knowledge they have to solve it, they build their conceptual knowledge and expand their mental flexibility independently. Choosing a “good” strategy is a skill learned by experience rather than instruction. Threlfall (2002) asserts that “there are no suggestions in the literature for direct teaching on how to make good strategy choices in mental calculations" (p. 38). Students must muddle through problems and figure out how to solve them on their own.

If an activity requires that students must determine how to solve a problem without direct instruction, many unique methods of solving that problem are likely to be used. Due to the immense number of strategies are used, four overlying categories of strategy methods have been named by various researchers. The categories seem "reasonable … [but] none of them is adequate to capture the diversity found in the calculations of a small sample of ordinary primary children" (Threlfall, 2002, p. 35). Though not all strategies can be labeled and identified specifically, it is possible to categorize them into the broadly described groups, as is done in the following section.

Strategies Used in Mental Calculations

Strategy choice is primarily based on which operation is in the problem. Addition and subtraction problems are often solved by similar techniques, while multiplication and division are solved by a different set of similar techniques. Threlfall (2002) compiled a list of the different strategies and methods used to solve addition and subtraction problems identified by various researchers. Other research has supported the categorization in Threlfall's (2002) work (Beishuizen, van Putten, & Van Mulken, 1997; Davis, 2009; LeFevre, Sadesky, & Bisanz, 1996). Solving multiplication and division problems relies on a different set of skills (Heirdsfield, Cooper, Mulligan, & Irons, 1999; Hickendorff, van Putten, Verhelst, & Heiser, 2010). While operation has the strongest influence on strategy choice, other factors, such as conceptual understanding, self-perceptions and beliefs in abilities, and the size of the numbers in the problem, also have an effect on strategy choice, accuracy, and efficiency (Gogus, 2013; Heirdsfield, 2001; LeFevre, Sadesky, & Bisanz, 1996).

The three central categories of strategies discussed in Threlfall's (2002) list include sequential, decomposition, and compensation strategies. At least seven researchers identified these types of strategies in their work, using various names to label them. Decomposition strategies are characterized by breaking down one or both numbers, often into place values, and combining the numbers based on place value. The "split-method" (Heuvuel-Panhuizen, 2001: Thompson, 1999), or "collections-based" method (Yackel, 2001) was used as labels for decomposition strategies. When problems are solved through a sequence of steps the strategy is labeled as sequential. Names such as the "jump" or "jumping" method (Heuvel-Panhuizen, 2001; Thompson, 1999), "counting-based" method (Yackel, 2001), "sequence" or "sequential" method (Fennema et al., 1998; Fuson et al., 1997), and "10s" or "N10" strategy (Beishuizen & Anghileri, 1998) have been given to such strategies. Researchers also listed variations of compensation methods, in which students altered both numbers in various ways to make solving the problem simpler. Rounding to the nearest ten, or "over-jumping" (Thompson, 1999), using number relations and proportions, called "flexible counting" (Heuvuel-Panhuizen, 2001), and using "near-doubles" by breaking numbers into easily-doubled parts (QCA, 1999) are examples of compensation methods. On the following page, Figure 3 provides examples of addition and subtraction strategies identified by Davis (2009). Threlfall (2002) reminds us that "there is too much diversity" in calculation strategies to have strict categories, but the majority of addition and subtraction strategies have characteristics that allow them to be classified into the above-described categories (pp. 33-35).

Addition strategies

Partition and recombine

• 23 + 12	23 + 10 = 33	33 + 2 = 35
or	20 + 10 = 30	3 + 2 = 5
		30 + 5 = 35

Add nearest multiple of 10 and adjust

• 27 + 31	27 + 30 = 57	57 + 1 = 58
• 27 + 39	27 + 40 = 67	67 – 1 = 66

Use 'near double' and adjust

• 44 + 45	45 x 2 = 90	90 – 1 = 89
• 139 + 142	140 x 2 = 280	280 + 1 = 281

Subtraction strategies

Partitioning

• 58 – 34	58 – 30 = 28	28 – 4 = 24

Find a small difference by counting up

• 82 – 79	79 + 3 = 82	
• 306 – 291	291 + 9 = 300	300 + 6 = 306
		9 + 6 = 15

Subtract nearest multiple of 10 and adjust

• 76 – 41	76 – 40 = 36	36 – 1 = 35
• 76 – 39	76 – 40 = 36	36 + 1 = 37

Figure 3. Mental addition and subtraction strategies. Davis, S. (2009). Oral and mental mathematics. *Mathematics Teaching, 215*(1), 46.

Categories for mental multiplication and division have been identified as well. Counting, basic fact recall, separation, and wholistic strategies are used to solve multiplication and division problems. Figure 4 on the following page depicts Heirdsfield, Cooper, Mulligan, and Irons' (1999) descriptions of these strategies.

Category	**Description**	**Examples**
<u>Multiplication</u>		
Counting (CO)	Any form of counting strategy, skip counting forwards and backwards, repeated addition and subtraction, and halving and doubling strategies.	5x8: 5, 10, 15, … 5x8: double 5, double 16, +8.
Basic fact (BF)	Using a known multiplication or division fact or a derived fact.	5x8: 10x8=80, so 5x8=40.
RL separated (RLS)	Numbers are separated into place values, then proceed right to left.	5x19: 5x9=45=40+5, 5x10=50, 50+40=90, 95.
LR separated (LRS)	Numbers are separated into place values, then proceed left to right.	5x19: 5x10=50, 5x9=45, 50+45=95.
Wholistic (WH)	Numbers are treated as wholes.	5x19: 5x20-5=100-5=95. 25x19: 4x25=100, 4x4=16, 4x100=400, add 3x25(75), so 475.
<u>Division</u>		
Counting (CO)	Any form of counting strategy, skip counting forwards and backwards, repeated addition and subtraction, and halving and doubling strategies.	24÷4: 4, 8, 12, … 24÷4: half of 24, half of 12.
Basic fact (BF)	Using a known division fact or a derived fact.	24÷4: 4x?=24. 6 24÷4: 5x4=20, so 6x4=24.
LR separated (LRS)	Numbers are separated into place values, then proceed left to right.	100÷5: 10÷5=2, 0÷5=0, 20.
RL separated (RLS)	Numbers are separated into place values, then proceed right to left.	100÷5:, 0÷5=0, 10÷5=2, 20.
Wholistic (WH)	Numbers are treated as wholes.	100÷5: 100÷10=10, 10x2=20. 168÷21: 5x21=100∴5x21=105, about 60 left, 3x20=60∴3x21=63, 63+105=168, ans. 5+3=8.

Figure 4. Mental multiplication and division strategies. Heirdsfield, A.M., Cooper, T.J., Mulligan, J., & Irons, C.J. (1999). *Proceedings of the 23rd Psychology of Mathematics Education Conference*: Children's mental multiplication and division strategies. Haifa, Israel: 91.

The counting and basic fact strategies are simpler methods, consisting of skip counting and repeated addition or subtraction and recalling multiplication or division facts, respectively. Separation strategies are slightly more involved, as numbers are separated into place values and computed either left to right (division) or right to left (multiplication). Wholistic strategies are specific to multiplication problems and require more number sense, as the numbers are left in whole form and multiplied through (Heirdsfield, Cooper, Mulligan, & Irons, 1999).

Multiple factors other than which choice of operation are involved in strategy choice and overall ability to accurately compute the answer to a problem. Of course, a problem's inherent difficulty affects one's accuracy and efficiency in solving problems. The perceived difficulty of the problem may also affect accuracy and efficiency, however. The problem size effect, or the idea that the larger the sum of two numbers, the more time it will take to find the sum, does have merit (LeFevre, Sadesky, & Bisanz, 1996). Though the difficulty of the addition problems was not altered much besides having more place values, adults take longer to solve problems that have a larger sum. Recall of information would make solving such problems easier, but adults often use different procedures instead, which take more time. The larger numbers used in the problem, the greater the perceived difficulty, and, possibly, the less faith people have in their own mathematical abilities.

Self-perception of mathematical abilities and knowledge base greatly affect accuracy and efficiency of mental calculations. Gogus (2013) found that expert mathematicians conceptualize mathematics differently than novice mathematicians; experience and knowledge of what one can do mathematically change how problems are perceived and which strategies are applied. More experience usually results in a more positive perception and the application of more effective strategies, which leads to more accurate and efficient computation. Additionally, more experience may cause stronger beliefs in mathematical ability, which also has a positive effect on mental calculation (Heirdsfield, 2001).

Many studies regarding mental mathematics take place in educational settings. These are beneficial for determining the effects of curriculum and instructional activities on students' mental computation abilities. However, mental calculations do not only occur in the classroom: they occur frequently in everyday life. The central goal of education is to prepare students to be productive members of society; teaching strategies to students, or allowing students to create their own strategies, follows this goal and prepares students for those everyday mental calculations.

Mental Mathematics Instruction

It is essential to expose students to real-world situations and provide them with some concept of how to solve problems they will face day-to-day. As with instruction of strategies to encourage mental flexibility, instruction on specific mental calculation strategies for solving everyday problems is debated. Some feel direct instruction of strategies is most beneficial for exposing students to multiple strategies (Allinger & Payne, 1986; Hazekamp, 1986; Weber, 1996) while others believe direct instruction limits creativity (Caney, 2004; Davis, 2009; Hazekamp, 1986; McIntosh, 1998).

