State University of New York at Fredonia
Department of Mathematical Sciences

CERTIFICATION OF PROJECT WORK

We, the undersigned, certify that this project entitled Eliminate the Substitution or Substitute the Elimination? A Study of Methods Used for Solving Systems of Equations in High School Mathematics Classrooms by Ricardo S. Hammond, Candidate for the Degree of Master of Science in Education, Mathematical Sciences, is acceptable in the form and content and demonstrates a satisfactory knowledge of the field covered by this project.

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ELIMINATE THE SUBSTITUTION OR SUBSTITUTE THE ELIMINATION? A STUDY OF METHODS USED FOR SOLVING SYSTEMS OF EQUATIONS IN HIGH SCHOOL MATHEMATICS CLASSROOMS

by

Ricardo S. Hammond

A Master's Project
Submitted in Partial Fulfillment
of the Requirements for the Degree of
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Abstract

This research examines the different methods used for solving systems of equations, how those methods are taught, and how they can be applied to real world situations. More specifically, this research examines which of these methods student tend to favor, as well as whether or not students can properly apply the concept of systems of equations to real world situations.

It is hypothesized that high school algebra students will use substitution over elimination when solving systems of equations, and that students who have not been previously introduced to any method will naturally use "guess-and-check." Furthermore, it is hypothesized that students' general approach to solving systems of equations is "procedural," causing them to score higher on algebraic-type problems than on word problems.

It was determined that students who had already been instructed on solving systems of equations were likely to favor substitution over other methods. It was also concluded that students who had not yet been instructed on solving systems of equations had a tendency to favor guess-and-check over other methods. Furthermore, students performed better on the algebraic problems than the word problems. Many students approached the word problems differently than how they approached the algebraic problems. Regardless of the methods used in the algebraic problems, many students abandoned those methods when attempting the word problems.

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Introduction

This research examines different methods of solving systems of equations by high school students, how these methods are taught, and how students apply the concept to non-algebraic contexts. More specifically, this research examines different algebraic methods for solving systems of equations in an attempt to determine which of these methods is most often taught, evaluate the pro and cons of each method, and determine which method is most preferred by students. For this study, a "system of equations" will be defined as two linear equations with two variables. Figure 1 shows an example of a system of equations.

$$3x + y = 14$$
$$y = 2 + 3x$$

Figure 1. An example of a system of equations.

It is a common belief that many areas of mathematics are taught in "procedural" ways that do not give students a truly deep understanding of what is happening mathematically. Teaching methods for systems of equations are often criticized in this manner. Students often resort to recalling a specific series of steps when solving these types of problems, often missing out on the "big picture" and losing an opportunity to appreciate what is actually happening mathematically. Furthermore, it seems that students often resort to favoring certain methods for solving these problems regardless of the way the problems are presented. Students may actually make the problem more difficult in order to use a preferred method, which may or may not be a method they

learned in class. This study seeks to determine whether or not these perceptions are true, as well as determine whether or not adjustments need to be made in New York State's mathematics curriculum.

I was interested in this topic because I currently teach courses that are centered on the Integrated Algebra curriculum. This curriculum requires that students be able to effectively solve systems of equations. More specifically, students are required to solve these problems in a specific way when directed to do so. It appears that algebraic methods are required the most often along with graphing methods. Because algebraic methods appear to be the leading requirement, I am interested in exploring the different algebraic methods for solving systems of equations in order to determine which methods are best for specific situations, which methods are preferred by students, and how well students are at applying those methods to non-algebraic contexts.

It is for this reason that I am also interested in exploring non-algebraic methods for solving linear systems in order to determine if the curriculum's current "algebra-heavy" approach is appropriate and beneficial for students. During my years of teaching I have noticed that students often prefer certain algebraic methods over others, usually to the point where they make the problem more difficult in order to use the "appropriate" method. Furthermore, I regularly find myself wondering if my students are gaining a truly deep understanding of what linear systems are and what they can be used for in the real world. I often question my methods, contemplating whether my approach is too "procedural" - enough to pass an exam without really learning anything. This has prompted me to not only question my own methods, but the curriculum in general.

My study focuses on determining which methods students prefer, how well they can apply those methods to other contexts, and whether or not they are benefiting from the current curriculum requirements regarding systems of equations.

> *It is hypothesized that high school algebra students will use substitution over elimination when solving systems of equations, and that students who have not been previously introduced to any method will naturally use "guess-and-check." Furthermore, it is hypothesized that students' general approach to solving systems of equations is "procedural," causing them to score higher on algebraic-type problems than on word problems.*

This hypothesis was tested with several high school Algebra, Geometry, and Business Math students. Some students had been taught different algebraic methods for solving systems of equations, while other students had not been introduced to systems of equations at all. Students were given a quiz with four problems that involve systems of equations. Some systems are biased with respect to a specific method while others are less partial. Evaluation of the quiz determined if students prefer certain methods and what those methods are. One aspect of the quiz evaluated students' abilities to apply the concept of systems of equations to other contexts, as well as make connections between various methods. The end of the quiz consists of a survey where students were asked more direct questions regarding the methods they preferred, as well as the difficulty of the problems.

Students who had not yet been instructed on systems of equations were given the same quiz. Since they had not been instructed on any of the "traditional" methods for solving linear systems, the evaluation consisted of determining what methods students were able to come up with on their own.

The next section focuses on research regarding different methods of instruction used when teaching the concept of systems of equations. More specifically, analysis will focus on the different methods used for solving systems of equations, how those methods relate to one another, and the approaches used by current textbooks and curriculum.

Literature Review

The purpose of this literature review is to examine the existing research pertaining to systems of linear equations, how they are taught in high school classrooms, and the applications that can be used in instruction. A more specific purpose is to examine methods used for solving systems of equations. The goal is to determine the most popular methods used in classrooms, evaluate their pros and cons, and determine which method is preferred among high school students. This literature review includes four subheadings. The first section is entitled "Applications, Hooks, and Projects," and discusses examples that are concrete and help engage students' interest. This section is followed by "Empirical Research, Data Collection, and Misconceptions," which defines and compares different methods for teaching systems of equations and evaluates their pros and cons. Following this section is "Current Textbook Approaches," which analyzes the most commonly used methods among current textbooks used in schools, as well as specific content textbooks are lacking. The fourth and final section, "Standards and

Assessments," defines and discusses the skills that students are required to obtain according to the New York State curriculum, as well as how these skills are assessed on Regents exams.

Applications, Hooks, and Projects

Many argue that the beauty of systems of equations is often undervalued in classrooms and that students often miss out on real-world applications as well as the connections among different methods used for solving them. Many teachers have begun to incorporate real-world applications, puzzles, projects and other types of "hooks" that successfully engage students in a way that gives them the opportunity to appreciate systems of equations in a less "procedural" way. Students have a great ability to situate mathematics in interesting and humorous, real-life contexts through their problem posing. Students will not naturally use variables. They may need to be forced to use them. The following section is devoted to a discussion on strategies teachers can use to provide students with a deeper appreciation for systems of equations as well as a detailed description of some specific examples of useful "hooks" (Berner, McLaughlin, & Verzoni, 1997; Reeves, 2000).

The Chicken Problem

The Chicken Problem (Reeves, 2000) is a simple yet effective way to demonstrate how systems of equations can be used in realistic contexts. It is an especially effective application because it can be solved in a variety of ways. Students can solve the problem using their own approaches, then compare and form connections between those different

approaches. While this problem is not exactly "hands-on," it provides students with a situation that is appealing to them. This particular application is nice because of its simplicity. As seen in Figure 2, it is a four-framed picture with limited numbers and words. It is this simplicity that, for many students, makes the problem "appear" less difficult, causing them to be more apt to genuinely attempt it.

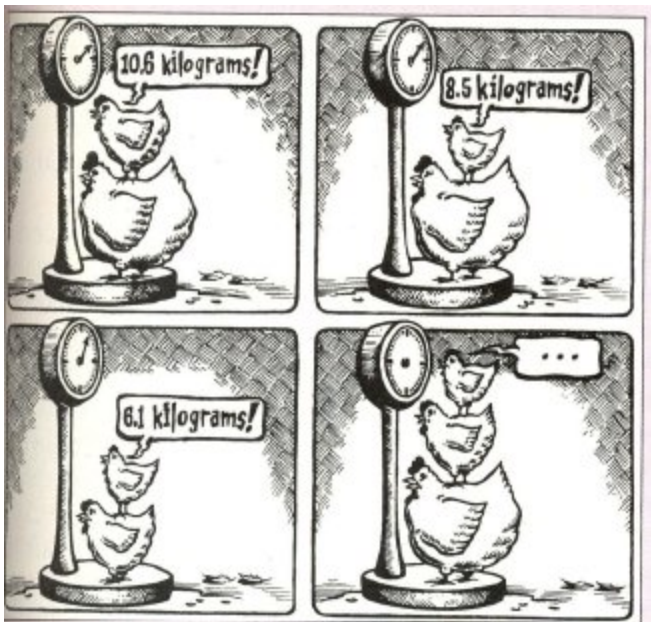


Figure 2. The Chicken Problem (Reeves, 2000, p. 399)

Rearranging the frames and changing the numbers to create different variations and extensions of the problem could prove useful. Perhaps students will eventually discover other methods, as well as make connections between those methods. Students who have already learned systems of equations may discover a connection that helps them see this mathematical concept in a different way, perhaps a way that will give them a deeper understanding of what systems of equations are and what they provide. This technique of allowing students to approach problems in a variety of ways and develop

their own methods suggests that **discovery** is a learning method that is undervalued in modern classrooms (Reeves, 2000).

Students in this experiment explained their strategies to the class (not just their computations). According to the current New York State Integrated Algebra curriculum, students are required to explain and justify their methods rather than simply derive the answers (New York State Department of Education, 2008-2012).

According to Reeves (2000), after presenting three variations of this problem to his students over the course of a school year, 30% of students had successfully used an easy-to-follow, systematic approach at least once. However the researcher was disappointed in the low number of students who used variables by the third variation, especially since students had explained their solutions to the class for the previous two variations. Student examples also suggest that some students actually created their own system of equations without even realizing it. While these systems of equations were not structured in the typical style, the remnants of a system can be clearly identified. Some of these students were in fact using algebra without even realizing it. Allowing students to develop their own approaches and then using their developments to guide them into more formal algebraic approaches is a teaching method that is discussed further in the next section.

Using Algebra Without Realizing It

The "algebra for all" philosophy is currently sweeping the nation. The National Council of Teachers of Mathematics suggests that teachers begin recognizing algebra as a strand for curriculum in very early grades as preparation for more abstract work that lies

ahead, offering methods that give young students a way of using algebra "without even realizing it." These methods are often in the form of "hooks" that engage students by giving them a real-world problem or a problem that has an interesting context. By keeping the methods simple, students can use algebraic approaches without actually using the algebra itself. This provides a gateway to more abstract algebra in future grade levels. Teachers should use manipulatives and pictures that make the problems appear easier and more engaging, eventually creating a transition to variables. This approach was also exhibited in The Chicken Problem (Reeves, 2000) that was previously mentioned, where students were given the opportunity to approach the problem in their own way with an eventual transition to variables (Maida, 2004; National Council of Teachers of Mathematics, 2000; Reeves, 2000).

One such example involves giving students different facts regarding the weights of a baseball, football, and soccer ball. Students must then use this information to determine the weight of each individual ball. As seen in figure 3, the situation is modeled using pictures rather than variables. The "system" is then solved using the pictures.

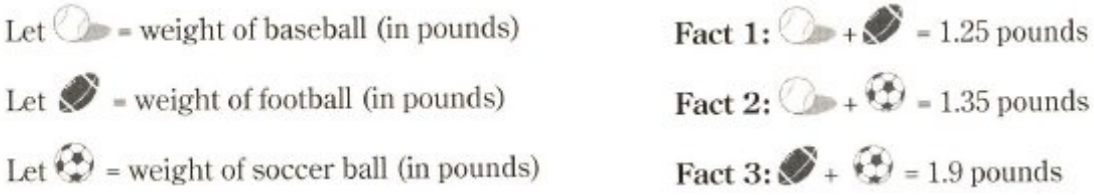


Figure 3. A system of equations that uses pictures (Maida, 2004, p. 485)

The use of logical reasoning is another very useful method that is underused in today's classrooms. Rather than always setting up a system of equations with variables (something that students know how to do but do not necessarily know why they are doing

it), using pictures and logical reasoning gives students a method that makes sense to them. Figure 4 shows a problem that can be solved using a system of equations. The problem is then followed by a solution using logical reasoning (Maida, 2004).

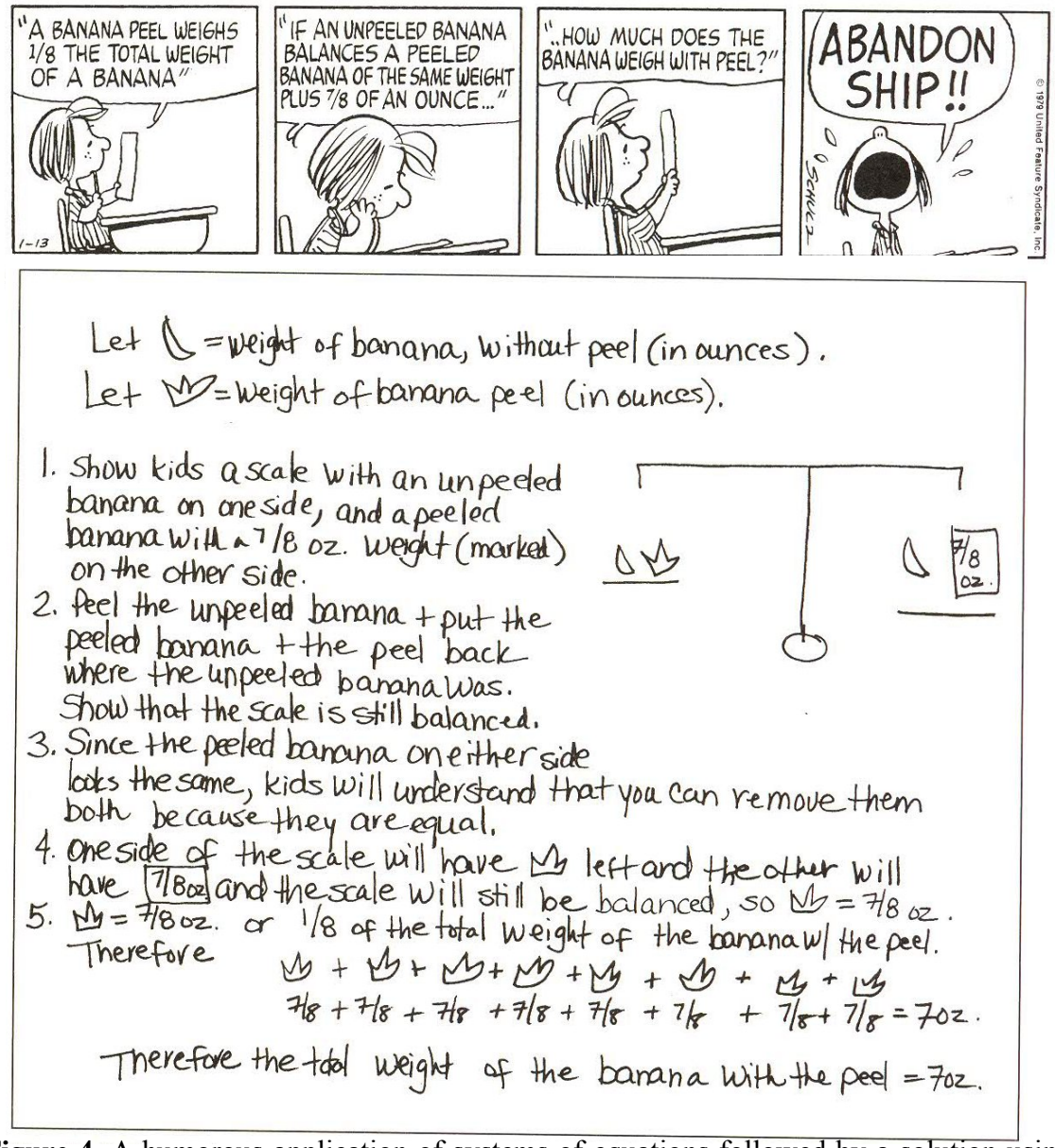


Figure 4. A humorous application of systems of equations followed by a solution using logical reasoning (Maida, 2004, p. 487)

As seen in these examples, there exist problems that allow students to use algebra without realizing it. The contexts of these problems often involve interesting or humorous situations that engage students in a way that motivates them to use their intellect and develop methods for solving the problems. The next section discusses the usefulness of these types of problems and how they can be used as learning tools in classrooms.

Algebra Affinity

Students have a great ability to situate mathematics in interesting and humorous, real-life contexts through their problem posing. For example, the use of recipes is a great way to explain the concept of independent and dependent variables. In one scenario, the first ingredient would be independent and all of the remaining ingredients would be dependent on the first one. Using examples like these, students end up solving systems of equations but not by the formal algebraic methods that are so often emphasized in modern classrooms. Instead, students ended up using line graphs and combination charts to solve the systems (Berner, McLaughlin, &Verzoni, 1997).

Another example involves using line graphs to compare imaginary pay scales for selling soft drinks at hockey games. Pay scales differ by commission rates and hourly wage, and pay scales can be different combinations of commission and hourly wage. Students graph the "what-if" lines in order to see the "big picture," helping them to determine which pay scales are better in certain situations. Having students use line-graph drawings and combination charts is yet another alternative method used for solving systems of equations. It seems that such alternatives, as well as the connections they

form with traditional and more formal methods, are often undervalued in the classroom, textbooks, and the curriculum in general (Berner, McLaughlin, & Verzoni, 1997).

Another interesting real world application discovered by Contino (1995) shows an out-of-the-ordinary connection between different methods used for solving systems of equations as well as exposes a misconception about traditional methods so often used for solving them. This particular application, which involves postal rates that are represented by linear functions, will be discussed more in the upcoming section of this literature review.

Empirical Research, Data Collection, and Misconceptions

There are a variety of ways to approach systems of equations including, but not limited to, substitution, elimination, comparison, graphing, and creating a table of values. According to Gannon and Shultz (2006), students are more likely to solve a system of equations using "substitution" as long as one of the variables has a coefficient of 1. However, if no variables have a coefficient of 1, students are more likely to solve the system using "elimination." However, the mere fact that only substitution and elimination were mentioned here reinforces the notion that current approaches are too algebra-heavy and students are not properly trained to look at systems of equations from a variety of other perspectives. The different methods are outlined in Figure 5.

Substitution: Isolate one variable in either equation and then substitute its equivalent expression into the other equation.

Elimination: Arrange both equations so that like terms are properly "lined up." Multiply one or both equations by a constant that will allow for the elimination of one of the variables when adding the equations together.

Comparison: Isolate the same variable in both equations and then set the corresponding expressions equal to each other.

Graphically: Graph both equations and determine the intersection of their resulting lines.

Table of Values: Also known as "guess-and-check." Substitute possible values for the variables, searching for a pattern and eventually "closing in" on the correct values.

Figure 5. Descriptions of methods used for solving systems of equations.

There exists a unique method that involves using elimination twice, once to obtain the value of one variable and a second time to obtain the value of the other variable. While this method may take a little more time and not be the ideal algebraic approach, more connections can be made between algebraic and graphing methods. Using this particular method, students will see that a unique solution can be found by obtaining two different intersecting lines, one horizontal and one vertical. This method allows students to witness more connections between systems of equations and geometry, abandoning the overused formal methods and examining systems of equations from a different perspective (Gannon & Shultz, 2006).

Another method for solving systems of equations involves the use of ratio and proportion. This method gives students yet another connection between systems of equations and other areas of mathematics, and provides an interesting discovery when

solving systems of equations involving nontraditional circumstances, such as a situation involving parallel lines (Katsaras, 1978).

One real-world application provides even further evidence that teachers are doing a disservice to students by only focusing on the traditional algebraic methods for solving systems of equations and neglecting to use different approaches and make connections with other strategies. This particular discovery involves a problem where students are comparing postal rates of two different companies. Creating two linear functions (one for each company) will yield one point of intersection, leaving students to conclude that one company has a favorable rate "below" that point and the other company has a favorable rate "above" that point. However, analyzing these functions as step functions rather than continuous functions reveals that there are in fact two points of intersection. Have two points of intersection (rather than one) concludes that there are actually three different scenarios (rather than two), with one company having favorable rates in two of those circumstances. Showing this type of connection to the real world gives students an even deeper appreciation for systems of equations and their relevance (Contino, 1995).

The unique examples mentioned in this section provide even further testimony that the current procedural teaching methods used by teachers and pushed by the New York State curriculum are not helping students develop a true, deep understanding of this mathematical concept, nor are they developing the connections among different areas of mathematics. Ironically, "mastery" and "connections" are two goals that the New York State curriculum is pushing for (New York State Board of Regents, 2007).

Making systems of equations accessible to young learners can be accomplished through the use of picture equations, guess-and-check, combination charts, and notebook building. These methods are often considered "gateways" for young students who will eventually be promoted to using more formal methods such as substitution, elimination, and line graphs (Berner, McLaughlin, & Verzoni, 1997; Reeuwijk, 1995; Streefland, 1995).

It is clear that there is much more to systems of equations than the formal, procedural methods that are so often encountered in modern classrooms. There exists an abundance of enriching real-world applications and alternative methods that are underutilized despite their ability to provide students with a deeper conceptual knowledge of systems of equations. This has spurred criticism of current teaching methods and caused many to rethink the teaching of systems of equations.

Rethinking the Teaching of Systems of Equations

According to Proulx, Beisiegel, Miranda, and Simmt (2009), there are four main areas of focus when it comes to teaching and learning systems of equations. They are 1) the meaning of a system of equations; 2) ways of representing a system of equations; 3) ways of solving a system of equations; and 4) interpreting solutions of a system of equations. The four areas of focus are illustrated in the "concept map" shown in figure 6.

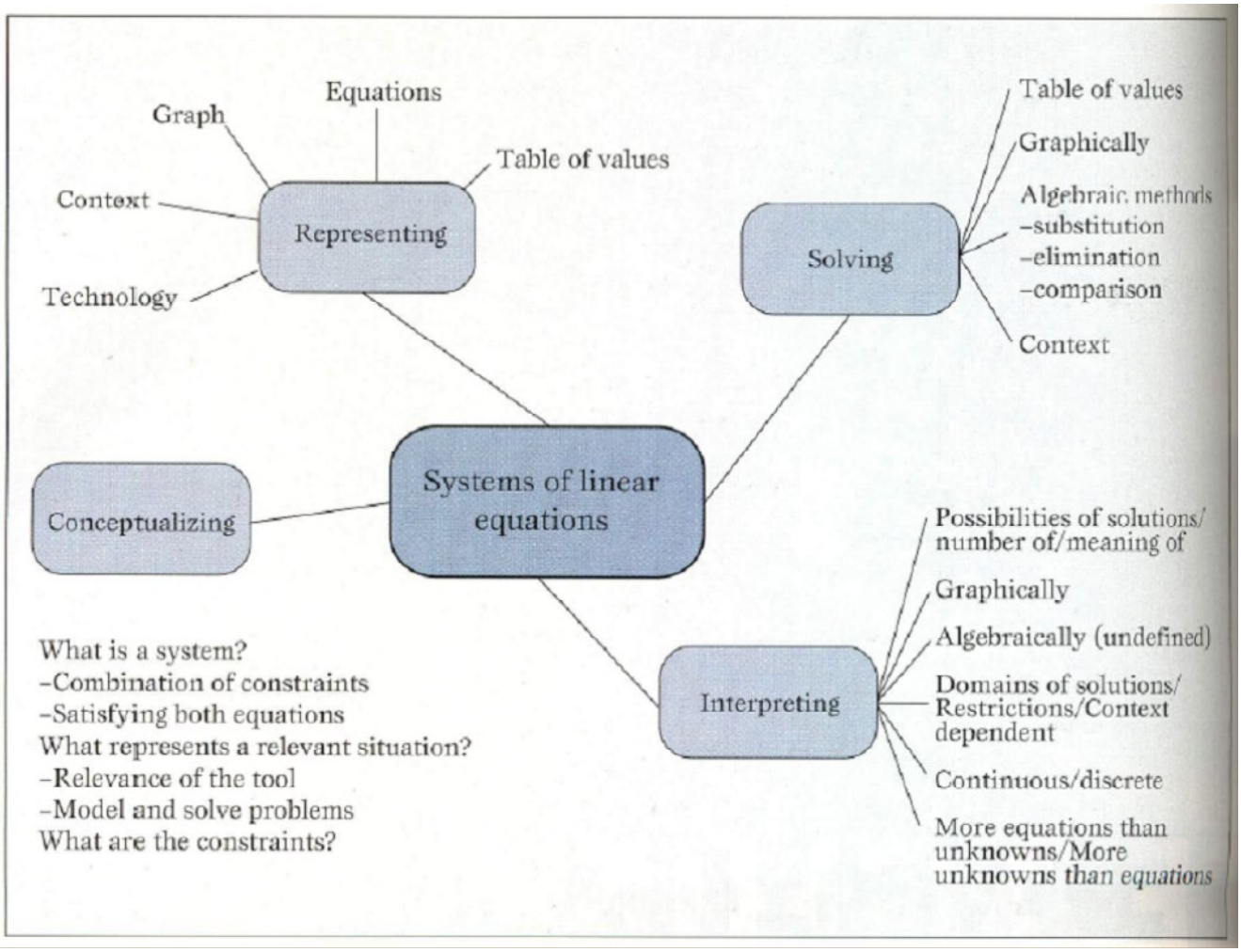


Figure 6. A concept map outlining the four areas of focus regarding the teaching and learning of systems of equations (Proulx, Beisiegel, Miranda, & Simmt, 2009, p. 528)

In today's classrooms, it appears that the "solving" strand receives most of the focus, with less emphasis on conceptualizing and interpreting systems of equations. Furthermore, textbooks and curriculum materials often focus mainly on algebraic manipulations and alternative algebraic procedures when solving systems of equations. While algebraic approaches are necessary, other approaches cannot be ignored if one wants to grasp a truly deep understanding of this powerful concept. Math is often inaccurately perceived as a field that offers only one method to solve a particular

problem. Rather, a "pluralistic perspective" should be used whereby problems are approached using a variety of methods. Multiple representations (algebraic, word problems, graphs, geometry, tables of values, etc.) can be used to enhance a student's perspective on systems of equations. In using this pluralistic approach students will recognize that where one method is lacking, another method could compensate. Failing to use such an approach will give students a very limited understanding of systems of equations. They will be familiar with many of the individual concepts, but will be unable to make the proper connections and understand how those different concepts work together to create a "big picture." Rather, students should be given the opportunity to navigate the strengths and weaknesses of each method. This will help them form connections and expand their comprehension of systems of equations (Cooney & Weigel, 2003; Proulx, Beisiegel, Miranda, & Simmt, 2009).

> For example, being able to interpret variations in the graph visually is mathematically significant for understanding the situation graphically, a unique perspective that cannot be assessed by algebraic methods. However, precise numeric values are often out of reach when one is using graphing methods. Alternatively, using a table of values enables a particular kind of attention to the ordered pairs that other methods do not afford. (Proulx, Beisiegel, Miranda, & Simmt, 2009, p. 530)

In today's classrooms students are not experiencing all of these different representations. Furthermore, of the ones they do experience, algebraic methods

dominate because they are seen as more rigorous. This creates an unbalanced approach that does not force students to make connections and develop a true understanding of what systems of equations really are and how they can be used in relevant situations (Berner, McLaughlin, & Verzoni, 2009; Gannon & Shultz, 2006).

> The desire for rigor can lead to incomplete and inadequate understanding of the concept of systems of equations, as Sfard and Linchevski's research (1994) illustrates. Hence, in addition to algebraic methods (of substitution, elimination, and comparison), tables of values and graphs are also important means for understanding and solving systems. They expand the student's conception of a system of equations. (Proulx, Beisiegel, Miranda, & Simmt, 2009, pp. 529-530)

More evidence for this claim can be witnessed in Contino's (1995) discovery regarding two points of intersection. As seen in the example involving postal rates, neither traditional graphing nor traditional algebraic methods would have resulted in a complete answer. Ironically, the method that yielded a complete answer to this "real world" problem is not part of the current Algebra curriculum, namely step functions.

This current algebra-heavy approach robs students of the beauty of mathematics by denying them the experience of understanding the geometry of what they are doing. Students should be given the opportunity to move back and forth between different representations. But while the integration of various methods will provide students with a much better conceptual understanding, it requires more than simply giving the students a list of methods from which to choose. Students need to be able to transition from one

method to another and understand how those methods relate to one another. This movement will enhance their ability to think more critically and make decisions based on applicable criteria rather than routine. For example, algebraic methods may come up short when trying to understand a system's solution in a certain "context." Examining the problem graphically as well as algebraically would give students a "bigger picture" and create connections that would otherwise be missed. This can be seen in the problem involving postal rates because it shows the connections that can be formed among different methods of solving systems of equations. Furthermore, this real-world application provides exposure to the flawed notion that systems of equations should be taught in a procedural manner despite the inability to generate complete solutions and illustrate the links between different methods. Perhaps one way to stress the connection between different methods, such as equations and their graphs, is to more consistently afford students the opportunity to see what is happening geometrically as they manipulate the algebra (Conney & Wiegel, 2003; Contino, 1995; Gannon & Shultz, 2006; Janvier, 1987).

While modern teaching methods have been criticized for being too procedural and algebra-heavy, the problem does not begin and end with classroom instruction. Textbook authors are also guilty of dismissing the pluralistic approach that is essential to deep conceptual understanding.

Current Textbook Approaches

As mentioned earlier, textbooks and curriculum materials often focus mainly on algebraic procedures when solving systems of equations, ignoring the pluralistic perspective that focuses on a variety of methods. Furthermore, another major criticism of current teaching methods for solving systems of equations is the procedural approach that is based solely on a "series of mechanical steps." It is often considered a disservice to students when teachers use a procedural approach to "conceptually rich" topics like systems of equations because they miss out on a potentially real, deep mathematical experience (Cooney & Weigel, 2003; Proulx, Beisiegel, Miranda, & Simmt, 2009).