The use of instructional materials designed to improve mental computation abilities has been found effective (Weber, 1996). More information was stored in students' conceptual networks, which allowed them to more effectively retrieve procedures to solve problems. This specific instruction is based on understanding, which "can influence the methods students use to compute and can help them develop more efficient procedures" (Weber, 1996, p. 20). Allinger and Payne (1986) make a similar claim to Weber's that specific instructional time should be dedicated to developing mental computation skills. Students must be encouraged to practice mental calculations in school and given procedures to choose from in order to be successful in these computations. Otherwise, they may not know where to start when a mental computation situation arises.

Direct instruction of mental computation strategies may be effective and emphasized, but students will choose whichever method they feel most comfortable with when solving problems. They may rely completely on number sense to create a solution method rather than solve through the application of taught strategies (McIntosh, 1998). For example, students learn and practice the basic written vertical addition algorithm time and time again. Thus, when presented with an addition problem to be solved mentally it would be expected that students find the answer by visualizing this algorithm. However, students do not necessarily do this—a variety of methods may be employed. Responses to the problem “36 plus 79” are shown in Figure 5 to demonstrate such variability.

“3 and 7 are 10; “6 and 9 are 15; that’s 115”
“30 and 70 are 100; 6 and 9 are 15; that’s 115”
“36 and 80 are 116; less 1 is 115”
“36 and 70 are 106; and 9 is 115”
“79 and 6 are 85; and 30 is 115”
“79 and 21 are 100; 36 less 21 is 15; 100 and 15 are 115”

Figure 5. Mental strategies used to solve 36 + 79. McIntosh, A. Teaching mental algorithms constructively. *The teaching and learning of algorithms in school mathematics*, 1998 NCTM Yearbook, 45.

It is clear from these responses that number sense was relied on to decompose each number and combine them in ways that made sense to the students. All were accurate in their responses even though they followed different paths to the answers.

Caney (2004) facilitates the use of different mental strategies by discussing students’ thought processes. She believes such instruction encourages flexibility and gives students a sense of pride in their mathematical abilities, especially if they were able to solve the problem in a completely different way than their peers. Students discuss answers to problems when they find the correct answer and when they find an incorrect answer. When an incorrect solution is calculated, students often identify their own mistakes through describing their thought processes. Such a focus on one’s own thoughts and computation strategies promotes metacognition and moves students past the “is that right?” stage (Caney, 2004, p. 11). Concern with whether or not students have found the correct answer clouds the conceptual understanding of the problem, as well as limits the confidence a student has in his or her mathematical ability. Discussion of steps taken to solve a problem undermines the expectation of immediate feedback on the accuracy of the answer and promotes conceptual understanding (Caney, 2004, pp. 10-14). Though immediate feedback is commonplace in present day society, it will not always be available in the everyday mathematical situations students may encounter. It is important for students to change their question from “is it right?” to “does my answer make sense?” in order for them to be successful in everyday calculations.

Everyday Mental Mathematics

The opportunity to perform mental calculations is everywhere in the real world. Whether one needs to calculate change while shopping, determine the gas mileage of a car, make a recipe for six people large enough for 30, figure out the discount or tax on an item, or compute how much of a tip should be left after a meal, the ability to perform mental calculations is a necessity (French, 2008; Hope & Sherrill, 1987). The situations in which calculations are presented determine the way in which problems are processed as well as which strategies are used. This deems mental calculations “situated processes” (Naresh & Presmeg, 2012; Silver, 1985; Swingler, 2014; Thaler, 1999). Either algorithmic or recursive thinking is used, based on the way problems are processed (Mingus & Grassl, 1998). Further, social contexts and background knowledge affect the perception of problems and how they are solved as well (Guberman, 1996; Naresh & Presmeg, 2012).

According to a study by Wandt and Brown cited in the article *Characteristics of Unskilled and Skilled Mental Calculators*, approximately 75 percent of calculations made throughout a day are done mentally (Hope & Sherrill, 1987, p. 46). This study was completed before the burst in technology, but the situations in which calculations are required continue to be similar. Such situations include, but are not limited to: shopping, finding discounts, determining the amount of material needed to complete a project, adaptations of recipes, performing exchange rates, leaving tips, and calculating gas mileage (French, 2008). An average college student will find him- or herself in many of these situations on a day-to-day basis. In current times, cellphones have many types of calculators, including those specifically designed to determine tax and tips. However, pulling out a phone to perform these computations may not be the most efficient approach.

French (2008) explains performing mental calculations in a series of four steps as listed in Figure 6.

Step Number	Step Name	Purpose
1	Estimate	Determine the expected answer and process
2	Calculate	Select a strategy and solve the problem
3	Check	Decide whether the answer matches the estimated answer
4	Round	Place the answer in the context of the question

Figure 6. Steps to solving in-context arithmetic problems mentally.
French, D. (2008). Estimate, calculate, check, and round. *Mathematics in School, 37*(4), 16.

The first step is to "estimate" what might happen in the problem. Which operation will be used? What is an approximation of the answer that should be found? This not only provides a starting point for the problem, it also provides a check for the answer. The "calculate" stage then follows, in which the exact calculations are performed using the chosen method. Next, the approximation made in the “estimate” step is used to “check” whether or not that exact answer makes sense for the problem, and if the calculations were done correctly. Finally, the exact answer is changed to an appropriate format in the “round” stage; for example, if money is involved, the answer will be rounded to the nearest hundredths place (French, 2008, p. 16). This final stage is dependent on the situation in which calculations are taking place.

Each situation requiring mental calculations initiates the use of unique methods to solve problems. Calculations involving money utilize different thought processes than computing gas mileage. When spending money, more cognition than simply performing a basic operation occurs. This set of cognitive skills is called “mental accounting” and includes analyzing value, determining worth, and deciding whether or not it is plausible to spend money on the items under consideration (Thaler, 1999). Such intricate thought processes would not be relevant to other mental calculation situations like adapting recipes or determining gas mileage. Furthermore, the ability to perform such calculations depends on conceptual understanding, domain-specific knowledge, and background knowledge.

Experience is shaped by social context and the cognitive functions used in mental calculations are shaped by experience. Problems involving money are solved quite differently depending on experience. While engineers automatically approximate an answer because they have learned to do so to solve complex math problems, Brazilian street children think of how they completed the errands assigned to them by their parents (Guberman, 1996; Swingler, 2014). The more experienced engineers are able to apply the mathematics instruction they received to approximate those answers and then find the correct answer, but Brazilian street children have received very little formal mathematics instruction. Instead, they need to apply their real-life experiences. As with levels of mental flexibility and conceptual understanding, different life experiences and background knowledge can be applied to the same problem to find the correct answer. The method of reaching that answer may vary, but accuracy is not compromised.

Perceptions of mathematical problems may also vary. Brazilian street children who are used to buying items for their parents conceptualize mathematics problems in terms of money, and in wholes. They are able to perform these types of calculations on a daily basis, and thus are comfortable and accurate in their computations (Guberman, 1996, p. 1621). Metropolitan bus conductors contextualize mathematical problems in terms of their wages or the traffic return of passengers (Naresh & Presmeg, 2012, pp. 55-6). In these specific contexts, problems can be solved accurately and efficiently. If problems cannot be easily contextualized, however, it is more difficult to apply taught strategies accurately.

In our current society, many calculations can be done using calculators readily available on cellphones or cash registers. People are not held as responsible for mental computations as in the past. The purpose of this study is to assess the ability of middle school and undergraduate college students to perform typical day-to-day mental calculations, and to connect past research with the present-day computer culture.

Experimental Design

This experiment was designed to test the hypotheses that non-major undergraduate students would use more efficient mental strategies than middle school students to solve in-context arithmetic problems though both groups of students would have equal accuracy, and that both groups of students would not demonstrate confidence in their mental computation abilities. The assessment consisted of four problems presented one at a time to students in a ten-minute interview session, followed by a written demographic survey. The interviews were recorded on an iPad. The four problems consisted of one of each of the following applications: change using coin values, percent tip, percent discount, and gas mileage. The problems were developed based on real-world experiences by the researcher and chosen due to their applicability to the everyday lives of the majority of prospective participants. Responses were analyzed to determine strategies used, accuracy, and time taken; the time and accuracy factors were then evaluated to determine efficiency of strategies used. An *efficient* strategy is defined as one that produces an accurate answer while taking a short amount of time.

Participants

This study was conducted in two locations: at a comprehensive, selective, public, residential liberal arts university located in the Northeast and at a selective private Catholic middle class preparatory school (grades 5-8) also located in the Northeast. The university has an estimated enrollment of approximately 5,000 students, including about 300 graduate students. The majority of the population are residents of the university's surrounding area, and the remaining population consists of students from other areas of the state and country and foreign countries. There were approximately 40 undergraduate students enrolled in the Pre-calculus class. All students had the opportunity to participate in the study but were not required to do so. The Catholic preparatory school has approximately 46 students: 10 in fifth grade, 10 in sixth grade, 12 in seventh grade, and 14 in eighth grade. These students are from the surrounding area and must apply to the school for admission. Forty of these students attended the elementary school connected to the preparatory school. Of the students in the Pre-Calculus class, 11 participated in the study with an age range of 18-28 years (6 male, mean age 20.1 years). Ten middle school students between the ages of 12 and 14 participated as well (4 male, mean age 12.8 years) for a total of 21 participants whose mental mathematical strategies were analyzed in the study as a whole.

Design

This assessment tested participants' applications of mental strategies to solve in-context problems. It consisted of four problems, each of which was printed individually on a five inch by eight inch notecard. One problem involving change with coins, one problem involving percent tip, one problem involving percent discount, and one problem involving gas mileage was used. Figure 7 below contains the phases through which the study proceeded.

Age Group	Phase 1	Phase 2	Phase 3
Middle School: Grades 5-8	► Obtain consent form signed by parent or guardian.	► 10 minute interview with each participant. The participant will narrate his/her thought processes to solve each presented in-context question. ►Participants will shuffle the question cards and place them in random order. ► Record each interview using an Apple iPad. ► Code each interview based on strategy used.	► Distribute the middle-school level surveys for participants to complete independently, though still on site, to gain information about each student for comparisons.
Undergraduate: Years 1-3	► Obtain consent form signed by participant (ages 18 and older).	► 10 minute interview with each participant. The participant will narrate his/her thought processes to solve each presented in-context question. ►Participants will shuffle the question cards and place them in random order. ► Record each interview using an Apple iPad. ► Code each interview based on strategy used.	► Distribute the undergraduate college level surveys for participants to complete independently, though still on site, to gain information about each student for comparisons.

Figure 7. Summary of the experimental design of the study.

During each ten-minute interview the notecards were flipped upside down so the problems were not visible and mixed up by the researcher. The participants then placed them in an order of their choosing, while still upside down, and returned them to the researcher. The researcher read the problems one at a time in the order of placement and displayed the numbers on the index cards to the participant while solving. Participants were asked to narrate their thought processes as they solved the problem. The researcher gave minimal feedback about each participant's answer, but refrained from commenting on the strategy used. This process continued until all problems were answered. Once the interview was completed, the participant received the survey pertaining to his or her age group (middle school or undergraduate) and completed it in written form.

Instrument Items and Justification

Two instruments were administered: the set of notecards associated with the assessment and the follow-up demographic survey. The set of notecards were administered in a 10 minute interview, and the follow-up survey was completed individually by the student with no time limit directly after the interview was completed. Students' narrations of their thought processes for solving the problems during the interview were recorded using an iPad.

The four notecards contained four different problems representing the basic arithmetic operations of addition, subtraction, multiplication, and division. Each problem was applied to a different real-world situation, including change with coin values, percent tip, percent tax, and gas mileage. The change with coins problem assessed how students reacted to paying a larger whole dollar amount with exact change for items (see Figure 8 below). The statements in quotations were read by the researcher, and indicated numbers were written on the notecard.

Problem #1. "You're at the store and your total bill is $38.67. You hand the cashier $50.67. How much money will you get back as change?"
Display:
Total Cost: $38.67
Amount Paid: $50.67

Figure 8. Problem one of the assessment.

Students were expected to subtract the 67 cents and determine the difference between 50 and 38 using one of a variety of strategies. They could subtract 38 from 50, count up to 50 from 38, or use a decomposition strategy. Customers often pay in whole dollar amounts despite the coin value of their purchases, so this problem may have been out of the ordinary for solving for change. However, it represents a skill that allows shoppers to get rid of coins and receive bills. The next problem was based on percent tip, which is often calculated using a tip calculator on cellphones (see Figure 9).

Problem #2. "Your family goes out to dinner and the bill comes to $120.00. Your group is so large you must leave a 20% tip. How much do you pay altogether?"
Display:
Cost: $120.00
Tip: 20%

Figure 9. Problem two of the assessment.

Students could have used a strategy to find 20% of 120 and then add it to the original amount of $120.00, or multiply 120 by 1.2 to find the total cost in one step. In order to find 20% of 120, students could multiply 120 by 0.2 or visualize the pen-and-paper strategy of cross multiplication $\left(\frac{x}{120.00} = \frac{20}{100}\right)$. The third type of problem participants received involves finding the discounted price of an item, a situation that often occurs while shopping (see Figure 10).

Problem #3. "You are shopping during a sale and find a $19.99 shirt 10% off. How much will you have to pay for the shirt?"

Display:

Price: $19.99

Discount: 10%

Figure 10. Problem three of the assessment.

A ten percent discount is a common sale price that is also simple to determine. One must know to move the decimal one place to the left to find ten percent of the original value. This can be subtracted from the original value to find the sale price. Participants may also multiply 19.99 by .9, or estimate the price using $20 as the initial amount instead of $19.99 and applying either of the aforementioned strategies. The final type of problem participants encountered concerns gas mileage. Cars are often filled with gas, sometimes more than once a week, and it is helpful to calculate the gas mileage one's car gets by dividing the number of miles driven by the number of gallons of gas that filled the gas tank (see Figure 11).

Problem #4. "You've driven 275 miles and need to fill your gas tank. It takes 13 gallons to fill your tank. What kind of gas mileage was your car getting?"

Display:

Miles: 225

Gas: 15 gal.

Figure 11. Problem four of the assessment.

Participants were expected to divide 225 by 15 either using a visualization of the paper-and-pencil algorithm, by repeated addition of 15 up to 225, repeated doubling of groups of 15, or by knowing the multiplication fact $15 \times 15 = 225$. Participants may have had difficulty identifying the basic fact, or they may be cautious because 25 is not a direct multiple of 15.

Context of Problem	Targeted Arithmetic Operation(s)
1. Solving for change when paying with partial dollar amounts.	► Subtraction
2. Determining how much money to leave as a tip for the waiter/waitress.	► Multiplication ► Addition
3. Determining the discounted price of an item.	► Multiplication ► Subtraction
4. Calculating the gas mileage of a car.	► Division

Figure 12. Summary of assessment problems.

A demographic survey was then distributed to participants. Two versions of this survey were created, one for the middle school participants and another for undergraduate participants. The undergraduate and middle school level surveys may be found in Appendices D and E respectively. Each version of the survey contained the same items but have varied wording; for instance, the undergraduate students were asked how many times they go shopping or fill up their gas tank, while the middle school students were asked how many times their family goes shopping or fills up their gas tanks. Middle school students are not legally able to drive a car to go shopping, eat meals out, or buy discounted items, though they may do these things independently when out with their families or friends.

The three open-ended items contained on the survey include “How many times per week do you go shopping?” or “How many times per week do you go shopping with your family?”; “How many times per month do you usually eat meals out at restaurants?” or “How many times per month does your family usually eat meals out at restaurants?”; “How often do you shop with percent discounts?” or “How often does your family shop with percent discounts?”; and finally “How often do you usually need to fill up your gas tank?” or “How often does your family need to fill up a gas tank?” for undergraduate and middle school students respectively. These questions developed a reference for the amount of experience participants had with each of the applied situations assessed during the interview. The following three items asked for demographic information including gender identification, age, and year in school. These items were the same on both survey versions, and the information collected from them were used for result comparison. Items seven through nine consisted of Likert scales to measure participants' confidence in their mathematics abilities in general, with the use of a calculator, and without the use of a calculator. One's perception of mathematical ability may have an effect on their actual ability, so this information was also used for comparison of perception, strategy use, accuracy, and efficiency. The final item prompted participants to choose which type of problem they felt most comfortable answering during the assessment and why in order to receive a better understanding of responses. This survey provided the researcher with important information for analyzing the data collected in the study.

Methods of Data Analysis

In order to test the hypothesis, the recordings of the interviews were replayed and coded to analyze the thought processes of each participant. Strategy, number of steps used, answer, amount of time taken, and efficiency were all indicated for each response. Efficiency scores were assigned to strategies and determined by calculating the product of the mean number of steps and mean time used to solve each problem. Results were analyzed to determine patterns in strategy use and the effect of strategy type on efficiency score. Demographic surveys were distributed after each interview to collect information about each participant, including year in school, gender, experience with situations similar to those presented in the study, and confidence in mathematical ability.

Data Collection and Scoring

Due to the varied order in which questions were asked to participants, data were compiled into separate charts for each question as displayed below in Figure 13.

Participant	Strategy	Number of Steps	Answer (correct or incorrect)	Time Taken	Efficiency Score

Figure 13. Example of charts used to compile results.

Participants were coded by number and level in order to maintain confidentiality. Middle school aged participants were identified by an “M” next to their number and undergraduate college students were labeled with a “C” next to their number. Strategy was indicated using a two- or three- character code indicating the general category of strategy used. The codes are as follows in Figure 14 on the next page.