Procedural approaches seem to occur more often at the high school level. According to Reeves (2000), algebraic reasoning problems are now appearing more often in middle school textbooks, giving students a chance to approach problems intuitively before making the link to more abstract methods used in algebra. However once in high school, intuition seems to become obsolete when it comes to solving systems of equations. Despite all of the methods that exist, algebraic and graphing methods dominate the textbooks. Furthermore, substitution and elimination are the center of discussion among algebraic methods. Maida (2004) does not discuss this isolation, but instead demonstrates it by only mentioning substitution and elimination as means to solving systems of equations algebraically. Whether intentional or not, this isolation of methods only further suggests that students are not being given the opportunity to explore a variety of methods and their connections, blinding them from the big picture and taking away the opportunity for them to experience a deep understanding of this rich mathematical concept (Bass, Hall, Johnson, & Wood, 2001; Bellman, Bragg, Charles,

Hall, Handlin, & Kennedy, 2007; Brown, Smith, & Dolciani, 2000; Bumby, Klutch, Collins, & Egbers, 1995; Gantert, 2007; Malloy, Price, Willard, & Sloan, 2006; Schultz, Kennedy, Ellis, & Hollowelly, 2001).

Textbooks and other curricular materials also fail to adequately provide students with problems that have different types of solutions. Problems involving a single, unique solution are encountered far more often than other possibilities (no solution, infinite solutions). While it is often the case that textbooks offer all three cases, the representation of each case is uneven. This in fact causes students to eventually expect one type of solution, causing confusion when other cases arise (Bass, Hall, Johnson, & Wood, 2001; Bellman, Bragg, Charles, Hall, Handlin, & Kennedy, 2007; Brown, Smith, & Dolciani, 2000; Bumby, Klutch, Collins, & Egbers, 1995; Gantert, 2007; Malloy, Price, Willard, & Sloan, 2006; Proulx, Beisiegel, Miranda, & Simmt, 2009; Schultz, Kennedy, Ellis, & Hollowelly, 2001; Sfard and Linchevski, 1994).

This problem is actually demonstrated by Gannon and Shultz (2006) as they discuss different methods for solving systems of equations. As the authors suggest, a system of linear equations results in two lines that "should intersect exactly once" (p. 189). While discussing the lack of a variety of methods, the authors actually demonstrate another problem in the teaching of systems of equations, which is an overuse of the unique solution. This thought process is so ingrained in students' heads that when a different type of solution is encountered they often become confused, believing that a mistake must have been made. This provides even further evidence that the case of the unique solution is the most dominant when it comes to teaching systems of equations (Proulx, Beisiegel, Miranda, & Simmt, 2009).

This same problem can also be observed in the current Integrated Algebra curriculum, namely Regents exams. Whenever a student is asked to solve a system of linear equations, either algebraically or graphically, there is almost certainly one unique solution. Rarely do students encounter a system of linear equations that yields no solution or infinite solutions (New York State Department of Education, 2008-2012).

Textbook Analysis

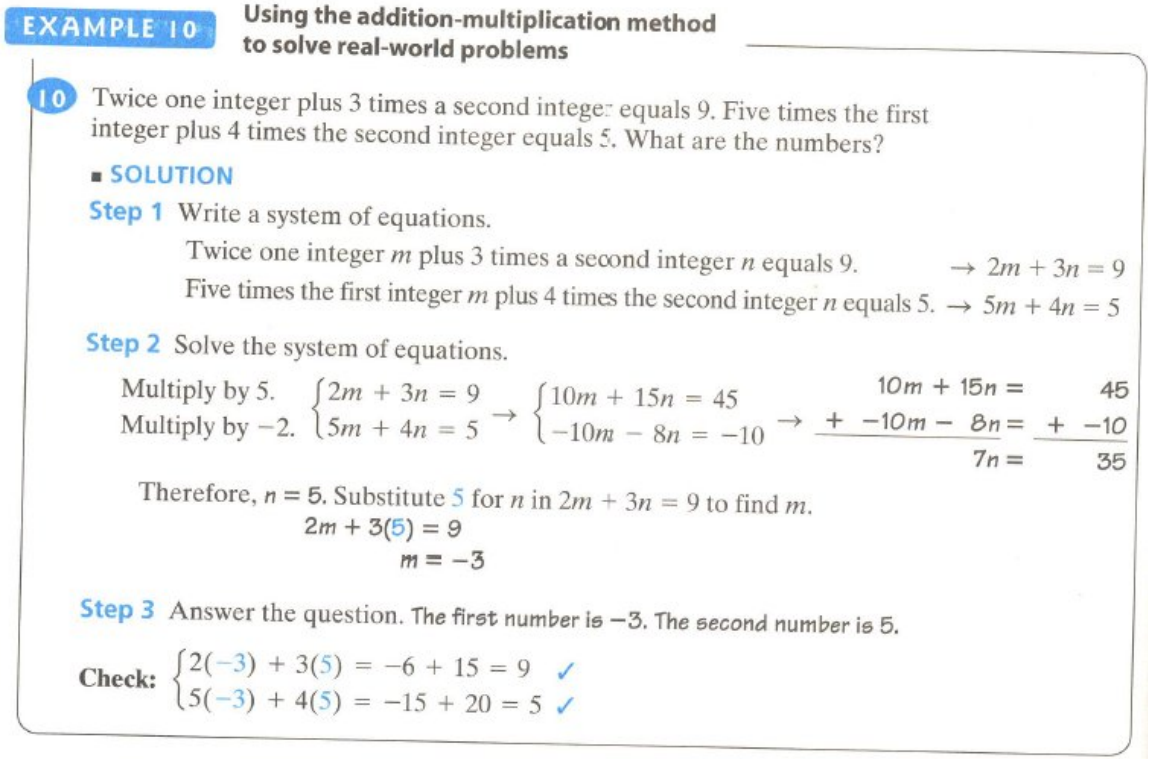
As noted earlier, textbooks and curricular materials fail to adequately provide students with problems with different types of solutions. A careful review of textbooks shows a lack of in-depth coverage of systems of equations. While the topic is covered in virtually all algebra texts, the content is uniform and plain. Most texts offer 3 methods for solving systems of equations: graphing, substitution, and elimination. High school mathematics textbooks that provide alternative methods such as tables, algebra tiles, and matrices do exist, but these textbooks are rare. Furthermore, when these alternative methods are mentioned, the reference is brief and with minimal connections to other methods. In many textbooks, the discussion of the "big picture" is minimal or nonexistent. The majority of the texts do provide examples other than "one-solution," namely "no solution" and "infinite solution." However many of the texts simply mention these "special" cases or provide limited examples (Bass, Hall, Johnson, & Wood, 2001; Bellman, Bragg, Charles, Hall, Handlin, & Kennedy, 2007; Brown, Smith, & Dolciani, 2000; Bumby, Klutch, Collins, & Egbers, 1995; Gantert, 2007; Malloy, Price, Willard, & Sloan, 2006; Proulx, Beisiegel, Miranda, & Simmt (2009); Schultz, Kennedy, Ellis, & Hollowelly, 2001).

Algebra II texts provide information on solving systems of equations that involve one linear function and one quadratic function. However they only offer graphing and algebraic methods for solving them. Furthermore, there are rarely any verbal or real-world investigations involving these types of systems. This reinforces the notion that current textbook approaches are failing to show students the real-world relevance of systems of equations. One possible reason for the lack of applications in textbooks is the lack of applications on the Regents exams. If the curriculum is not mandating that students apply mathematical concepts to the real world, textbooks are less likely to include those types of problems (Gantert, 2009; Keenan & Gantert, 1991; Keenan, Gantert, & Dressler, 2002; New York State Department of Education, 2007-2012). This leads to the following question that is addressed throughout the experiment: Can students effectively apply learned algebraic concepts to real-world contexts?

Many texts claim to offer "real-world" applications to systems of equations, but it can be argued that many of these texts fall short of their claim. While the majority of them do provide "verbal" problems and puzzles, suggesting that these are "real-world" problems is often considered deceiving by educators since these problems usually have little relevance to the real world. Figure 7 shows an example of a "real-world" problem taken from a textbook.

The authors of this text claim that this is a "real-world" problem, but one can plainly see that it is no more than a generic word problem whereby the student is given the information necessary to set up a system of equations and solve it. Little evidence can be found from this problem that suggests a student would have to truly "think critically" or "analyze" the problem in a way that requires conceptual understanding of

systems of equations and their real-world relevance and usefulness. To claim that this is a "real-world" problem only reinforces the assertion that students are being robbed of the beauty of mathematics, the relevance of systems of equations in the real world, and the connections that exist between different areas of mathematics (Gannon & Shultz, 2006; Proulx, Beisiegel, Miranda, & Simmt, 2009).



EXAMPLE 10 **Using the addition-multiplication method to solve real-world problems**

10 Twice one integer plus 3 times a second integer equals 9. Five times the first integer plus 4 times the second integer equals 5. What are the numbers?

■ **SOLUTION**

Step 1 Write a system of equations.

Twice one integer m plus 3 times a second integer n equals 9. $\rightarrow 2m + 3n = 9$

Five times the first integer m plus 4 times the second integer n equals 5. $\rightarrow 5m + 4n = 5$

Step 2 Solve the system of equations.

Multiply by 5. $\begin{cases} 2m + 3n = 9 \\ 5m + 4n = 5 \end{cases} \rightarrow \begin{cases} 10m + 15n = 45 \\ -10m - 8n = -10 \end{cases} \rightarrow \begin{array}{rr} 10m + 15n = & 45 \\ + \ -10m - \ 8n = & + \ -10 \\ \hline 7n = & 35 \end{array}$

Multiply by -2.

Therefore, $n = 5$. Substitute 5 for n in $2m + 3n = 9$ to find m.

$$2m + 3(5) = 9$$
$$m = -3$$

Step 3 Answer the question. The first number is -3. The second number is 5.

Check: $\begin{cases} 2(-3) + 3(5) = -6 + 15 = 9 \ \checkmark \\ 5(-3) + 4(5) = -15 + 20 = 5 \ \checkmark \end{cases}$

Figure 7. An example of a "real-world" textbook problem (Primiani & Caroscio, 2010, p. 94)

Some texts do provide less generic applications. Figures 8 and 9 show examples of textbook applications that are relevant to the real world. Notice that the first example provides students with a realistic situation to which systems of applications can be applied. The second example shows students how this particular mathematical concept relates to a different subject, namely chemistry. One could argue that these types of connections between different subject areas are undervalued in modern classrooms.

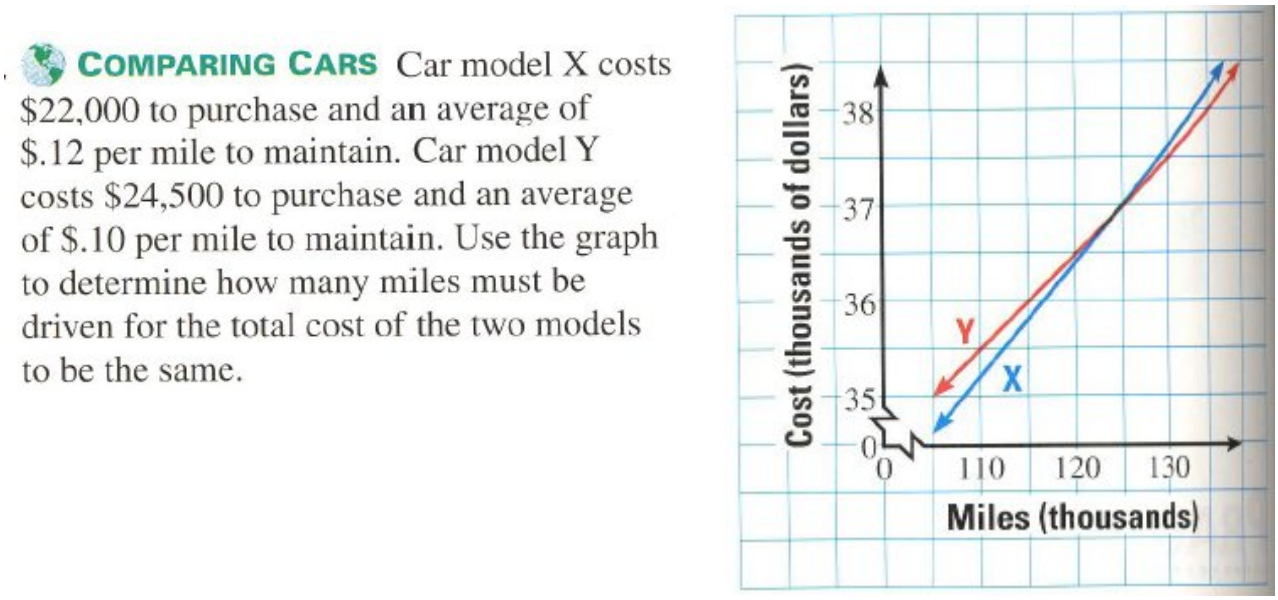


Figure 8. Example of a real-world application for systems of equations (Larson, Boswell, Kanold, & Stiff, 2001, p. 402)

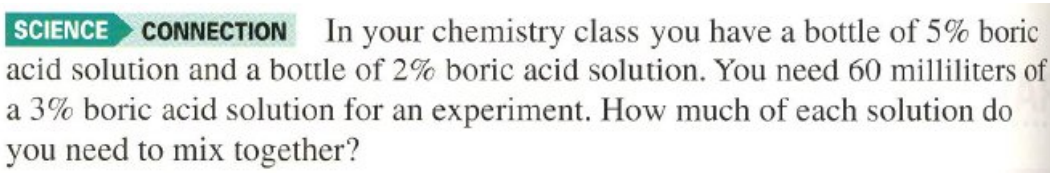


Figure 9. Example of a real-world application for systems of equations (Larson, Boswell, Kanold, & Stiff, 2001, p. 422)

Despite the majority of texts focusing on graphing, substitution, and elimination, there do exist texts that provide alternative methods such as tables, algebra tiles, and matrices. But these texts are rare. A likely reason for this rarity is that New York State's curriculum provides little focus on such methods. When combined with the coverage-mastery dilemma that is so prevalent in the New York State curriculum, textbook writers often resort to focusing on the specific methods that students are required to know (Bass, Hall, Johnson, & Wood, 2001; Bellman, Bragg, Charles, Hall, Handlin, & Kennedy, 2007; Brown, Smith, & Dolciani, 2000; Bumby, Klutch, Collins, & Egbers, 1995;

Gantert, 2007; Malloy, Price, Willard, & Sloan, 2006; Schultz, Kennedy, Ellis, & Hollowelly, 2001; New York State Board of Regents, 2007; University of the State of New York, 2011).

Thus far, the literature review has included an examination of methods for solving systems of equations, how these methods are taught, and how they are presented in textbooks. Analysis suggests that these methods are flawed, often involving too much procedure and a lack of variety. However it seems that the current New York State standards are a primary cause of flawed instruction and textbooks.

Standards and Assessments

There is currently an "algebra for all" goal that is sweeping the country. This can be seen in the "Algebraic Thinking" strand of New York State's current curriculum, as well as the new Common Core curriculum that is being initiated nationwide. The current New York State Standards state that students should be able to "analyze and solve verbal problems whose solution requires solving systems of linear equations in two variables." The new Common Core standards are even more generic and simply state "solve systems of equations." This leads one to wonder whether or not the curriculum for solving systems of equations, and other math topics, is headed in an appropriate direction (Maida, 2004; New York State Board of Regents, 2007; Reeves, 2000; University of the State of New York, 2011).

Despite this heavy "push" for "algebraic thinking," New York State has taken a very procedural approach to instruction on systems of equations, as suggested by past Regents examinations and standards. Regents exams often consist of questions that ask

students to solve a system of equations algebraically, by substitution or elimination, or graphically. Alternative methods are often ignored, and the methods that are used are rarely done so in ways that will force students to understand and appreciate the connections between the methods. Furthermore, while Regents examinations and standards often attempt to emphasize the notion of having students work with real-world applications, the Regents examinations often have very generic "real-world" problems. The curriculum states that students should be able to analyze and solve "verbal" problems, which hardly qualify as "real-world." These problems often involve a context that applies certain ***characteristics*** of the real world (such as names, environment, etc), but they are usually worded in a way so that students can easily derive the system of equations without actually having the need to analyze the situation fully and understand why a system of equations is needed in the first place. Additionally, the "real-world" problems often seen on Regents examinations rarely involve realistic situations for which one would actually need a system of equations (New York State Department of Education, 2008-2012). Figures 10 and 11 each show an example of such problems.