Code	Strategy
BF	Basic Fact- applying knowledge of basic addition, subtraction, multiplication, or division facts. For example, $10 + 10 + 20$ or $5 \times 7 + 35$
SA	Standard Algorithm- visualizing the standard addition, subtraction, multiplication, or division algorithm, i.e. lining numbers up based on place value and adding, using long division.
D1	Decomposition- breaking down the numbers into numbers that are more easily manipulated. For example, if the question was 57×6, the student would break down 57 into $50 + 7$ and multiply each component by 6, and then add the products.
Comp	Compensation- manipulating a number to one that is more easily used for calculations. For example, rounding $14.99 to $15 and then multiplying. The difference would be accounted for as the final solution step.
CSK	Counting Strategies- often skip counting, where students begin with a number, such as 5, and count by fives up until the desired value. Repeated doubling was also included in this strategy category.

Figure 14. Codes for strategies used to solve given problems.

Data from the surveys was also organized into separate tables for the middle school participants and the undergraduate college participants. The table is comprised of the information about age, gender, experience with each type of situation assessed in the study, and confidence scores about mathematics skills from the Likert Scale prompts indicated by each participant. Confidence scores are directly related to level of confidence—the higher the score (on a scale of one to five) the more confident the participant feels in his or her ability to perform that mathematics skill. Data was also analyzed to determine which question participants felt was the easiest and which was the most difficult.

Descriptive Statistics

Mean scores for each problem on the assessment and responses on the survey were calculated and analyzed, with five key comparisons between the data sets:

1. Age and strategy use
2. Age and accuracy
3. Age and efficiency
4. Strategy and accuracy
5. Age and confidence level

Participant responses to each individual problem on the assessment were analyzed to determine strategies used, time taken, number of steps used, accuracy, and number of attempts. Data for each of these categories was organized and compared based on age to determine differences between middle school and undergraduate students. Efficiency scores were determined for each strategy on each problem by multiplying the mean amount of time taken to solve the problem using that strategy by the mean number of steps used to solve the problem using that strategy. The lowest product was considered the most efficient and given an efficiency score of 1, with the next lowest product assigned a score of 2, and so on. Scores were then compared based on age and accuracy. Finally, the information accumulated from the surveys was organized and compared based on age, with a focus on Likert Scale confidence ratings for mathematics abilities using a pencil and paper, calculator, mental calculation, and overall mathematics skills. These confidence ratings were compared by age and overall accuracy on the instrument.

Inferential Statistics

General linear models were completed to determine statistically significant comparisons at the 0.05 significance level as a supplement of the descriptive statistics calculated for each data set (this included the mean and standard deviation). These models provided the information needed to analyze the data and determine the results of the study, as described in the following section.

Results

At the conclusion of this analysis, it was evident that the number of participants in the study was rather small. More participants would be needed for stronger conclusions relative to the hypothesis. Three major results were identified for addressing the hypothesis from the data collected in this study. They include:

1. **There is no statistical difference in accuracy between undergraduate ($\bar{x} = 6.9$) and middle school ($\bar{x} = 5.3$) students when solving in-context problems mentally (p-value = 0.065), out of 8 possible points.**
2. **Undergraduate students use more efficient strategies to solve in-context problems mentally (p-value = 0.019).**
3. **Undergraduate and middle school students were both confident in their mathematics skills on a scale of 1-5 ($\bar{x} = 4.00$ and $\bar{x} = 4.30$ for middle school and college students respectively), though confidence level did not affect accuracy (p-value = 0.298).**

The following figure contains the analyzed data used to make the comparisons that determined the results listed above. Means for time and accuracy score, percentage of participants who used the most efficient strategy and a list of the strategies that were used are displayed for each problem at each level, middle school versus undergraduate.

		Middle School	College
Problem 1 You're at the store and your total bill is $38.67. You hand the cashier $50.67. How much money will you get back as change?	Mean Time	27.8 seconds	15.09 seconds
	Strategies Used	**Basic Fact**, Standard Algorithm	**Basic Fact**, Standard Algorithm
	% Use of Efficient Strategy	50%	91%
	Mean Accuracy	1.5	2
Problem 2 Your family goes out to dinner and the bill comes to $120.00. Your group is so large you must leave a 20% tip. How much do you pay altogether?	Mean Time	80.9 seconds	54.27 seconds
	Strategies Used	**Basic Fact**, Standard Algorithm, Decomposition, Counting	Standard Algorithm, Decomposition
	% Use of Efficient Strategy	20%	0%
	Mean Accuracy	1.3	1.45
Problem 3 You are shopping during a sale and find a $19.99 shirt 10% off. How much will you have to pay for the shirt?	Mean Time	82.1 seconds	34.91 seconds
	Strategies Used	**Standard Algorithm**, Compensation	**Standard Algorithm**, Compensation
	% Use of Efficient Strategy	30%	36%
	Mean Accuracy	1.2	1.82
Problem 4 You've driven 225 miles and need to fill your gas tank. It takes 15 gallons to fill your tank. What kind of gas mileage was your car getting?	Mean Time	68.2 seconds	75.91 seconds
	Strategies Used	Decomposition, Counting	**Basic Fact**, Standard Algorithm, Counting
	% Use of Efficient Strategy	0%	36%
	Mean Accuracy	1.2	1.73

Figure 15. Descriptive statistics for middle school and undergraduate levels. Bold-facing indicates the most efficient strategy to use when solving that problem.

As displayed in the table, age (level) did not seem to determine which strategies participants applied. The undergraduate participants tended to utilize two or three strategies for each problem, and the middle school students often attempted to use these strategies as well as several others. However, participants at both levels used similar strategies, though not necessarily in the same fashion. Additionally, the use of the most efficient strategy was affected by age overall, though this effect was not necessarily apparent when each question was analyzed individually. None of the undergraduate participants used the most efficient basic fact strategy to solve problem two though 20% of middle school participants used it; none of the middle school participants used the most effective basic fact strategy to solve the fourth problem, though 36% of undergraduate students did. The largest difference in percentage of participants using the most efficient strategy was for problem one, where 50% of middle school participants recognized and applied the strategy compared to its use by 91% of undergraduate students.

Accuracy was also fairly constant for each problem, though mean accuracy scores stayed between 1.2 and 1.5 for middle school students and between 1.4 and 2 for undergraduate participants. Middle school participants had lower mean accuracy scores than undergraduates, but there was no statistical significance for overall accuracy based on level.

Overall, undergraduate students were also quicker to solve all problems than the middle school students ($\bar{x} = 45.03$ seconds- college; $\bar{x} = 64.75$ seconds- middle school). After analyzing the interactions between mean time and age level, it was noted that undergraduate participants took a lower mean amount of time to solve each individual problem than the middle school participants as well. A one-way ANOVA at the 0.05 significance level was calculated to determine if college students were significantly faster than their middle school counterparts. A p-value of 0.088 was found, indicating that in fact, they were not, despite the differences in mean times

Result 1: There is no statistical difference in accuracy when solving in-context problems mentally (p-value = 0.073).

Problems were assigned an accuracy score of 0 (did not attempt), 1 (incorrect), or 2 (correct) points. Eight accuracy points were available overall on the assessment, so participants earned an overall accuracy score out of eight points. Mean overall accuracy was determined for middle school students ($\bar{x} = 5.3$) and undergraduate students ($\bar{x} = 6.9$) and then compared using a two-sample t-test at the 0.05 significance level (p-value = 0.073). This p-value indicates that there is no significant difference in accuracy on the assessment between middle school and undergraduate students, though the mean accuracy score for undergraduate students was 1.6 points higher than that of the middle school students. This result supports the hypothesis that participants at both levels would be accurate in solving the posed problems.

Accuracy scores were then broken down and compared by level for each problem. It was determined that undergraduate students had a higher mean accuracy score on problems 1, 3, and 4, but middle school students achieved a higher accuracy score on problem 2. This problem required students to determine how much money needed to be left on a large bill for a meal at a restaurant. It could be that the middle school students were familiar with the standard algorithm for solving percent problems due to their current instruction at the time of the study.

Result 2: Undergraduate students use more efficient strategies to solve in-context problems mentally (p-value = 0.019).

Efficiency scores differed based on the individual problems and the strategies used by participants to solve them. The mean time taken to solve the problem using a specific strategy was multiplied by the number of steps taken to solve the problem using that same strategy. The products for each strategy were then ranked from lowest to highest, indicating the least amount of time and fewest number of steps were used to solve the problem with that strategy. Efficiency scores were distributed as follows in Figure 16, and strategy use occurred as depicted in Figure 16.

Problem	Strategies (Efficiency Scores)
1	1. BF- Basic Fact (1) 2. SA- Standard Algorithm (2)
2	1. BF- Basic Fact (1) 2. SA- Standard Algorithm (2) 3. D1- Decomposition (3) 4. Comp- Compensation (4) 5. CSK- Counting (5)
3	1. SA- Standard Algorithm (1) 2. Comp- Compensation (2)
4	1. BF- Basic Fact (1) 2. D1- Decomposition (2) 3. SA- Standard Algorithm (3) 4. CSK- Counting (4)

Figure 16. Distribution of efficiency scores by problem.

Each strategy used was ranked one through four for efficiency. A ranking of one indicated that that was the most efficient strategy used on the specific problem; in other words, participants who employed that strategy did so with the most accuracy in the least amount of time. As rankings increased the efficiency of the strategy decreased in accordance with the performance of the participants. Each accuracy score matches the ranking (i.e. a rank of one translated to an efficiency score of one, a rank of two translated to an efficiency score of two, and so on). As the strategies used varied for each problem, the efficiency score for each strategy varied for each problem. Specific strategies used by middle school and college students for each problem are expressed in Figure 17.

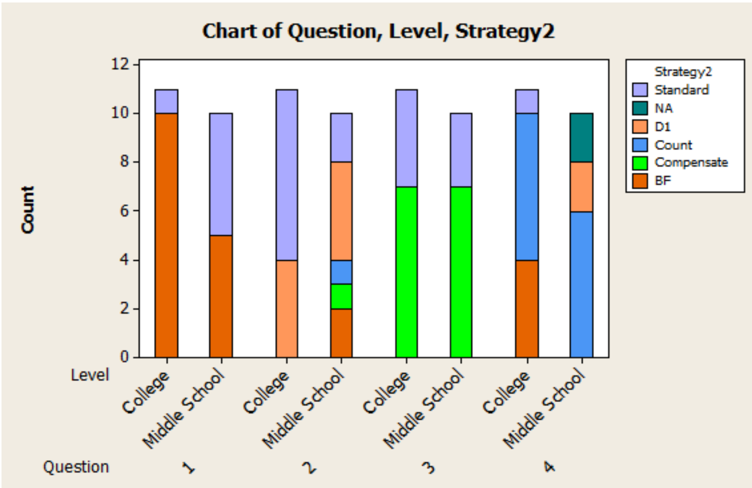


Figure 17. Strategies employed for individual problems based on level.

Notice from this figure that college students typically used one of two strategies for each problem while there was more variation in strategies used by middle school students. Further, the strategy most frequently used by participants from both levels was not always the most efficient strategy; however, college students were more likely to use the most efficient strategy. The methods used to solve problems were specific and each had certain prevalent misconceptions as well.

Overall, the undergraduate participants tended to utilize more efficient strategies than the middle school participants. This may be due to their increased amount of experience with numbers and the contexts of the problems, or with their understanding of how to manipulate the numbers to create more simple calculations.

Result 3: Undergraduate and middle school students were both confident in their mathematics skills on a scale of 1 through 5 ($\bar{x} = 4.00$ and $\bar{x} = 4.30$ for middle school and college students respectively), though confidence level did not affect accuracy (p-value: 0.298).

During the interviews, the confidence levels of participants seemed low. One participant in particular remarked "I know the process, I'm just bad at math" multiple times. The surveys distributed to students after the interviews contained Likert Scale items in which participants rated their confidence level from 1 (not confident) to 5 (very confident) in solving mathematics problems with a pencil and paper, with a calculator, mentally, and lastly their confidence in overall mathematical abilities. Participants reported that they were most confident in their ability to solve mathematics problems with a calculator ($\bar{x} = 5.0$ middle school; $\bar{x} =$4.90 undergraduate), and least confident in their ability to perform calculations mentally ($\bar{x} = 3.3$ middle school; $\bar{x} = 3.82$ undergraduate). However, their overall confidence was high ($\bar{x} = 4.30$ middle school; $\bar{x} = 4.00$ undergraduate) on the five-point scale. This did not support the hypothesis that students in both middle school and college would not be confident in their mathematical abilities. Further, there was no statistical significance between the level of confidence and accuracy, according to the general linear model calculated for the data sets (p-value = 0.298). Participants' confidence in their mathematical abilities may have increased during the interview, which could serve as an explanation for the high confidence ratings.

The strategies employed and overall performance of participants varied on each problem. Middle school and undergraduate students utilized the same or similar strategies on some problems and employed quite different strategies on others. In either case, the analysis of these responses provides data associated with the primary purpose of this study—determining how students solve in-context mathematics problems mentally.

Analysis of Assessment Problems

Each problem targeted the use of certain operations and described specific everyday situations that involve the use of mental mathematics. The responses received for each problem were as unique as the problem, though many participants utilized the same or similar solution methods as their peers when answering. Common misconceptions for each problem also occurred. Many factors may have played roles in these common solution methods and misconceptions, including, but not limited to, level, age, experience, and confidence in mathematics skills. Overall summaries of participant performance on the problems are included below, naming the common solution strategies and misconceptions for the individual problems.

Problem 1: You're at the store and your total bill is $38.67. You hand the cashier $50.67. How much money will you get back as change?

Only two strategies were used by participants to solve problem one concerning change—knowledge of the basic fact $50 - 38 = 12$ and the standard algorithm of setting 38.67 underneath 50.67 and subtracting vertically using regrouping. The majority of participants completely ignored the 67 cents, as they knew those would subtract away. The most common misconception for this problem was an answer of $22.00, because students counted up from 8 to 10 as well as from 3 to 5 without accounting for the regrouped 10 in the ones place.

Problem 2: Your family goes out to dinner and the bill comes to $120.00. Your group is so large you must leave a 20% tip. How much do you pay altogether?

College students employed either the decomposition or standard algorithm strategy to solve this question, while middle school students attempted using both of these as well as compensation, basic fact, and counting strategies. Participants decomposed $120.00 into $100 and $20 and took 20% of each of those, finding $20 and $4, so $24 was the tip. They would then add $24 to $120 to find a total of $144 left as a tip. Those who used the standard algorithm found 10% of 120 by moving the decimal point one place to the left to find $12, then doubled that for a tip of $24, and ultimately $144. Common misconceptions included mistaking $5 as 20% of $20 and $25 as 20% of $100, or adding the tip to the bill incorrectly. Some participants added $24 to $100 and found $124 to be the answer. They were able to determine the correct tip but did not use the available resources to add the tip to the correct bill.

Problem 3: You are shopping during a sale and find a $19.99 shirt 10% off. How much will you have to pay for the shirt?

The most common strategy used on this problem for undergraduate students was to round $19.99 to $20, know that $2 is 10% of $20, and subtract that $2 from the $20 to find an answer of $18.00. Many middle school students continued to employ the standard algorithm by moving the decimal point one place to the left in $19.99 to find $1.99, then subtracting $1.99 from $19.99 to determine an answer of $18.00. Three middle school participants found the answer of $18.00 and then subtracted a penny to account for the missing penny in $19.99. Of the 25 participants, 16 provided a solution to the problem. The other five described the process they would use but did not attempt to use that process because the number was not exact. Fourteen of the 16 participants found the correct answer. The two who did not determine the correct answer proceeded by subtracting 1.99 from 19.99 incorrectly, resulting in an answer that was less than the correct answer.

Problem 4: You've driven 225 miles and need to fill your gas tank. It takes 15 gallons to fill your tank. What kind of gas mileage was your car getting?

This problem was designed for participants to utilize their knowledge of division and assess to determine whether they understand the connection between division and multiplication/repeated doubling or counting. The most commonly used strategy was counting—either skip counting by 15s or repeated doubling of multiples of 15 (for example, $15 \times 4 = 60$, so students would double 60 to 120, then add 60 more to find 180, etc.). The most common problem with repeated doubling of multiples of 15 was that participants became unsure once their sum reached close to 225. Some stopped to think saying, "it's hard because it's not exact," or "I can't mentally divide 225 by 15 because the divisors aren't even." When participants who used this strategy made mistakes, it was due to the incorrect counting by 15s or by losing track of how many 15s had been accounted for. Only one participant used the standard algorithm, two knew the basic multiplication fact, and two decomposed 225 into 150 (10×15) and 75 (5×15). Two middle school participants described the process of long division but did not attempt to apply it and solve the problem.

The analysis of each individual problem provided insightful information about the mental mathematics strategies employed by participants at the middle school and undergraduate educational level. This information supported the hypothesis that middle school and undergraduate students had similar accuracy on each of the problems, though undergraduates employed more efficient strategies to solve each problem. These older students demonstrated more experience with the types of situations used to design the assessment problems. In order to gather supporting information for this indication, the surveys asked participants to approximate the number of times they experienced each type of situation per month. The analysis of survey results further explores potential explanations for the participants' responses to the problems.

Analysis of Survey Results

All survey data was collected in an Excel table. The mean values or distribution of responses separated by college and middle school level are indicated in Figure 18.

Survey Item	Middle School	Undergraduate
Age	$\bar{x} = 12.8$ years	$\bar{x} = 20.1$ years
Gender	M: 4 F:6	M: 6 F: 5
Cash	$\bar{x} = 12.2$ times/month	$\bar{x} = 6.7$ times/month
Eat	$\bar{x} = 8.0$ times/month	$\bar{x} = 5.7$ times/month
Discounts	$\bar{x} = 9.1$ times/month	$\bar{x} = 5.4$ times/month
Gas	$\bar{x} = 16.7$ times/month	$\bar{x} = 3.6$ times/month
Easiest	1(6); 2(2); 3(1); 4(1)	1(6); 2(2); 3(1); 4(2)
Hardest	1(0); 2(3); 3(2); 4(5)	1(0); 2(3); 3(1); 4(7)
Pencil and Paper	$\bar{x} = 4.5$	$\bar{x} = 4.5$
Calculator	$\bar{x} = 5.0$	$\bar{x} = 4.9$
Mental	$\bar{x} = 3.3$	$\bar{x} = 3.8$
Overall	$\bar{x} = 4.3$	$\bar{x} = 4.0$
Overall Accuracy	$\bar{x} = 5.3$ points	$\bar{x} = 6.9$ points

Figure 18. Summary of survey data by educational level of participants.