As seen in Contino's (1995) example involving postal rates, there do exist real-world applications that can inspire students to think critically and appreciate the real-world relevance of systems of equations. However it is evident through vague standards and generic word problems that New York State's curriculum is taking the teaching of systems of equations in a different direction.

31 Chad complained to his friend that he had five equations to solve for homework. Are all of the homework problems equations? Justify your answer.

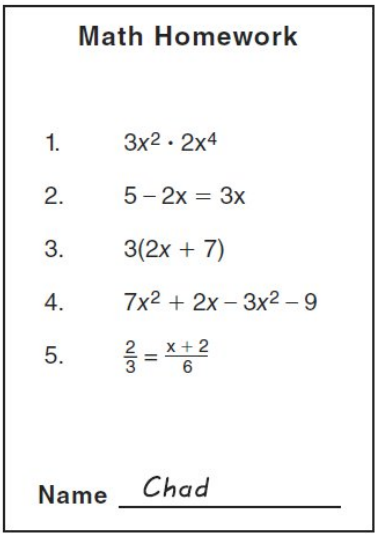


Figure 10. Example of a "real-world" Regents problem (New York State Education Department, 2009, p. 12)

37 The cost of 3 markers and 2 pencils is \$1.80. The cost of 4 markers and 6 pencils is \$2.90. What is the cost of *each* item? Include appropriate units in your answer.

Figure 11. Example of a "real-world" Regents problem (New York State Education Department, 2008, p. 19)

The concept of systems of equations has the potential to provide students with a rich, deep, and relevant mathematical experience. While there exist a variety of approaches that can be used when working with systems of equations, the real learning occurs when utilizing the connections between those different methods. Furthermore, systems of equations maintain real-world relevance, something that can be difficult to find among high school mathematical concepts. Analysis of the current curriculum leads one to believe that students are being robbed of this great mathematical experience through the use of "procedural" approaches, minimal real-world applications, and a lack

of connections among different methods used for solving systems of equations. The remainder of the study seeks to provide further evidence of these claims as well as the previously mentioned hypothesis.

Experimental Design and Data Collection

This experiment is designed to test whether high school students will show preference to a certain method when solving systems of equations algebraically, as well as determine which methods students will use if they have not been formally taught the traditional algebraic methods to solving systems of equations. Furthermore, this experiment seeks to determine whether or not students can effectively apply the concept of systems of equations to non-algebraic contexts.

Participants

The individuals participating in this study are students from a small, rural school district in New York State. It consists of 3 schools: two elementary schools and one middle/high school. The district is fairly small in terms of population, yet it is one of the largest in New York State in terms of square mileage. According to information from "School Report Card" (2009-2010 school year), the middle/high school serves 596 students in grades 7-12. Of these students, 2% are black or African American, 2% are Hispanic or Latino, and 96% are white. The average class size for Mathematics is 17 students. About 45% of the students are eligible for free or reduced school lunch prices.

The subjects of the study are enrolled in the middle/high school, which includes grades 6 through 12 and graduates approximately 80 students per class. Classes meet 40

minutes per day with the average math class consisting of approximately 20 students. This study includes 57 students from grades 9-12 who are enrolled in different mathematics courses, including Integrated Algebra 1A, Geometry, and Business Math. The sample of students consists of 26 males, 31 females, 15 freshmen, 10 sophomores, 7 juniors, and 25 seniors.

Students enrolled in the researcher's Integrated Algebra 1A, Geometry, and Business Math classes were directly involved in the experiment. All are morning classes. The Integrated Algebra 1A class runs from 10:50 to 11:36 and consists of 18 students who are mostly freshmen and sophomores. The Geometry class runs from 9:22 to 10:02 and consists of 14 students who are mostly freshmen, sophomores and juniors. The Business Math class runs from 8:38 to 9:18 and consists of 28 students who are mostly seniors.

Before the study began, the researcher obtained consent from the students and their parents/guardians. The study and the consent forms were also explained to students during their regular class time. Students were informed that participation is not mandatory and that they would not be penalized in any way if they chose not to participate. Students were also informed of the possibility of a quiz that would be a regular assessment for their particular class. In this case, students who chose not to participate in the study would still take this quiz. However, their grades would only be recorded in the regular grade book and would not be used for the study. This information was made clear in the letter to the parents/guardians of the students.

Design

The experiment was conducted during the second semester of the 2011-2012 school year. During this time, students were divided into three groups: level one, level two, and level three. Level one consists of students who had been taught both substitution and elimination, the two traditional algebraic methods used for solving systems of equations, and had been reviewing the topic throughout the current academic year. Level two consists of students who had been taught both substitution and elimination in recent years, but had not currently reviewed the topic. Level three consists of students who had never been introduced to systems of equations. All three groups were given a quiz with a variety of problems involving systems of equations. Some problems are strongly suited for a specific method while other problems are less biased. Quizzes were analyzed in an attempt to find a preferred method. The end of the quiz consisted of a survey that asked students more direct questions regarding the methods they preferred, as well as the difficulty of the problems.

Quizzes were graded using a specific rubric in order to determine the correctness of answers. Also, the specific methods used by the students were tallied. The goal was to determine which methods were most successful in producing correct answers, which methods were the most popular, and how well students were able to apply those methods to word problems. The goal in quizzing level one and level two students was to analyze both the methods used as well as the performance. The goal in quizzing level three students was to focus more analysis on the methods derived while still taking note of the correctness of the final answers.

Quizzing Level One and Level Two Students

The same quiz was given to all of the students and consists of 4 different questions involving systems of equations. The first part of the quiz consists of two "algebraic" problems. Both of these problems involve a system of equations that is already set up using variables. The first problem is suited for substitution by having one variable isolated in one of the equations. The second problem is suited for elimination by having both equations "lined up," no variable isolated, and all coefficients greater than one. The goal was to determine whether students are apt to using a non-favorable method simply because it's easier for them, as well as analyze which methods tend to yield better performance.

The second part of the quiz consists of two "word problems." Both of these problems involve a non-algebraic context that is fully described in words. Students are then asked to answer a specific question based on the information they were given. Although the problems can be easily solved using a system of equations, no system of equations is set up for the students, nor are the students instructed that they must use a system of equations. The first goal was to determine whether or not students would ***recognize*** that systems of equations can be used for these problems, as well as determine whether or not students could properly ***apply*** the concept of systems of equations once they recognized the connection. The second goal was to analyze methods that students derive on their own if they choose not to use a system of equations. The third goal was to determine if there was any connection between the methods students used in the algebraic problems and the methods used in the word problems.

Quizzing Level Three Students

Level three students were given the same quiz as the level one and level two students. However the goal was to determine which methods students derive on their own. Since these students had not been formally taught the traditional algebraic methods for solving systems of equations, analysis focused on what methods they came up with, as well as whether or not they used one of the traditional methods without even realizing it. For the word problems, analysis focused on which methods students came up with, as well as any consistency with the methods they used for the algebraic problems.

Instrument Items and Justification of Items

The quiz questions were designed to assess various student abilities regarding systems of equations. The goal was to determine which setup (suited for either substitution or elimination) yielded better results, which method was preferred, and which "type" of problem (algebraic or word) yielded better results. The objective was to create a quiz that not only addressed each of these criteria, but would do so in a short, simple manner that would not overcomplicate analysis and would not overburden the participating students.

The quiz consists of four problems. The first two problems are "algebraic," meaning that there is no context. The systems of equations have already been set up using variables and the students are simply asked to solve them. The first problem is heavily suited for substitution with one variable already isolated in the first equation. The second problem is heavily suited for elimination with both equations properly "lined up," no variable isolated, and all coefficients greater than one. The purpose of these two

problems was to determine whether substitution or elimination yields better results, determine how often students resort to using the opposing method that does not fit the setup of the problem, and analyze the lengths that students will go to in order to use that opposing method.

The last two problems are "word problems." Both of these problems involve non-algebraic contexts that are described to the student in words. The student must then use the given information to determine the answer. Although each question is designed in a way that is favorable to using a system of equations, students are not required to do so. The problem simply explains the situation and then asks a question. The first of these problems is slightly suited for substitution and the second problem is slightly suited for elimination. The purpose of these two problems was to compare the results with the first two problems and determine whether students are more successful on algebraic problems that involve procedure or word problems that involve application. The goal was to determine whether or not students could take the concept they used in the first two questions and actually ***apply*** it to a situation where the strategy is not perfectly clear to them. Since these questions are slightly suited for either substitution or elimination, they were also used in determining which method yields better results.

All problems were specifically designed to "work out" so that all answers are integer values. The problems were created in this manner to prevent excessive computational mistakes. Furthermore, all questions were formed to "work out" so that if a student decides to use a method other than the one the problem is suited for, they will be able to do so without experiencing an over abundance of computational problems. The goal of the quiz was not only to determine which methods are preferred and which

question "types" yield better results, but also to determine to which "types" of questions students have a better overall understanding. Putting too much emphasis on computation could hinder our analysis of the students' overall understanding and skew the results. Therefore the analysis of computational mistakes, while not completely irrelevant, was kept to a minimum.

The last section of the quiz was a survey. The survey consists of five multiple-choice questions that were designed to analyze the students' opinions of the quiz problems and systems of equations in general. The first question seeks to determine which "type" of problem (algebraic or word) was considered the most difficult. The second question asks students which "type" of problem they liked the most. The third and fourth questions ask students which problem was considered to be the easiest and which problem was considered to be the most difficult, respectively. The final question simply asks students to select the method for solving systems of equations that they like the most. These questions are designed to provide further insight as to which "type" of problems (algebraic problems or word problems) yield better results, as well as determine which method for solving systems of equations is most preferred by students.

Data Collection

There were two main components to the data collection: determining the correctness of answers and determining the preferred method used for solving a system of equations. All problems were assigned a score and method used as outlined below.

Correctness of answers

The following grading procedure was used for determining the correctness of answers for all of the quizzes. All problems were given a score of 1, 2, 3, or 4 based on the rubric shown in figure 12.

SCORE	RESPONSE
4	Answer is correct. Student displays full understanding of the problem.
3	Answer is incorrect. Student displays a reasonable understanding of the problem despite computational or conceptual mistakes.
2	Answer is incorrect. Student displays a limited understanding of the problem.
1	Answer is incorrect. Student displays minimal or no understanding of the problem.

Figure 12. Rubric for scoring quizzes.

Preferred Method

For each problem of the quiz, the methods used for solving the systems of equations and word problems were tallied into five categories: substitution, elimination, guess-and-check, other, and no response. Methods that were slightly based on substitution or elimination still qualified for that category.

Follow-up Survey

Students also filled out a follow-up survey which was used to gain further insight into each student's approach to the problems on the quiz, as well as how they felt about the difficulty of the problems. The survey consists of five multiple choice questions. The first two questions ask students which "type" of problem (algebraic problems or

word problems) they thought was more difficult and least liked. The next two questions seek to determine which individual problem was the easiest and which was the most difficult. The fifth and final question of the survey attempts to verify which method for solving systems of equations students like the most.

At the conclusion of this study, nearly all graded material was returned to those students participating. Some examples of student work were retained to add to the completed study. However, all identifying features were removed in order to ensure anonymity. Assessment scores were kept on file, without identifying features, for the purposes of analysis within the study. All confidential materials were kept in a locked cabinet, to which the principal investigator had the only key.

Methods of Data Analysis

The study has both quantitative and qualitative aspects. The quantitative aspect includes assigning numerical scores to each problem and determining the average of those scores. Furthermore, the frequency of each particular method was recorded and then used in determining which method is the most popular. The qualitative aspect includes a follow-up survey that each student filled out upon completion of the quiz. The goal was to gain an understanding of each student's opinion in an attempt to understand which methods they prefer to use when solving systems of equations, as well as gain insight into their confidence regarding algebraic problems and word problems.

Analyzing Performance

Analyzing Quizzes

Once students' tests had been graded according to the rubric, the problems were analyzed in various ways to determine the students' performance under different circumstances. The most successful method was determined by comparing student scores with the methods they chose. Since each problem is suited for a specific method, scores were also compared among individual problems in order to determine which individual problem students were more successful at completing. One possible issue with this method of analysis is that some students chose a more difficult method for some problems, possibly causing them to get the problem wrong. This made some methods appear to be less successful because they were "improperly" used and thus skewed the results.

Scores were also compared to the "problem type" in order to determine which type of problem (algebraic problems or word problems) students understand better, as well as determine whether or not students' general approach to systems of equations is "procedural."

Analyzing Preference

Students' preferences were determined by analyzing both the quizzes and surveys.

Analyzing Quizzes

Since the problems in the quiz were equally suited for the two most dominant traditional methods for solving systems of equations, substitution and elimination, the

methods used by students throughout the entire quiz were totaled in order to determine which method was the most popular. However because of the differences in experience and ability across the three levels of students, the methods chosen were analyzed in more specific cases as well. The methods chosen were divided according to student level, problem type, and the way the problem was suited. For example, the preferred method may vary greatly when comparing level one students with level three students, or when comparing algebraic problems with word problems. Dividing the analysis into smaller groups helped explain the broader trends (or lack of trends) that were seen among the entire sample of students. Furthermore, analyzing the methods used in algebraic problems and comparing those with the methods used in word problems provided insight into whether or not students can apply algebraic concepts to non-algebraic contexts.

Analyzing Surveys

Students filled out a follow-up survey which was used to gain further insight on each student's approach to the problems on the quiz, as well as how they felt about the difficulty of the problems. The survey consists of five multiple choice questions as previously explained. The responses for each question were assigned percentage values. However, the responses for each question were divided according to student level as well. Analysis of each individual student level provided explanation of any trends (or lack of trends) that existed within the overall results for each question.

Students' responses to the survey questions were also compared with the quiz results in order to find trends and consistencies that may or may not exist. For example, students who believe the word problems to be more difficult probably (but not

necessarily) scored lower on the word problems than the algebraic problems. However it is also possible that a student's perception of the difficulty of a problem was not consistent with how well they actually understood it.

After the students completed the quizzes and surveys, the above methods of data analysis were performed. The following section is devoted to reporting the results that were found upon analysis of the gathered data.

Results

Quantitative Quiz Results

Analysis of only the methods used for solving systems of equations provided results that support the hypothesis to various degrees:

> *It is hypothesized that high school algebra students will use substitution over elimination when solving systems of equations, and that students who have not been previously introduced to any method will naturally use "guess-and-check." Furthermore, it is hypothesized that students' general approach to solving systems of equations is "procedural," causing them to score higher on algebraic-type problems than on word problems.*

It was hypothesized that students would use substitution over elimination. This part of the hypothesis is supported by the first two results.

- Among students who have recently had instruction on systems of equations and the different methods used for solving them, substitution was the most popular method for algebraic problems.
- Among students who have recently had instruction on systems of equations and the different methods used for solving them, substitution was used in problems that were suited for elimination almost half of the time.

The third result supports the second part of the hypothesis, that students who had never been introduced to systems of equations would use guess-and-check.

- Among students who have never been introduced to systems of equations, "guess-and-check" was the most popular method for algebraic problems, and was used 72% of the time.