The rows labeled "Cash," "Eat," "Discount," and "Gas" hold data for the mean number of times each participant or each participant's family shopped using cash, ate meals at restaurants, shopped for discounts, and filled their gas tank per month. Rows labeled "Easiest" and "Hardest" indicate which problems the participants thought were the easiest and most difficult to solve. Mean Likert Scale ratings for confidence in calculations with "Pencil and Paper," with a "Calculator," without technology ("Mental"), and for "Overall" mathematics abilities are listed in the rows with those names. Finally, mean overall accuracy scores out of eight points are indicated in the row labeled "Overall Accuracy."

The majority of participants at both levels rated the first problem as the easiest and the fourth problem as the most difficult. The first problem required subtraction, while the fourth problem required division, so this ranking makes sense. This follows the mean accuracy scores calculated for each problem—accuracy was highest for problem one at both levels ($\bar{x} = 1.5$ for middle school; $\bar{x} = 2$ for undergraduate), and mean accuracy score was lowest for middle school participants for problem four ($\bar{x} = 1.2$). Undergraduate participants, on the other hand, scored the lowest on the second problem ($\bar{x} = 1.45$); the mean accuracy score on the fourth problem was the second highest of all four problems ($\bar{x} = 1.73$).

Low levels of experience with the situations used to contextualize the assessment problems are evident for the middle school participants when reviewing the responses to those items on the surveys. Some indicated that their parents fill their gas tank more than 10 times per month; the mean indicated number of times middle school students' families filled their gas tank was 16.7 times per month. While this may occur, it seems unlikely that a car tank would need to be filled that many times in one month. This implies that students most likely solved the problems by associating the problems with information they have learned during mathematics instruction, rather than through life experience. Since the study was designed to isolate drastic differences in life experience, such implications support that design.

Implications for Teaching

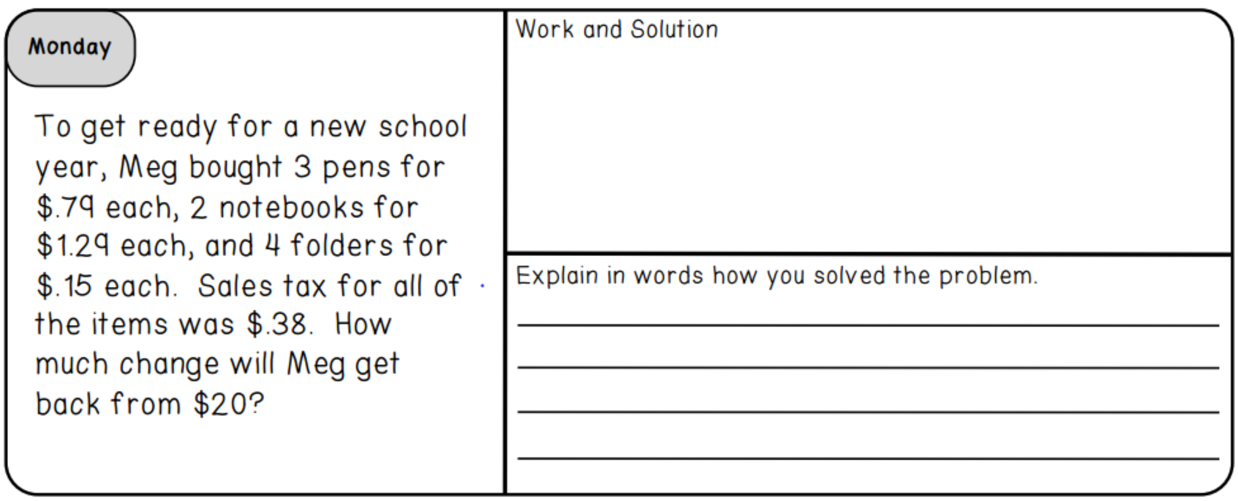
The design of this study was based on a desire to gain more knowledge about the thought processes students use when solving mathematics-based problems mentally. Furthermore, these problems were written in specific everyday contexts in order to analyze the effect experience has on efficient and accurate mental calculations. Participants included middle school students and undergraduate college students; a fairly dramatic range of ages to truly isolate the effect of experience. To further this research, high school students and adults above the age of 25 could be included as well. The inclusion of these age groups would expand understanding of the development of mental mathematics skills and give more information about the effect of experience on those skills. Results of this study revealed that both middle school and undergraduate students were able to accurately solve the posed questions but undergraduate students employed more efficient strategies to do so.

As light was shed on these research questions through the study procedures, academic needs for students to more successfully solve such everyday problems surfaced and developed into implications for teaching. The majority of middle school students relied on the standard algorithm to determine their answer while undergraduate students manipulated numbers to make calculations more simplistic. Students who used the standard algorithm took more time and used more steps to find their answers than those who applied an alternate method. These observations indicate that instructional methods need to focus more on problem solving and encouraging students to approach problems from different perspectives. More specifically, the implications for teaching include:

1. Problem solving activities need to be incorporated in the classroom on a regular basis.

Students of this generation have the world at their fingertips with technology. Perseverance is not a necessary skill when they need to find an answer—they can simply type the question into an internet browser and have hundreds of thousands of results in less than a second. If they are confused about how to solve a mathematics problem they can enter what they know into a calculator and have an answer generated for them just as quickly. This easy access to information is wonderful, but it is also limiting. In the study, many middle school students insisted on using the standard algorithm strategy to solve most problems though other, easier methods would have solved the problem as well. A push to expand the mind and think outside of the box is needed in preparation for efficiency in real world situations.

Problem solving activities are easy to find and implement in the classroom. Simple Internet or Pinterest.com searches result in a plethora of resources within seconds. Websites such as TeachersPayTeachers.com also contain numerous problem solving activities. Common problem solving questions such as the one in Figure 19 can be displayed on the board or distributed in paper form to all students at the beginning of the week, the day, or the class period. This set of problems was discovered on TeachersPayTeachers.com downloaded for free. Its intended audience is sixth graders, but problems could be adapted for any grade level.


Monday

To get ready for a new school year, Meg bought 3 pens for $.79 each, 2 notebooks for $1.29 each, and 4 folders for $.15 each. Sales tax for all of the items was $.38. How much change will Meg get back from $20?

Work and Solution

Explain in words how you solved the problem.

Figure 19. Sixth grade example of a problem solving activity. Math on the Move. (2016). *Daily problems for middle school math: Decimal and fraction mini-set.*

This problem was designed for students to solve individually—in fact, it is the first problem in a three-week set of daily problems that is printed in packet format and distributed to all students. Students begin their school day or mathematics class by completing the designated problem individually, though educators adapt the resource as best fits their classroom and the learning styles of the students.

The example problem does not require extra resources, though problem solving activities may be designed so they do require other resources. Activities may also be designed so students must work in collaborative groups as opposed to individually. Other than posing the problems and helping to guide thinking if needed, problem solving tasks are student-centered and hands-off for teachers. A weekly or daily problem solving activity like the one above, would give teachers time to complete their own work while students continue to engage in learning.

The purpose of the inclusion of problem solving tasks on a regular basis is to provide opportunities for students to apply their own experiences and background knowledge to questions that concern mathematics. In the current curriculum, students are required to learn different strategies for solving problems and are often told which strategy to use and when. Again, this diminishes the need to persevere through problems, unless students truly do not understand the assigned strategy. There need to be opportunities for students to develop their own strategies in classrooms. Students need to learn the age-old saying "if at first you don't succeed, try, try again." Such instances where students work through a confusing or difficult problem provide the most rewarding feelings when a solution is discovered. While finding time to include out of curriculum activities is difficult, it is necessary to prepare students for further education and to build their self-confidence in academic, and mathematical) abilities.

Problem-based learning often maintains higher levels of student engagement than the traditional style of instruction. Problem solving should be an important skill highlighted in the classroom, not only for activities but also for assessment. New concepts are introduced in terms of problems that require students to utilize the skills they learn to master in the associated unit. This approach provides students with hands-on learning opportunities, experience discovering content, and a sense of ownership in learning. Students understand that meaningful learning occurs when they persevere to solve problems rather than obtain answers from the teacher. A simple way to introduce more problem-based learning in the classroom is to assign a Problem of the Week each Monday. Depending on the level of difficulty and depth of the problem, allow students to either work individually or in assigned groups during specific work times throughout the week. On Friday, hold a whole-class discussion of the problem and the various solution methods the students employed to solve the problem. Implementation of a discussion encourages students to understand their peers' thought processes, which in turn facilitates mental flexibility. Have the class as a whole generate an outline of the most effective and accurate method of solving the problem of the week to complete the class discussion. Then students are not only required to problem solve in these activities but also to collaborate with their peers: another skill that should be reinforced in the classroom.

2. Encourage students to engage in collaboration.

Many current instructional methods focus on collaboration, such as the Kagan Cooperative Learning strategies. Such strategies provide templates for teachers to utilize when organizing their classroom so students are set up in mixed-ability groups, along with protocols for cooperative learning activities in which students work with peers in their group to complete a task together through collaboration of knowledge and skills (Kagan, 2016). These should be utilized! Students learn more through conversation and thinking about how their peers solved the same problem. At the beginning of each interview given in this study, an example question was read and solved by the interviewer. Often the participants commented on whether or not they had thought of solving the problem that way—some remarked "That is exactly how I was going to solve it!" or "I never would have thought to do that, but it seems easier." It is common to learn a single strategy well and employ it for any given problem; however, that strategy may not be the most efficient method to solve every problem. As students expand their problem solving skills through regularly assigned problem solving tasks, they should simultaneously expand their collaboration skills by discussing solution methods with their peers. This can be done as a whole group activity or through the placement of students in small groups based on ability. In either case, students at any age should learn to listen to their peers and be able to explain their own solution methods.

Some favorite Kagan Cooperative Structures include *Think-Pair-Share* and *Numbered Heads Together*. Begin a lesson with a *Think-Pair-Share* problem. This problem may be as simple or as complex as you like and could be related to the content of the lesson and act as a hook or could be unrelated to the content and act as an activity to kick-start student thinking. Students must solve the problem individually, then turn to a partner to share their solution, and finally discuss the problem as an entire class. Design heterogeneous-ability groups of four or five students, designating numbers to students in each group. Distribute problem sets and have groups work through them collaboratively for a certain amount of time, then ask all “ones” to break into their own group and discuss a certain problem, all “twos” to discuss another problem as a group, and so on. Students collaborate to solve all of the problems and then collaborate again to discuss their group’s solution methods with other groups. *Numbered Heads Together* activities require more planning, while *Think-Pair-Share* activities can be implemented last-minute and do not require much planning. These and many other activities that foster cooperation and discourse are effective methods for encouraging collaboration.

3. Spend time practicing and applying mental mathematics skills.

Technology of some kind exists in any context one might possibly experience in today’s society. This has decreased the level of necessity of mental mathematics skills when shopping, getting gas, or paying for meals eaten at restaurants, but it has not necessarily decreased the necessity of having those mental mathematics skills. Results from the surveys indicated that middle school students do not have much experience with the situations posed in the questions from the study. Some indicated that their parents fill their car's gas tank 30 to 40 times a month—this misunderstanding affects the ability to solve the mathematical problem associated with getting gas. Awareness of everyday activities that require mathematics skills needs to be built while practicing mental mathematics skills.

Simulations of shopping or paying for meals at restaurants in the classroom are fun, interactive ways to develop mental mathematics skills in context. The development of a classroom economy complete with jobs and incomes, rent, and taxes, fosters the real-world mathematics environment of everyday living. Students so often wonder "when in my life will I ever use this?" A classroom economy with connections to the mathematical concepts being taught in class builds the students' experience and shows them where they will use that concept. Rafe Esquith (2007) describes a very successful classroom economy setup in his book *Teach like Your Hair is on Fire*. Build the system into the classroom procedures to ensure that time is spent practicing the targeted skills.

On a less formal level, quick games of *Around the World* are always fun and interactive for developing basic skills rather than applied skills. Begin the year with mental mathematics practice of those basic skills and gradually incorporate context. These activities can directly follow any problem solving activity, or serve as a "warm up" for problem solving. My sixth grade mathematics teacher had *Around the World* competitions every Friday afternoon where we could win peanuts for each question we answered correctly. He kept track of how many questions we answered correctly week by week to track progress. While it is not plausible to give peanuts away as prizes anymore, the competitive edge and the self-satisfaction that came from answering more questions correctly than the week before were enough to keep us all engaged and excited for those Friday afternoons. All of that teacher's students had more developed mental mathematics skills as well, and nobody even realized that learning was taking place. Institute some sort of activity like this in your classroom and hold competitions weekly, biweekly, monthly, or at whatever interval suits your classroom best. Switch or incorporate more topics as well—basic multiplication or division facts, exponents, solving simple equations, deriving equations, and many more are all skills students must learn to master. These competitions provide data about what the students do and do not know; use the data wisely and to design instruction.

Concluding Remarks

Overall, the purpose of this study was to analyze the thought processes used by students to answer in-context problems concerning the basic operations mentally, and to compare that information for middle school students and undergraduates. The data gained from the study indicates that students of all ages need more practice with mental mathematics. This can start as young as elementary school, where the foundations of mathematics skills are laid. It is true that time is difficult to assign to any activities other than those written in the curriculum, it is necessary to take some time on a regular basis and focus on developing mental mathematics skills. This will set students up for success as future members of society.

References

Allinger, G.D., & Payne, J. N. (1986). Estimation and mental arithmetic with percent. In H. L. Schoen & M. J. Zweng (Eds.), *1986 Yearbook: National Council of Teachers of Mathematics.* Reston, VA: NCTM.

Beishuizen, M., & Anghileri, J. (1998). Counting, chunking, and the division algorithm. *Mathematics in School, 27*(1), 2-4.

Beishuizen, M., van Putten, C. M., & Van Mulken, F. (1997). Mental arithmetic and strategy use with indirect number problems up to one hundred. *Learning and Instruction, 7*(1), 87-106.

Blöte, A.W., Klein, A.S., & Beishuizen, M. (2000). Mental computation and conceptual understanding. *Learning and Instruction, 10*(3), 221-247.

Caney, A. (2004). Numbers + magic = answer. Students explaining: Make the most of mental computation. *Australian Primary Mathematics Classroom*, *9*(3), 10-14.

Childers, T. L., Houston, M. J., & Heckler, S. E. (1985). Measurement of individual differences in visual vs. verbal information processing. *Journal of Consumer Research, 12*(2), 125-134.

Davis, S. (2009). Oral and mental mathematics. *Mathematics Teaching*, 215, 45-48.

Esquith, R. (2007). *Teach like your hair's on fire*. Penguin Books.

Fennema, E., Carpenter, T.P., Jacobs, V. R., Frank, M.L., & Levi, L.W. (1998). A longitudinal study of gender differences in young children's mathematical thinking. *Educational Researcher, 27*(5), 6-11.

French, D. (2008). Estimate, calculate, check, and round. *Mathematics in School, 37*(4), 15-18.

Fürst, A. J., & Hitch, G. J. (2000). Separate roles for executive and phonological components of working memory in mental arithmetic. *Memory & Cognition, 28*(5), 774-782.

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., ... & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130-162.

Gogus, A (2013). Evaluating mental models in mathematics: a comparison of methods. *Educational Technology Research and Development, 61*(2), 171-195.

Guberman, S. R. (1996). The development of everyday mathematics in Brazilian children with limited formal education. *Child Development, 67*(4), 1609-1623.

Hazekamp, D. W. (1986). Components of mental multiplying. In H. L. Schoen & M. J. Zweng (Eds.), *1986 Yearbook: National Council of Teachers of Mathematics.* Reston, VA: NCTM.

Heirdsfield, A. M. (2001). Integration and compensation in accurate mental computation. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and Beyond, 2*(1), 273-280.

Heirdsfeld, A. M., & Cooper, T. J. (2004). Factors affecting the process of proficient mental addition and subtraction: case studies of flexible and inflexible computers. *Journal of Mathematical Behavior, 23*(1), 443-463.

Heirdsfield, A. M., Cooper, T. J., Mulligan, J., & Irons, C. J. (1999). Children's mental multiplication and division strategies. *Proceedings of the 23rd Psychology of Mathematics Education Conference.* Haifa, Israel.

Hickendorff, M., van Putten, C. M., Verhelst, N. D., & Heiser, W. J. (2010). Individual differences in strategy use on division problems: Mental versus written computation. *Journal of Educational Psychology, 102*(2), 438-452.

Hope, J. A., & Sherrill, J. M. (1987). Characteristics of unskilled and skilled mental calculators. *Journal for Research in Mathematics Education, 18*(2), 98-111.

Heuvuel-Panhuizen, M. (2001). Realistic mathematics education as work in progress. [PDF Document]. Retrieved from http://www.fisme.science.uu.nl/staff/marjah/documents/Marja_Work-in-progress.pdf

Kagan, S. (2016). About kagan publishing & professional development. *Kagan Online*.

LeFevre, J.A., Sadesky, G. S., & Bisanz, J. (1996). Selection of procedures in mental addition: Reassessing the problem size effect in adults. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 22*(1), 216-230.

McIntosh, A. (1998). Teaching mental algorithms constructively. *The teaching and learning of algorithms in school mathematics*, 1998 NCTM Yearbook, 44-48.

Miller, H., & Bischel, J. (2004). Anxiety, working memory, gender, and math performance. *Personality and Individual Differences, 37*(3), 591-606.

Mingus, T. T., & Grassl, R. M. (1998). Algorithmic and recursive thinking: Current beliefs and their implications for the future. *The teaching and learning of algorithms in school mathematics*, 1998 NCTM Yearbook, 32-43.

Naresh, N., & Presmeg, N. (2012). The role of context-related parameters in adults' mental computational acts. *PNA, 6*(2), 51-60.