The fourth and fifth results support the final part of the hypothesis, that students' approach to these problems is "procedural," causing them to struggle on application-type problems.

- Word problems received a lower score than algebraic problems and were also considered to be the most difficult. The mean score for algebraic problems was 3.06 while the mean score for word problems was 2.15.
- When it came to solving the word problems, many students abandoned the methods they used on the algebraic problems.

Result 1: Among students who have recently had instruction on systems of equations and the different methods used for solving them, substitution was the most popular method for algebraic problems.

This result refers to students who are already familiar with solving systems of equations. Level one students are Geometry students who have been practicing solving systems of equations throughout the current school year. Level two students are Business Math students who have been instructed on solving systems of equations in previous years, but have not reviewed them during the current school year.

All level one students used one of the traditional methods for solving systems of equations (substitution or elimination) on the algebraic problems. Despite the fact that there was one problem suited for substitution and one problem suited for elimination, 71% of them used substitution and 29% used elimination. Eighty-one percent of level two students chose one of the traditional methods for solving systems of equations. They also favored substitution, but by a lesser margin than the level one students. Forty-six percent of them used substitution and 35% used elimination.

When analyzing all students who have been instructed on systems of equations at one time or another (all level one and two students), 88% of them chose a traditional method for solving the system. Fifty-five percent chose substitution, 33% chose elimination, and 12% did something other than one of the traditional methods. Figure 13 shows the methods used and is broken down by problem type and student level.

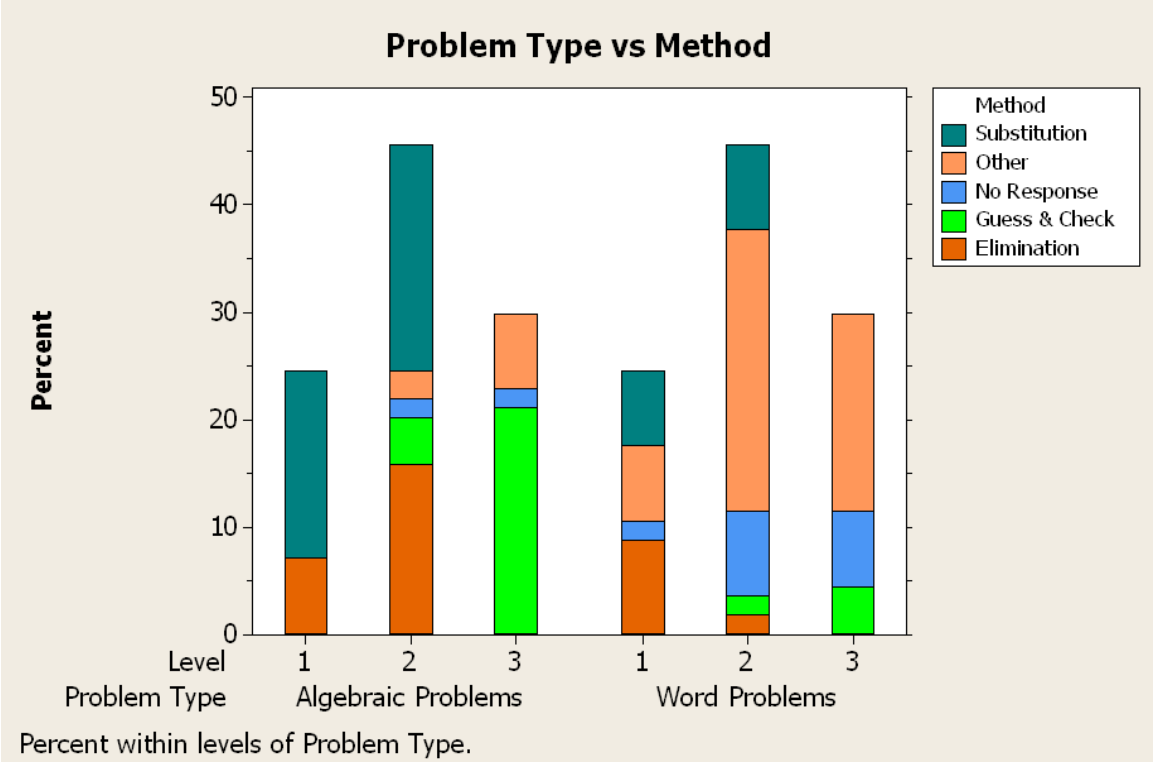


Figure 13. A bar chart comparing method and problem type. Problem types are divided into individual student levels.

Result 2: Among students who have recently had instruction on systems of equations and the different methods used for solving them, substitution was used in problems suited for elimination almost half of the time.

This result refers to level one and level two students and the methods those students used for the algebraic problems. Both levels of students showed a preference for substitution, although this preference was much stronger among level one students. As seen in figure 14, this particular student took the time to solve for x in terms of y and then substitute the corresponding expression into the other equation. Following this process

required the use of a higher variety of algebra skills, such as dividing a binomial by a monomial and distributing after the substitution.

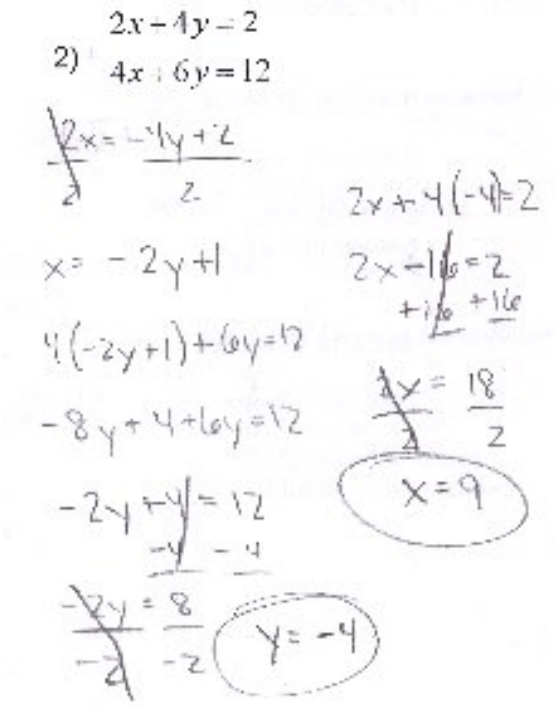


Figure 14. A sample of work done by a student who chose to use substitution in a problem that was suited for elimination.

Among level one students, 44% of them used substitution and 56% used elimination for the problem that was suited for elimination. Yet 0% of them used elimination and 100% used substitution in the problem that was suited for substitution.

Among level two students, 28% of them used substitution and 50% used elimination in the problem that was suited for elimination. For the problem that was suited for substitution, 20% used elimination and 68% used substitution. Although the percentages for substitution and elimination are much closer here, it is important to recognize that in the problem suited for elimination, only 50% of the students chose to actually use elimination. In contrast, more than two-thirds of the students chose to use

substitution in the problem that was suited for substitution. In figure 15, question number one is suited for substitution and question number two is suited for elimination.

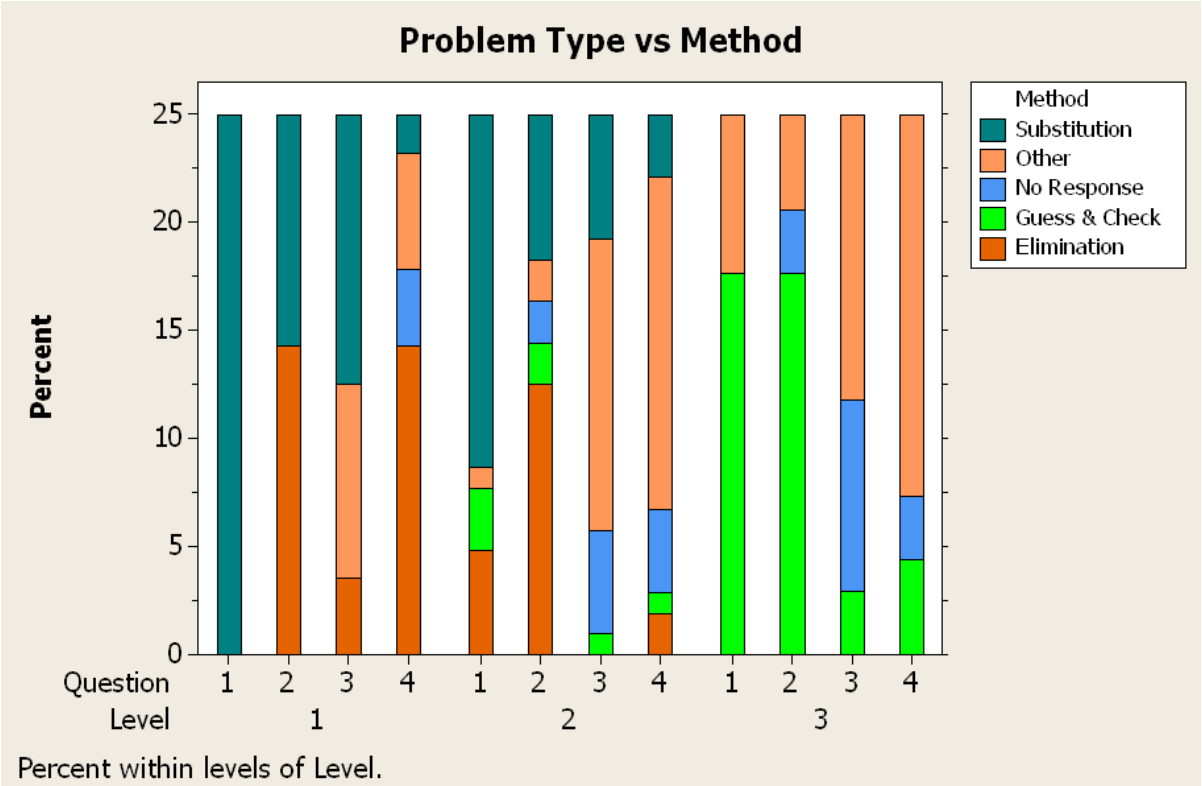


Figure 15. A bar chart showing the methods used for all four problems.

Result 3: Among students who have never been introduced to systems of equations, "guess-and-check" was the most popular method for algebraic problems.

This result refers to level three students and the methods those students used for the algebraic problems. Level three students are students in Algebra 1A who have never been introduced to systems of equations and have no knowledge of the traditional methods for solving them.

For each of the algebraic problems (one suited for substitution and the other suited for elimination), 72% of level three students used guess-and-check. None of the students used elimination or substitution. It is clear that students who have no prior knowledge to solving systems of equations are most comfortable performing guess-and-check. However not all students performed guess-and-check in the same manner. While a few students performed guess-and-check in a systematic way, many used a more random approach for guessing values. One student actually wrote on the quiz that he/she was "multiplying random numbers to fit with the answer."

The sample in figure 16 shows the work of a student who attempted a systematic approach yet was unable to come up with the answer. For this problem, the student created a table with three rows. The top row has different guesses for Thor's strength, the middle row has different guesses for Hulks strength, and the bottom row is the combined total. Each of the student's guesses had numbers that added to 35, showing that he/she understood how to make the guesses satisfy the first condition. However this student was unable to come up with the pair of guesses that satisfied the second condition as well.

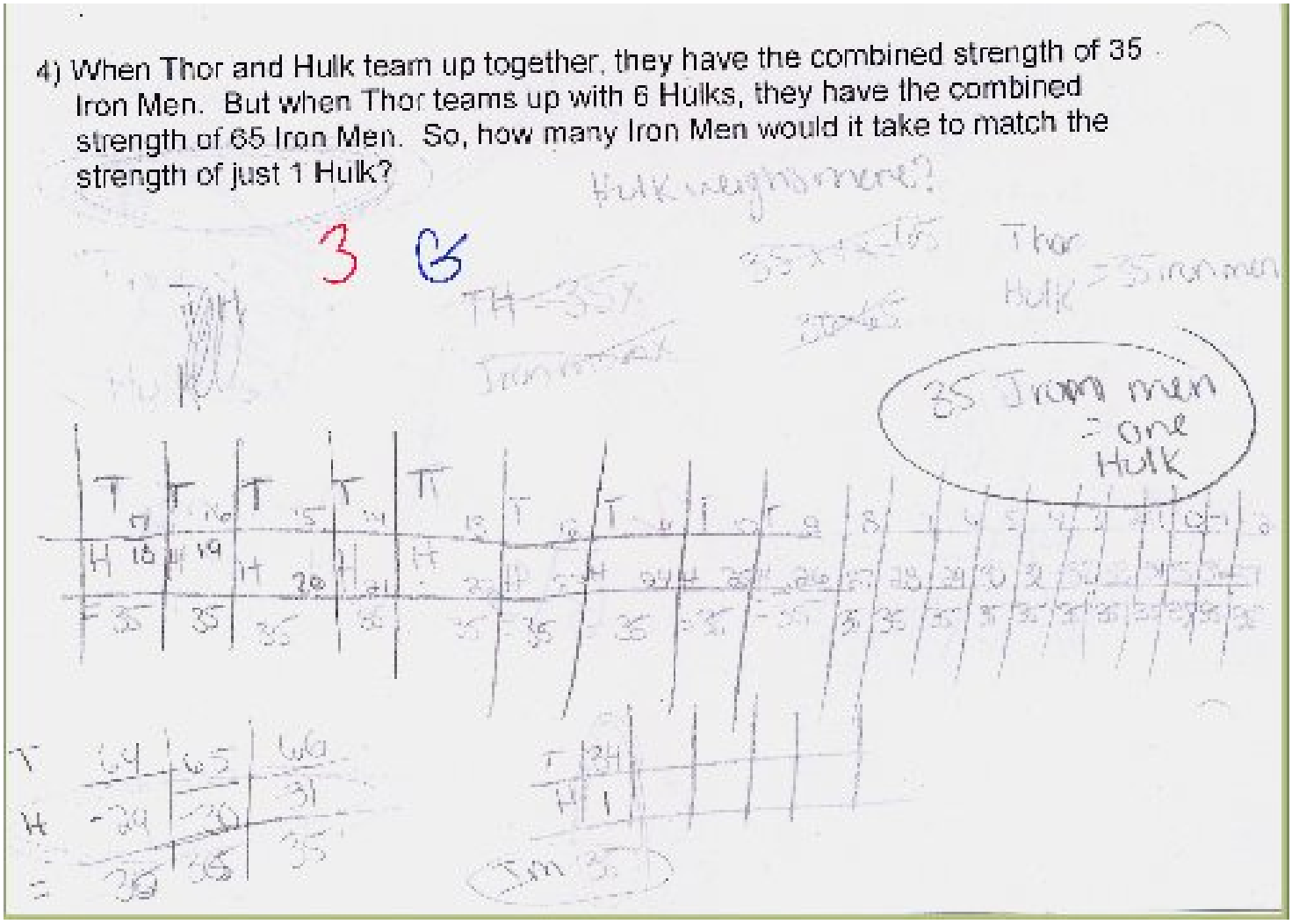


Figure 16. A sample of work done by a student who attempted to use systematic guess-and-check in a word problem.

Result 4: Word problems received a lower score than algebraic problems and were also considered to be the most difficult.

The average score for the algebraic problems was 3.06 while the average score for the word problems was 2.15. It is clear that students performed better on the "procedural" problems than the problems that gave a specific context. It is also

interesting to note that students had a tendency to approach the word problems differently than the algebraic problems and often did not set up a system of equations for the word problems. This is discussed in more detail in the upcoming sections.

While the word problems averaged almost one point lower than the algebraic problems, the discrepancy may be even greater when viewing it from a different perspective. For example, many students who scored a three still had a good overall knowledge of the problem. Since a response needed to be completely correct in order to receive a score of 4, many students received a score of three for simply having one or two computational mistakes even though they clearly demonstrated a thorough understanding of the problem. Perhaps if the scoring rubric was altered there would be an even higher discrepancy between the scores on algebraic problems and the scores on word problems.

Another way to analyze the scores is to examine the percentage of students who scored either a three or four and compare that to the percentage of students who scored either a one or two. Based on the rubric, any student who scored at least a three demonstrated a reasonable understanding of the problem, while any student who scored less than three did not demonstrate a reasonable understanding of the problem. For the algebraic problems, 73% scored a three or a four while only 27% scored a one or a two. For the word problems, 39% scored a three or a four while 61% scored a one or a two. So while 73% of students demonstrated a reasonable understanding of the algebraic problems, only 39% of students demonstrated a reasonable understanding of the word problems.

Word problems were also considered by the students to be the most difficult. Survey question number one asked students which problems seemed the most difficult.

Students chose one of the following: the word problems, the algebraic problems, all of the problems, or none of the problems. 59% of students chose the word problems while only 13% chose the algebraic problems. This discrepancy may have been even higher had the students only chosen between algebraic and word problems.

The fact that students considered the word problems to be the most difficult shows that they actually knew they did not understand the problems. It is not uncommon for students to think they are doing a problem correctly when they in fact are not. In this case it is obvious that the students were fully aware of their lack of understanding of the word problems. Figure 17 shows student scores according to the problem "type."

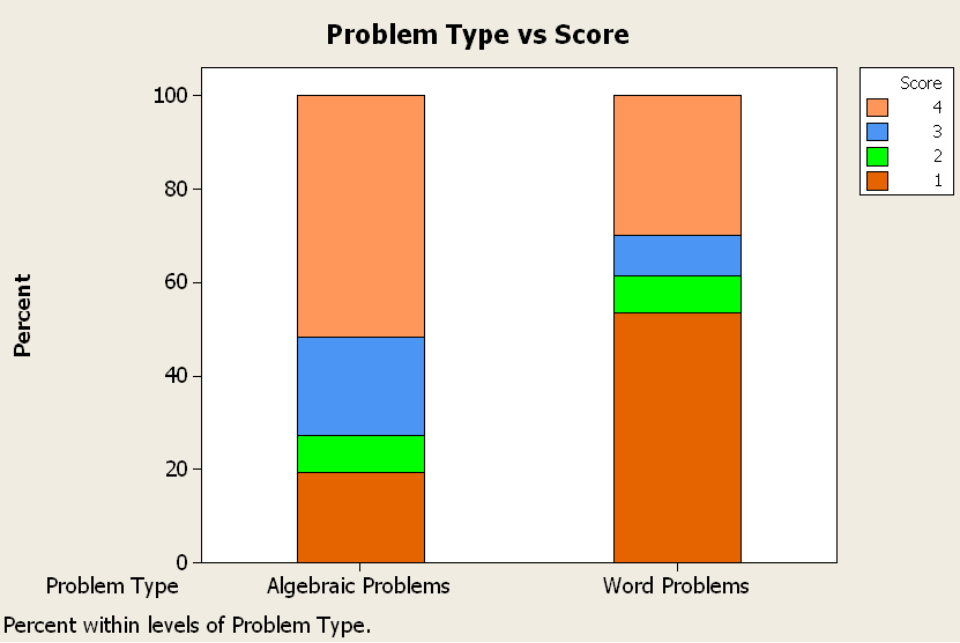


Figure 17. A bar chart that separates the scores according to the "type" of problem.

Result 5: When it came to solving the word problems, many students abandoned the methods they used on the algebraic problems.

Substitution and elimination were the favorite methods used when solving the algebraic problems. For level one and two students, 54% used substitution and 33% used elimination, which means that 87% chose one of the two traditional methods. For the word problems, however, 22% used substitution and 15% used elimination, which means that only 37% chose one of the two traditional methods.

Many students did not even set up a system of equations for the word problems, which leads one to believe that students who have been instructed on solving systems of equations still have difficulty **applying** the concept to non-procedural contexts. Figure 18 shows the methods that level one and level two students used according to problem "type."

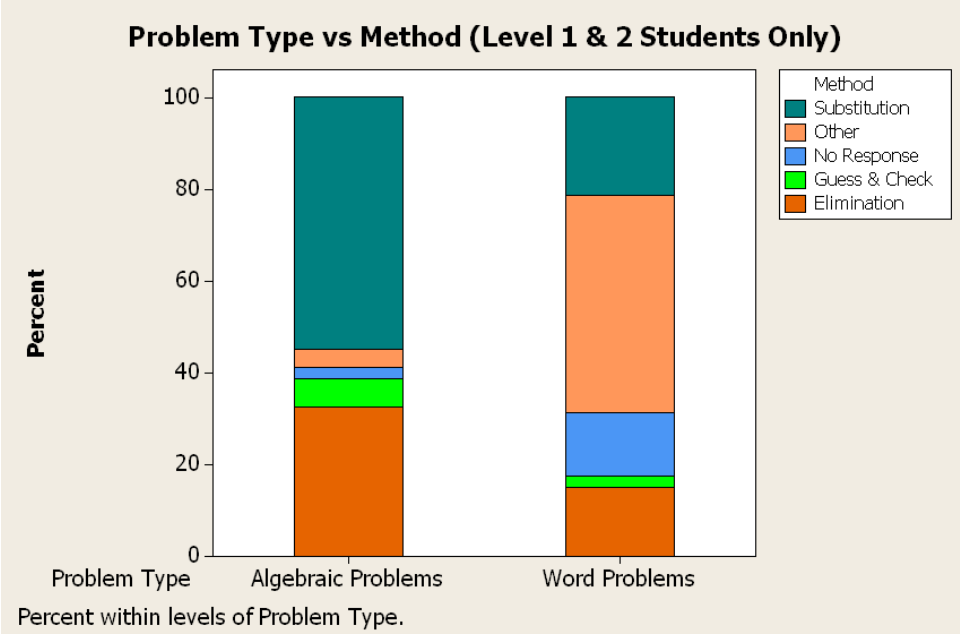


Figure 18. A bar chart that separates the methods used according to problem "type."

Level three students also changed their method when doing the word problems. Seventy-one percent of them used "guess-and-check" for the algebraic problems but only 14% of them used "guess-and-check" when it came to the word problems. Figure 19 shows the methods that level three students used according to problem "type."

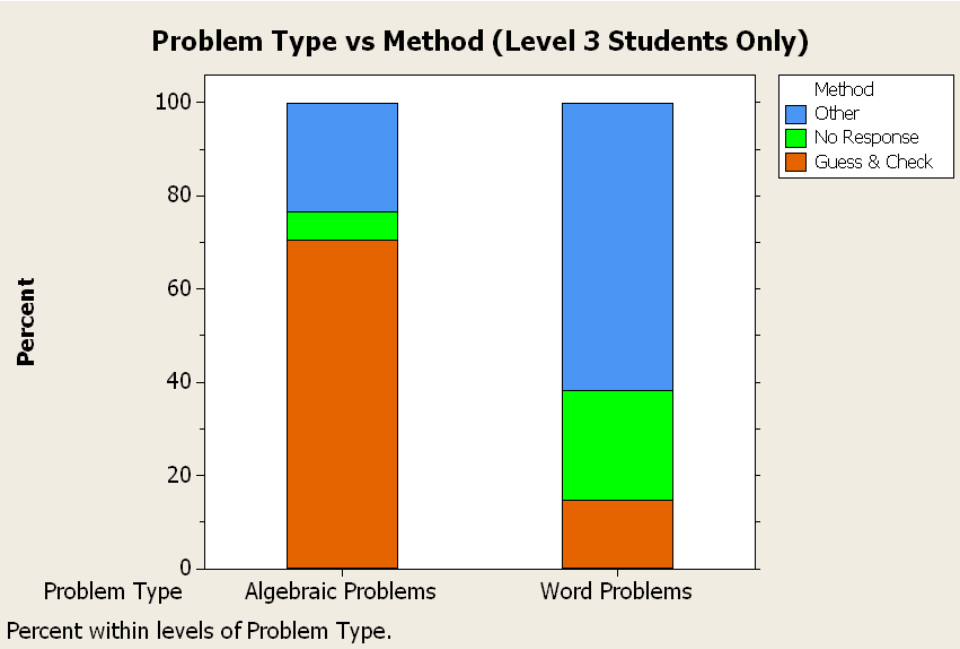


Figure 19. A bar chart that separates the methods used according to problem "type." This bar chart refers to level 3 students only.

The student samples in figures 20 and 21 show some interesting methods used to solve the word problems. It should also be noted that these particular students successfully used both substitution and elimination on the algebraic problems but did not resort to these exact methods for the word problems.

Figure 20 shows the work of a student who constructed pictures of the different situations and altered them to come up with the answer. It seems that this particular student did some mental reasoning without communicating all of his/her thoughts on paper.

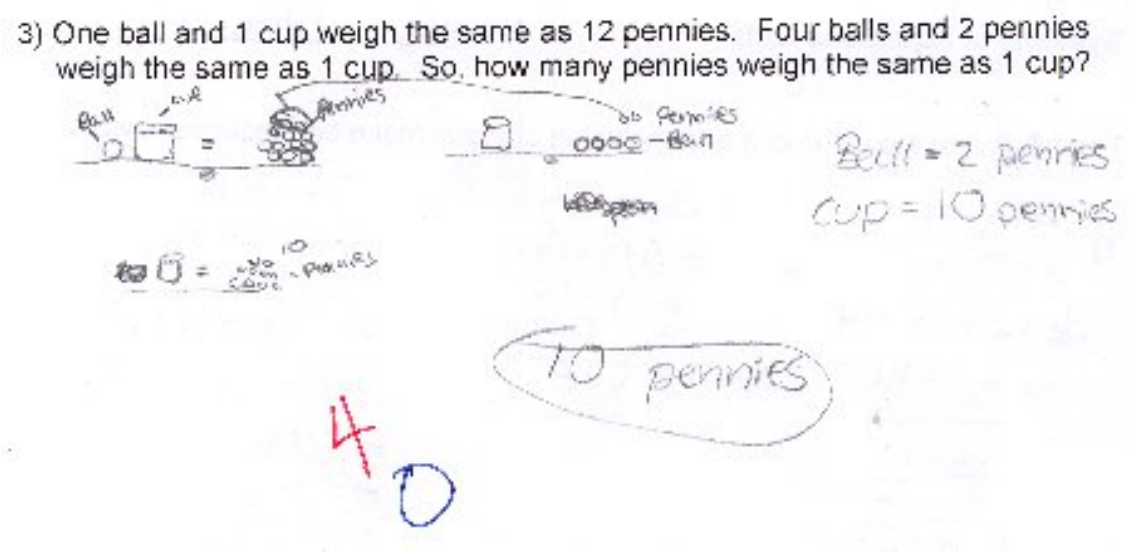


Figure 20. A sample of work done by a student who used traditional methods on the algebraic problems but resorted to different methods on the word problems.

Figure 21 shows the work of a student who used variables and substitution but did not adhere to the traditional "procedure" for substitution. This student initially used substitution to create an equation in terms of balls and pennies and then manipulated that equation to conclude that five balls weigh the same as ten pennies. But rather than continuing the traditional approach and finding the number of pennies that weigh the same as one ball, he/she used a proportion to determine how many pennies weigh the same as four balls. This value was then substituted into the second equation that already contained a "4*b*" term. So instead of substituting a variable, this student actually substituted an entire term. The student then finished the problem algebraically and obtained the correct answer.

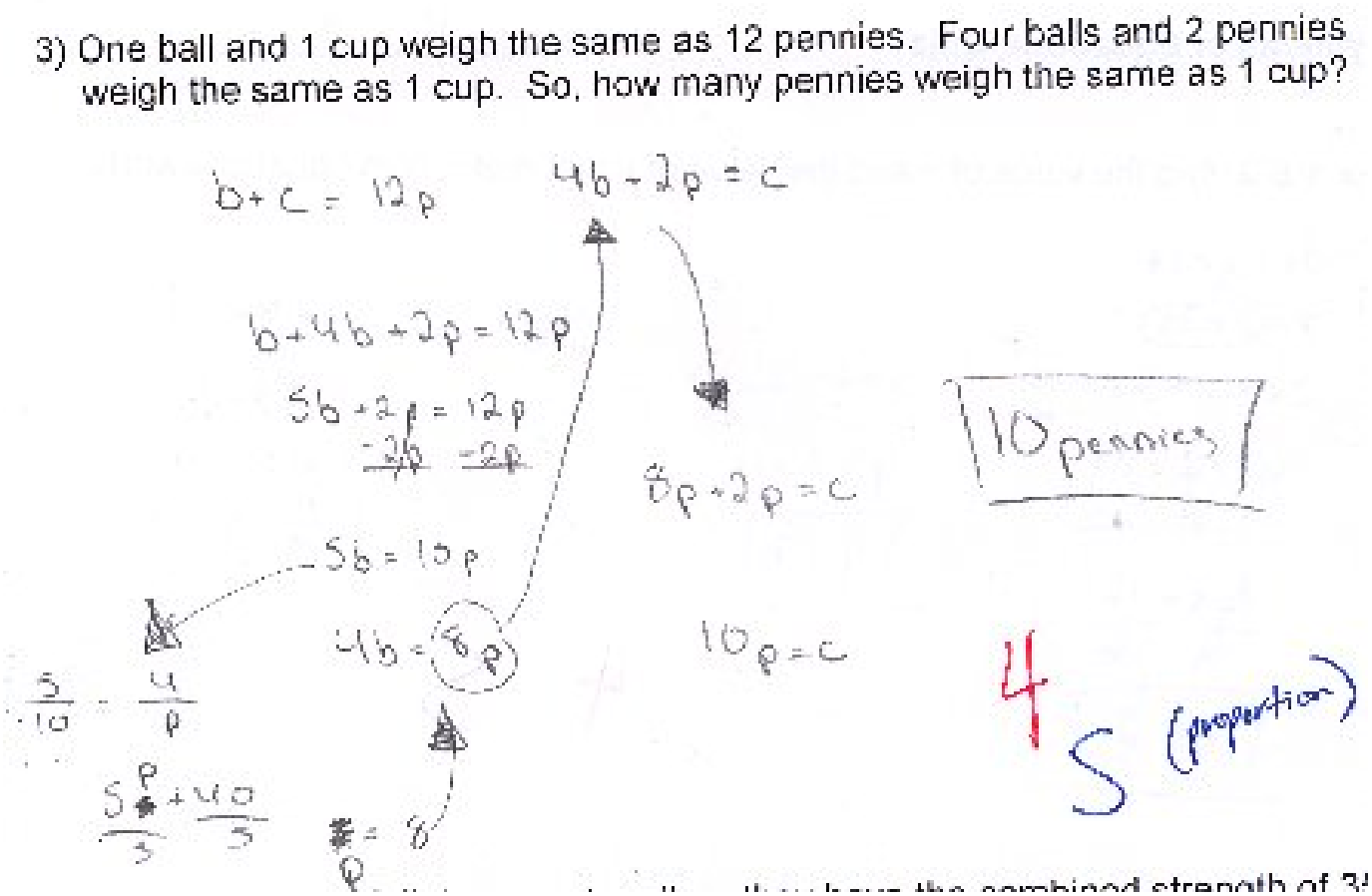


Figure 21. A sample of work done by a student who used a non-traditional adaptation of substitution by substituting for an entire term rather than just one variable.