Nisbett, R. E., & Wilson, T. D. (1977). Telling more than we can know: Verbal reports on mental processes. *Psychological Review, 84*(3), 231-259.

Nys, J., & Content, A. (2010). Complex mental arithmetic: The contribution of number sense. *Canadian Journal of Experimental Psychology, 64*(3), 215-220.

Pittsburgh University Learning Research and Development Center. (1990). *Treating estimation and mental computation as situated mathematical processes* (1st ed.). Pittsburgh, PA: Silver.

Rickard, T. C. et. al. (2000). The calculating brain: An fMRI study. *Neuropsychologia, 38*(3), 325-335.

Simon, H. A. (1979). Information processing models of cognition. *Annual Review of Psychology, 30,* 363-396.

Silver, E. A. (Ed.). (1985). *Teaching and learning mathematical problem solving: Multiple research perspectives.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Swingler, D. N (2014). A rule-of-thumb approximation for time value of money calculations. *Journal of Personal Finance, 13*(2), 57-61.

Thaler, R. H. (1999). Mental accounting matters. *Journal of Behavioral Decision Making, 12*(3), 183.

Thompson, I. (1999). Getting your head around mental calculation. *Issues in Teaching Numeracy in Primary Schools.* Open University Press: Buckingham. 145-156.

Threlfall, J. (2002). Flexible mental calculations. *Educational Studies in Mathematics, 50*(2), 29-47.

Weber Jr, W. B. (1996). *Paper presented at the Annual Meeting of the American Educational Research Association:* Filling in the Gaps: An Experimental Study on Mental Computation Achievement and Strategies. New York, NY, 1-27.

Yackel, E. (2001). Explanation, justification, and argumentation in the mathematics classroom. *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education.* Utrecht, The Netherlands.

Zarnhofer et. al. (2012). The influence of verbalization on the pattern of cortical activation during mental arithmetic. *Behavioral and Brain Functions, 8*(1), 13-27.

Appendix A
Letter and Consent Form for Middle School Participants

Dear Parents/Guardians and Students,

My name is Ellie Brion and I am currently the Mathematics Specialist at Northern Chautauqua School. I am also working on my Master's degree in Mathematics Education at the State University of New York at Fredonia. My name may seem familiar, as I student taught in Mrs. Long's classroom at the beginning of 2015.

I am writing because I have to complete a project for my Master's Thesis and I was hoping to be able to have your children participate in it. I will be studying mental mathematics and the strategies used by students of different ages to solve problems related to real-world situations. I am also asking my undergraduate students at Fredonia to participate. My project will require about fifteen minutes of your child's time sometime during the school day, and will take place at NCCS.

Your child's participation will consist of:

- A ten minute interview with me that will be audio-recorded
- Completion of a survey with questions about the assessment and the student's age, gender, and feelings about mathematics.

During the interview, each participant will be asked four questions related to real-world situations. These include calculating tips, discounts, gas mileage, and change. Participants will be read a question and shown the numbers and asked to solve the problem mentally, narrating their thought processes as they solve. Their narrations will be recorded on an iPad. They will then be asked to complete the survey at their own pace.

All data collected during this study will remain anonymous and safely locked in my office in Fenton Hall at Fredonia. Further, it will be destroyed as soon as possible once the study is complete.

No part of this study will affect your child's grade or education in any way. This is a topic that I have been interested in since I was in fourth grade and my father had me mentally calculate the change when we went shopping. If I found the correct change he gave me the coins to keep, and one day I made almost twenty dollars in change! I am hoping to learn a lot about how students solve real-world problems mentally, and would greatly appreciate the participation of your child in my project.

Thank you for your time and consideration!

Sincerely,

Ellie Brion

Parental Consent Form

State University of New York at Fredonia

Your child's participation in this important study would be greatly appreciated. Please print and sign your name below to indicate your agreement to allow your child to participate. Feel free to keep a copy of the letter for your records. Thank you for your full consideration regarding this request!

Voluntary Consent: I have read this memo and have been given all important information about this study. My signature indicates that my child has permission to participate in this study. I understand that no part of my child's education will be affected by this study, and that I may withdraw him/her at any time. I understand that if I have any questions about this study I may contact Elizabeth Brion at 585-727-7042 or brio4228@fredonia.edu.

Please return this original, completed consent form as soon as possible. Thank you for your cooperation and consideration!

Parent/Guardian Name (please print):____

Parent/Guardian Signature:____

Date:___________________

Parent/Guardian Contact Information (not required):

Phone:____________________________

Email:_____________________________

Appendix B
Letter and Consent Form for Undergraduate College Students

Dear Students,

My name is Ellie Brion and I am currently a Teaching Assistant for Pre-calculus classes at Fredonia University. I am working on my Master's degree in Mathematics Education at Fredonia.

I am writing because I have to complete a project for my Master's Thesis and I was hoping to be able to have your children participate in it. I will be studying mental mathematics and the strategies used by students of different ages to solve problems related to real-world situations. I am also asking my undergraduate students at Fredonia to participate. My project will require about fifteen minutes of your time sometime during the semester.

Your participation will consist of:

- A ten minute interview with me that will be audio-recorded
- Completion of a survey with questions about the assessment and your age, gender, and feelings about mathematics.

During the interview, each participant will be asked four questions related to real-world situations. These include calculating tips, discounts, gas mileage, and change. Participants will be read a question and shown the numbers and asked to solve the problem mentally, narrating their thought processes as they solve. Their narrations will be recorded on an iPad. They will then be asked to complete the survey at their own pace.

All data collected during this study will remain anonymous and safely locked in my office in Fenton Hall at Fredonia. Further, it will be destroyed as soon as possible once the study is complete.

No part of this study will affect your grade or education in any way. This is a topic that I have been interested in since I was in fourth grade and my father had me mentally calculate the change when we went shopping. If I found the correct change he gave me the coins to keep, and one day I made almost twenty dollars in change! I am hoping to learn a lot about how students solve real-world problems mentally, and would greatly appreciate your participation in my project.

Thank you for your time and consideration!

Sincerely,

Ellie Brion

Participant Consent Form

State University of New York at Fredonia

Your participation in this important study would be greatly appreciated. Please print and sign your name below to indicate your agreement to participate. Feel free to keep a copy of the letter for your records. Thank you for your full consideration regarding this request!

Voluntary Consent: I have read this memo and have been given all important information about this study. My signature indicates that I have agreed to participate in this study. I understand that no part of my education will be affected by this study, and that I may withdraw myself at any time. I understand that if I have any questions about this study I may contact Elizabeth Brion at brio4228@fredonia.edu.

Please return this original, completed consent form as soon as possible. Thank you for your cooperation and consideration!

Participant Name (please print):____

Participant Signature:____

Date:___________________

Appendix C

Assessment Problems

Each question will be printed on a 5x8 inch notecard. What is in quotations will be read for each question. They may be read in a different order than shown below.

Problem #1. "You're at the store and your total bill is $38.67. You hand the cashier $50.67. How much money will you get back as change?"

Printed on the notecard: Bill: $38.67

Amount Given: $50.67

Problem #2. "Your family goes out to dinner and the bill comes to $120.00. Your group is so large you must leave a 20% tip. How much do you pay altogether?"

Printed on the notecard: Bill: $120.00

Percent Tip: 20%

Problem #3. "You are shopping during a sale and find a $19.99 shirt 10% off. How much will you have to pay for the shirt?"

Printed on the notecard: Price: $19.99

Discount: 10% off

Problem #4. "You've driven 275 miles and need to fill your gas tank. It takes 13 gallons to fill your tank. What kind of gas mileage was your car getting?"

Printed on the notecard: Number of Miles Driven: 225

Number of gallons bought: 15

Appendix D
Undergraduate Level Survey

Please answer the following questions.

What is your:
Age?
Year in school?
Identified gender? M F

About how often (per month) do you do the following?

Shop with cash

Go out to eat

Shop for discounts

Put gas in your car

Which of the questions from the interview did you feel most comfortable with?

Which of the questions from the interview did you feel the least comfortable with?

How confident do you feel in your mathematical abilities using a pencil and paper?

Not Very 1 2 3 4 5 Very

How confident do you feel in your mathematical abilities using a calculator?

Not Very 1 2 3 4 5 Very

How confident do you feel in your mental mathematical abilities?

Not Very 1 2 3 4 5 Very

How confident do you feel in your mathematical abilities overall?

Not Very 1 2 3 4 5 Very

Appendix E
Middle school level survey

Please answer the following questions.

What is your:

Age?
Year in school?
Gender? M F

About how often (per month) does your family do the following?

Shop with cash

Go out to eat

Shop for discounts

Put gas in your car

Which of the questions from the interview did you feel most comfortable with?

Which of the questions from the interview did you feel the least comfortable with?

How confident do you feel in your mathematical abilities using a pencil and paper?

Not Very 1 2 3 4 5 Very

How confident do you feel in your mathematical abilities using a calculator?

Not Very 1 2 3 4 5 Very

How confident do you feel in your mental mathematical abilities?

Not Very 1 2 3 4 5 Very

How confident do you feel in your mathematical abilities overall?

Not Very 1 2 3 4 5 Very