Even though students used varying traditional methods for the algebraic problems, the data shows that students veered from those methods and tried non-traditional methods when it came to word problems. Furthermore, there were more instances of "no response" in the word problems than in the algebraic problems. Figure 22 shows the methods used and is divided by student level and problem "type."

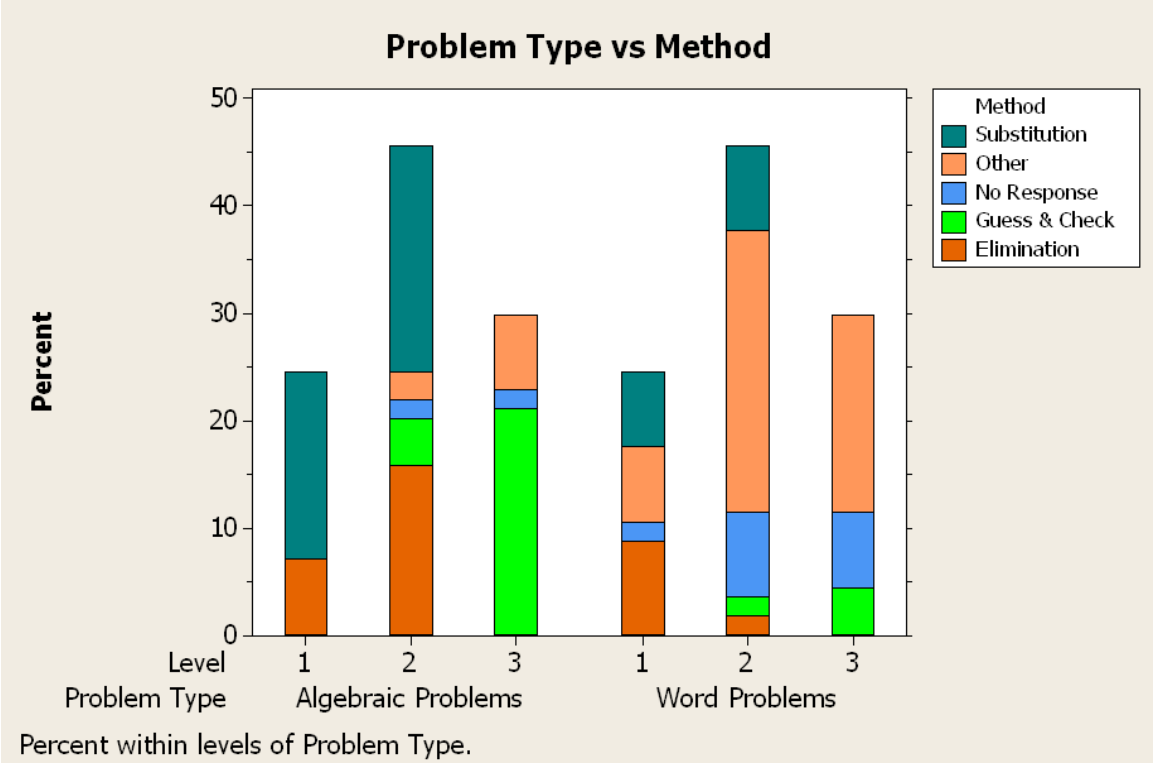


Figure 22. A bar chart comparing method and problem type. Problem types are divided into individual student levels.

It is evident that students who have had instruction on systems of equations favor substitution over elimination and that students who have not been taught systems of equations naturally resorted to guess-and-check. Furthermore, students had a more difficult time with the word problems than the algebraic problems and often resorted to other, non-traditional methods when working on the word problems. These results are compared with the results of the survey in the next section.

Qualitative Survey Results

The survey consisted of five multiple choice questions that asked students how they felt about the problems on the quiz. The survey is important because it provides

insight into what the students were thinking. Knowing the students' feelings about the quiz problems aided in the analysis of the quiz results.

Question 1: Which problems were more difficult?

	Level One		***Level Two***		***Level Three***		***All Students***	
Choice	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**
Algebraic problems	0	0%	5	20%	2	13%	7	13%
Word problems	12	86%	16	64%	4	25%	32	58%
None of the problems	2	14%	2	8%	0	0%	4	7%
All problems	0	0%	2	8%	10	63%	12	22%

Figure 23. The results of question 1, separated by student level. Questions that were left blank by the students were not included in the analysis.

The responses to question one show that students considered the word problems to be more difficult than the algebraic problems. This is consistent across each of the levels of students that were examined. Notice that level one students (who are the most advanced mathematically) felt the strongest about the difficulty of the word problems and had no issues with the algebraic problems. In fact, the more experience the students had with solving systems of equations, the more difficult they believed the word problems were. It is also important to note that a high percentage of level three students believed that all of the problems were difficult. This is likely because they had not yet been introduced to systems of equations and their applications. The level three students were the main cause in the 22% of total students that believed all of the problems were difficult.

Question 2: Which problems did you like the most?

	Level One		*Level Two*		*Level Three*		*All Students*	
Choice	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**
Word problems	2	14%	4	16%	3	19%	9	16%
Algebraic problems	12	86%	13	52%	7	44%	32	58%
All problems	0	0%	5	20%	0	0%	5	9%
None of the problems	0	0%	3	12%	6	38%	9	16%

Figure 24. The results of questions 2, separated by student level. Questions that were left blank by the students were not included in the analysis.

The responses to question two show that students liked the algebraic problems more than the word problems. This seems to be consistent with the responses to question one, where students said that the word problems were more difficult than the algebraic problems. It should also be noted that despite the interesting contexts used in the word problems, students still claimed to like them less than the algebraic problems. This suggests that students based their choice for question two on how difficult the problems were and not on how interesting the contexts were.

Question 3: Which problem seemed the easiest to solve?

	Level One		*Level Two*		*Level Three*		*All Students*	
Choice	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**
Problem # 1	12	86%	17	71%	8	57%	37	71%
Problem # 2	0	0%	3	13%	0	0%	3	6%
Problem # 3	0	0%	4	17%	2	14%	6	12%
Problem # 4	2	14%	0	0%	4	29%	6	12%

Figure 25. The results of question 3, separated by student level. Questions that were left blank by the students were not included in the analysis.

The first problem was clearly considered by students to be the easiest. The first problem is not a word problem (which should receive favor from the students based on their responses to the first two questions), and it is also a substitution problem (the method that students seem to favor based on the results of the quiz as well as survey question # 5). Interestingly, despite their dislike for the word problems, students still said that both word problems were slightly easier than the algebraic problem that was suited for elimination. The reason for this may be due to all of the level three students who found "guess-and-check" to work well for the first problem but not well for the second problem.

Question 4: Which problem seemed the most difficult to solve?

	Level One		*Level Two*		*Level Three*		*All Students*	
Choice	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**
Problem # 1	0	0%	3	12%	5	31%	8	15%
Problem # 2	1	7%	4	16%	6	38%	11	20%
Problem # 3	5	36%	10	4%	3	19%	18	33%
Problem # 4	8	57%	8	32%	2	13%	18	33%

Figure 26. The results of question 4, separated by student level. Questions that were left blank by the students were not included in the analysis.

These results are somewhat consistent with the other survey questions. The two word problems were equally considered to be the most difficult while the algebraic problem suited for substitution was selected the least in terms of difficulty. It is also interesting to note that the level three students' responses were not consistent with the level one and level two students' responses. When asked which problem was the most difficult, level three students had a tendency to select more of the algebraic problems while level one and level two students had a tendency to select more of the word problems. This may have caused the overall results to be less drastic.

Question 5: Which method for solving these problems do you like the most?

	Level One		*Level Two*		*Level Three*		*All Students*	
Choice	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**
Substitution	11	79%	10	42%	0	0%	21	39%
Elimination	1	7%	5	21%	2	13%	8	15%
Guess & Check	1	7%	7	29%	10	63%	18	33%
Other	1	7%	2	8%	4	25%	7	13%

Figure 27. The results of question 5, separated by student level. Questions that were left blank by the students were not included in the analysis.

When analyzing all three levels of students together, substitution was the most popular method, but not by a large margin over guess-and-check. However, analyzing each level of students individually reveals different results. Level one students, who have been working with systems of equations throughout the current school year, strongly favored substitution over the other methods. Level three students, who have never worked with systems of equations before, strongly favored guess-and-check over the other methods. This suggests that maybe guess-and-check comes more naturally to students who have not been trained in any other way. It is also very important to note the results for the level two students. Level two students have received instruction on systems of equations in the past, but have not worked with them during the current school year. These students placed somewhere between level one and level three students. While substitution was still the favorite, it was not favored by the same margin as with level one students. Level two students also favored guess-and-check more than level one

students. This could be because some level two students have simply forgotten how to perform the traditional methods for solving systems of equations. Another possibility is that some of the level two students, who have not recently had instruction with traditional methods, have reverted back to what comes naturally to them, namely guess-and-check.

Overall, the survey results are consistent with the quiz results. Students who have received instruction on solving systems of equations favored substitution over elimination, and students who have had no instruction on systems of equations naturally favored guess-and-check. Furthermore, students had a more difficult time with word problems and confirmed this in the survey. While many of them could complete the algebraic problems using a procedural approach, many had difficulty applying those concepts to the word problems. These results suggest that changes need to be made in modern teaching techniques as well as the curriculum in general. Teaching implications based on these results are discussed in the next section.

Implications For Teaching

The original hypothesis for this study suggested that students would favor substitution over elimination when solving systems of equations. Secondly, the hypothesis suggested that students who have never been introduced to solving systems of equations would naturally use guess-and-check to solve them. Finally, the hypothesis suggested that students' overall approach to solving systems of equations is procedural and that students would be less successful on applications.

The results moderately favored the first part of the hypothesis. Students generally chose substitution over elimination when given the option to use any method. The results

strongly supported the second part of the hypothesis. Students who had never been introduced to solving systems of equations naturally used guess-and-check more than any other method, although they rarely did so in a systematic way. Finally, the results strongly supported the third part of the hypothesis. Students scored much lower on the word problems than the algebraic problems. In addition, students often abandoned the methods they used for the algebraic problems and used some other method for solving the word problems. Many students did not even set up a system of equations for the word problems. The following implications for teaching were developed upon analysis of these results.

- The curriculum needs less focus on non-contextual problems that force students to demonstrate algebraic concepts in procedural ways and more focus on how to **apply** those concepts to non-algebraic contexts.
- Students need more instruction on how to effectively utilize guess-and-check by approaching it in a systematic way.
- Teachers should focus on developing the skills that come to students more naturally (such as substitution, guess-and-check) rather than continually force skills that students do not readily grasp.
- The curriculum should be designed in a way that provides students with the opportunity to see a variety of methods and how they work together.

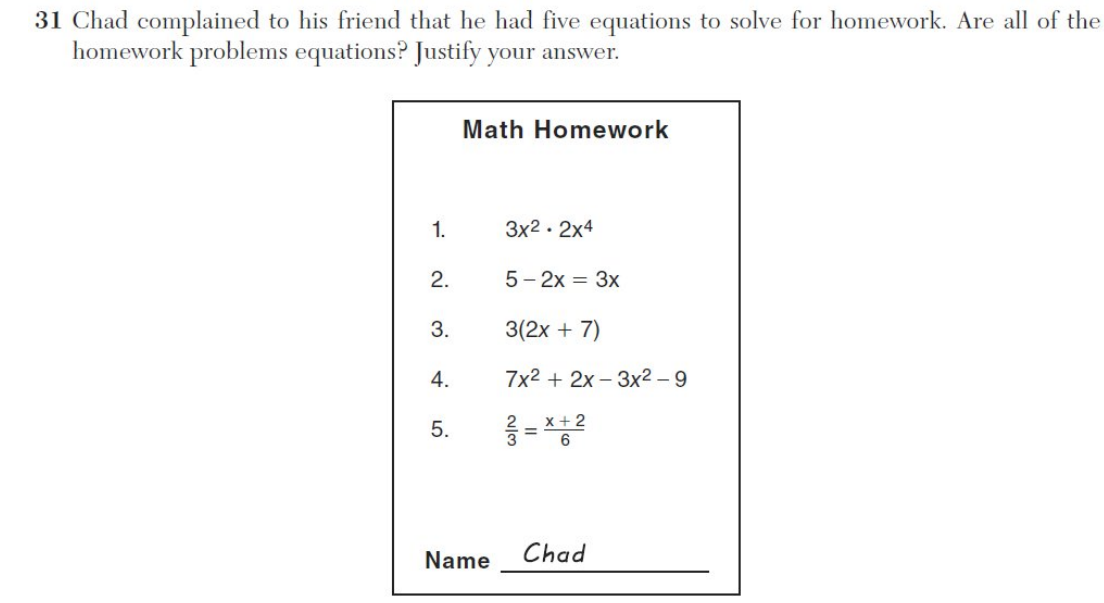
Implication 1: The curriculum needs less focus on non-contextual problems that force students to demonstrate algebraic concepts in procedural ways and more focus on how to apply those concepts to non-algebraic contexts.

It seems as though students become more confused on word problems and generally cannot apply learned mathematical concepts to contexts other than the procedural-friendly problems they are used to encountering. This calls for changes in the curriculum.

The current New York State curriculum focuses on problems that can be mastered by remembering a series of steps. Students often learn how to complete these problems by memorizing and carrying out a specific protocol. Protocol is often enough to do well on the Regents exam, since the exam often consists of these types of problems. Most of the problems have the same format and offer no surprise to the students when they encounter them. They recognize the problem and then carry out the appropriate steps. Students are not required to truly think. They are simply being asked to recall a specific protocol rather than actually **apply** a supposedly "learned" mathematical concept. The students are never given the opportunity to understand and appreciate what is actually happening mathematically.

One way to help students be able to truly understand and appreciate mathematical concepts is to show students how those concepts can be applied to other contexts. Whether those contexts are simply other areas of mathematics or real-world applications, students should be given the opportunity to gain a deeper understanding of the mathematical concepts they have "learned." The current New York State curriculum

would have one believe that it is in fact pushing for such applications. Yet the curriculum tells us otherwise. Even when a Regents exam does contain an application-type problem, it is often worded in a way such that students can easily set up the correct algebraic situation without having to really think about it. Sometimes the "real-world" aspect of the problem is completely irrelevant, as shown in figure 28. While this particular example does not involve systems of equations, it is an excellent example of a "real-world" problem whose real-world aspect is completely irrelevant to the task. In this problem, students are being tested on their ability to distinguish equations from expressions. So why not just ask them directly? Do the real-world ***characteristics*** of this problem actually make it relevant to the real world? It seems highly unlikely that two students would get into a discussion about whether or not the homework problems were equations or expressions. But even if they did, it still doesn't show students how these mathematical concepts are ***relevant*** to the real world. Rather, it just shows that they simply ***exist*** in the real world.



31 Chad complained to his friend that he had five equations to solve for homework. Are all of the homework problems equations? Justify your answer.

Math Homework

1. $3x^2 \cdot 2x^4$
2. $5 - 2x = 3x$
3. $3(2x + 7)$
4. $7x^2 + 2x - 3x^2 - 9$
5. $\frac{2}{3} = \frac{x+2}{6}$

Name *Chad*

Figure 28. Example of a "real-world" Regents problem (New York State Education Department, 2009, p. 12)

Despite the obvious flaws, it is difficult to adjust instruction so that mathematical concepts can be applied to other areas of mathematics and real-world situations. Unfortunately, teachers are currently bound to a curriculum that does not promote deep understanding of mathematical concepts. Instead, students are expected to master a wide range of topics, taking away the time that would provide them the opportunity to delve deeper into those topics. Because of the mastery/coverage dilemma, teachers are usually forced into choosing between the two. In New York State, the choice is coverage. Since the curriculum demands students have knowledge of a wide range of topics, teachers simply do not have the time to focus on mastery and deep understanding. In fact, it can be argued that the New York State curriculum not only expects students to understand a wide range of topics, but also ***master*** those topics. It seems that New York State has in fact attempted defeat the coverage/mastery dilemma by raising one side of the bar without the other side lowering. Of course this is impossible, and the result is that scores for most Regents exams are heavily curved in order to compensate for the flawed curriculum. Until changes are made, teachers will be forced to continue teaching with their hands tied, and students will continue to "learn" math by memorizing and recalling steps rather than applying mathematical concepts to relevant situations.

Implication 2: Students need more instruction on how to effectively utilize guess-and-check by approaching it in a systematic way.

Guess-and-check is truly undervalued in schools as a method for solving mathematical problems. Unfortunately, it is often perceived as juvenile, primitive, and

not as rigorous as algebraic methods (Proulx, Beisiegel, Miranda, & Simmt, 2009). While students are allowed to use guess-and-check for certain problems, the method is not required in the curriculum and is rarely taught and promoted in high school classrooms. Students are simply told that if they choose to use guess-and-check, they must "show at least three guesses." This is a true injustice to students, especially students who have difficulty understanding other, more "sophisticated" methods.

The results showed that students who have not been introduced to a specific method for solving systems of equations naturally chose guess-and-check more than any other method. Since this method seemed the most instinctive to students, it makes sense to promote, elaborate, and master such a method. This is not to say that this method should receive all of the focus and that all other methods should be dismissed. After all, approaching problems with a variety of methods helps students see the "big picture" and gain a deeper understanding of the area they are studying. But certainly guess-and-check should be given more consideration in classrooms rather than be completely dismissed.

> The desire for rigor can lead to incomplete and inadequate understanding of the concept of systems of equations, as Sfard and Linchevski's research (1994) illustrates. Hence, in addition to algebraic methods (of substitution, elimination, and comparison), tables of values and graphs are also important means for understanding and solving systems. They expand the student's conception of a system of equations. (Proulx, Beisiegel, Miranda, & Simmt, 2009, pp. 529-530)

Guess-and-check is not juvenile. It is a mathematical concept that can provide students with a different way of looking at a problem. Considering the fact that our education system is often promoting multiple representations, it makes sense to instruct students on mastering the art of guess-and-check. Guess-and-check is not a method that promotes randomness and luck. It is systematic and teaches students how to "close in" on an answer (something that is actually done in some upper level mathematics courses). Figure 29 shows the work of a student who wanted to use guess-and-check but lacked the ability to systematically do so. The student created tables with three columns, one for each variable in the problem. Each of the two tables represented guesswork for one of the two equations in the problem. It appears that the student understood that guess-and-check can in fact be conducted systematically, but he/she did not know how to actually do it. The fact that some students can recognize the systematic nature of guess-and-check but are unable to complete it provides more evidence that this method should be given more emphasis in classrooms (Lange, Rousos, & Mason, 1998).

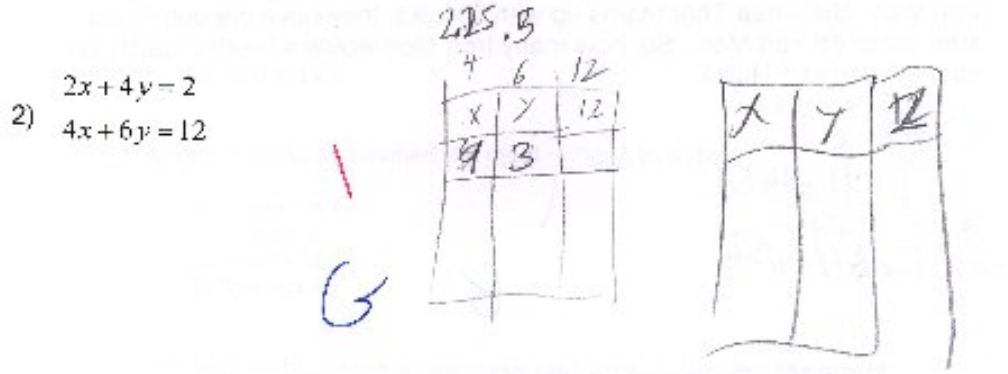


Figure 29. A sample of work done by a student who began to use guess-and-check in a systematic way but was unable to carry out the procedure.

Supporters of the current curriculum would have one believe that it does in fact promote a systematic approach to guess-and-check. According to the rubric for the

Regents exam, students may use guess-and-check but are required to show at least three guesses. But does simply showing three guesses necessarily demonstrate true knowledge of guess-and-check? Students who happen to guess correctly the first time can fulfill this requirement by jotting down two wrong guesses to accompany the correct one. Students who guess at random until they discover the answer will have fulfilled the requirement as well. But in neither case does the student show any true knowledge of systematic guess-and-check. Students in both scenarios would receive full credit when in fact they should receive no credit at all since they had only demonstrated minimal knowledge of the mathematical concept. Rather, students should be required to clearly demonstrate systematic guesswork that will ***inevitably*** reveal the answer.

Implication 3: Teachers should focus on developing the skills that come to students more naturally (substitution, guess-and-check) rather than continually force skills that students don't readily grasp.

As previously stated, the results showed that students who have not been introduced to a specific method for solving systems of equations naturally chose guess-and-check more than any other method. Since this method seemed the most instinctive to students, it makes sense to promote, elaborate, and master it while still taking the time to engage in other methods.

In addition, substitution seems to come more naturally to students than elimination. The concept of replacing one thing with its equal is a concept that is used in everyday life and students are already familiar with. It is easy to understand and can

even be done with physical objects. The mere concept of trading one equal for another, whether exchanging money for a product or trading baseball cards, is something that students already have experience in. They have been familiar with it since they were young.

One argument against the "overuse" of substitution is that it is not always the "ideal" method. Using substitution in problems that are suited for elimination can actually make the problem more difficult. However if teachers would put more focus into working through these "more difficult" problems, students would make less mistakes and actually enhance their algebra skills. Working through these "more difficult" problems would actually provide opportunity to practice more complicated algebraic concepts that are encountered in such problems.

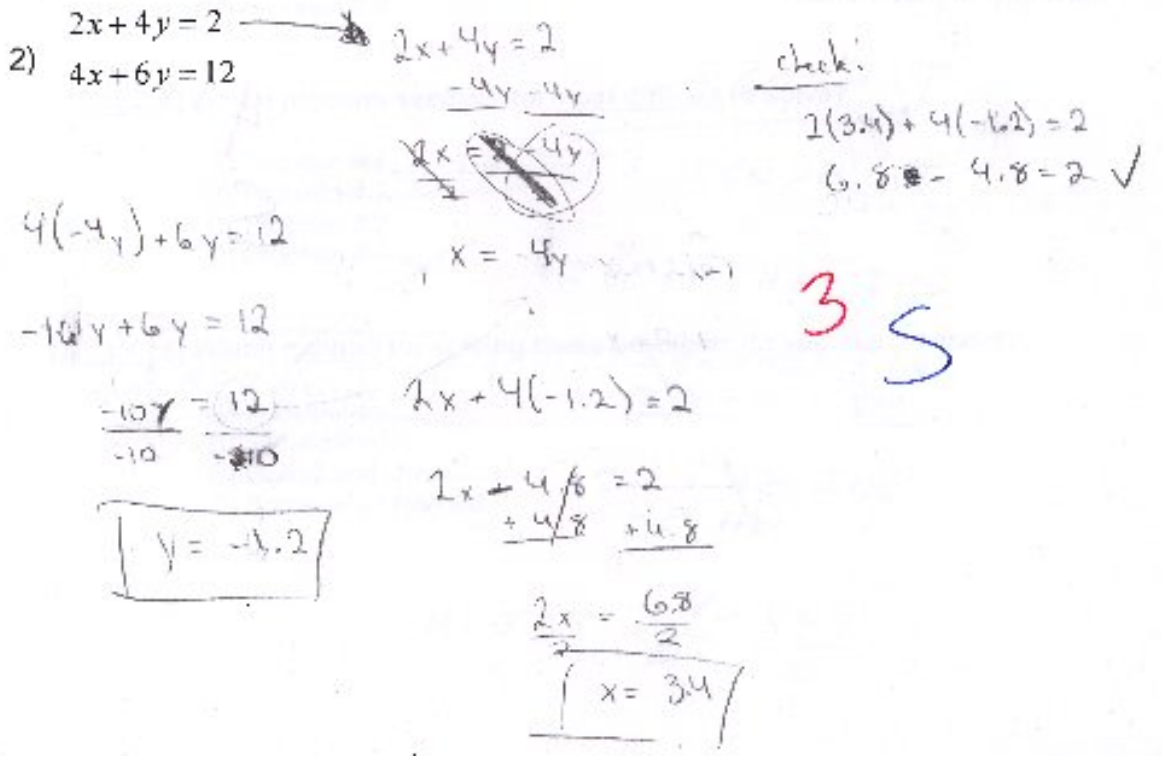


Figure 30. A sample of work done by a student who attempted to use substitution in a problem that was suited for elimination. The student made an algebra error while solving for *x* in terms of *y*, causing the solution to be incorrect.

Rather than use another method that makes the problem "easier," working through more difficult problems (but using the same method) could actually enhance students' basic math skills. After all, many of these basic skills are often left unattended while learning such a broad range of methods and topics. Perhaps the student whose work is shown in figure 30 would not have made the mistake of only dividing one term by 2 if these "more difficult" problems were encountered and practiced more often.

Implication 4: The curriculum should be designed in a way that provides students with the opportunity to see a variety of methods and how they work together.

Continuously working with a variety of methods would give students a deeper grasp on the concept of systems of equations and help them be able to properly apply the concept to other contexts. According to Gannon and Shultz (2006), an algebra-heavy approach "robs them [students] of seeing some of the beauty of mathematics by denying them the experience of understanding the geometry of what they are doing." But it's more than simply giving the students methods to choose from. Students should be given the opportunity to move back and forth between the different representations and understand how those methods relate to one another (Conney & Wiegel, 2003; Janvier, 1987). This will force students to ***deviate from routine*** and learn how to properly apply the concept of systems of equations in relevant but unfamiliar situations. A perfect example of this is Contino's (1995) postal rate problem, as previously mentioned.

> Students must be able to explore the variety of methods for solving systems in order to understand each method's strengths and weaknesses; then, when the students work with systems of equations, they can make critical decisions in their mathematical modeling. (Proulx, Beisiegel, Miranda, & Simmt, 2009, p. 530)

Systems of equations can be made accessible to younger learners through the use of less rigorous methods (Berner, McLaughlin, and Verzoni, 1997; Streefland, 1995; Reeuwijk, 1995). But if these "gateway" methods are not emphasized enough with younger students, they may lack the proper foundation necessary to properly apply the concept to more abstract situations in high school and college.

Furthermore, these methods should not only be used as "gateways" for young students who will eventually work with more formal methods. Although more formal methods are introduced in high school, the informal methods that were used to lay the foundation should not be completely abandoned, since losing this foundation could cause students to have difficulty with the more complicated variations they will inevitably encounter.

Conclusion

The concept of systems of equations is a very mathematically rich topic, with real-world relevance and connections with many other areas of mathematics. Rather than limiting students to the traditional approaches currently used in classrooms, a "pluralist" approach should be used that provides students the opportunity to experience connections

among different methods and see the "big picture." For example, students should be given the opportunity to master non-traditional methods such as guess-and-check, or use substitution in problems that are suited for elimination. This would give students the opportunity to see the connections among the different methods as well. Furthermore, students should be engaged in more applications that help them see the real-world relevance to the concepts they are learning in class. This would require a change in the curriculum that would take focus off of the "procedural" contexts that students have grown so accustomed to.

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Appendix

TO: Parents/Guardians of CVCS Students
FROM: Rick Hammond
DATE: 3/5/12
RE: Consent Form

- Your child is being asked to participate in a research project.
- To participate, he/she may need to take a brief quiz outside of regular class time.
- By signing the consent form, you are allowing Mr. Hammond to use your child's grade information and data in the study. If you do not sign, your child is still required to participate in all classroom activities as he/she usually would.
- This project will take place for about ten weeks during the second and third marking periods. Your child will be given quizzes that will be the focus of the project.
- Your child's name will never be used in any way. The study will not identify your child personally. Only his/her grades will be used.
- Mr. Hammond may ask to keep examples of your child's work. This is only by your child's choice. To make sure his/her identity is safe, your child's name would be removed from the example before it is included in the project.
- There are no risks involved in this project.
- The principal is aware of the project and is available if you would like to speak with him about it.
- Remember, this project is an attempt to understand students better and make learning easier and more interesting.
- If you do not sign the consent form, your child will not be penalized in any way.
- Your child will not be paid or given any rewards for participation in the project.
- If you have any questions or concerns, please feel free to contact Mr. Hammond at any time by phone: 716-962-8581, ext. 215 or by e-mail: rhammond@cvcs.wnyric.org
- You may also contact Mr. Hammond's college advisor, Dr. Keary Howard, at SUNY Fredonia by phone: 716-673-3873 or by e-mail: keary.howard@fredoia.edu
- Or you may contact the Human Subjects Protection Administrator, Maggie Bryan-Peterson, at SUNY Fredonia by phone: 716-673-3528 or by e-mail: petersmb@fredonia.edu

Please discuss this with your child.
If you choose to participate, please sign and return the consent form as soon as possible.

Parental Consent Form

SUNY Fredonia

Your participation in this important study is greatly appreciated. Please print and sign your name below to indicate your agreement to participate in the study. Be advised that your consent authorizes the use of your child's grades for the data required in the study. Students that choose not to participate in the study are still required to participate in class as usual.

Voluntary Consent: I have read this memo and am fully aware of the study and all that it entails. My signature indicates that I agree to allow my son/daughter to participate in this study. If I withdraw my son/daughter from the study, I understand that there will be no penalty assessed to him/her. I understand that my son's/daughter's confidentiality will be maintained. I understand that if I have any questions about the study, I may contact Rick Hammond at 716-962-8581 or rhammond@cvcs.wnyric.org

Please return this consent form as soon as possible. Thank you very much for your cooperation.

Parent/Guardian Name (please print):_____

Parent/Guardian Signature:_____

Student Name (please print):_____

Date:_____

Parent/Guardian Contact Information (optional):

Phone:_____

E-mail:_____

TO: Students in Mr. Hammond's Math Class
FROM: Mr. Hammond
DATE: 3/5/12
RE: Consent Form

- You are being asked to participate in a research project.
- To participate, you may need to take a brief quiz outside of regular class time.
- By signing the consent form, you are allowing Mr. Hammond to use your grade information and data in the study. If you do not sign, you are still required to participate in all classroom activities as you usually would.
- This project will take place for about ten weeks during the second and third marking periods.
- You will be given quizzes that will be the focus of the project.
- Your name will never be used in any way. The study will not identify you personally. Only your grades will be used.
- Mr. Hammond may ask to keep examples of your work. This is only by your choice. To make sure your identity is safe, your name would be removed from the example before it is included in the project.
- There are no risks involved in this project.
- The principal is aware of the project and is available if you would like to speak with him about it.
- Remember, this project is an attempt to understand students better and make learning easier and more interesting.
- If you do not sign the consent form, you will not be penalized in any way.
- You will not be paid or given any rewards for participation in the project.

Please discuss this with your parent or guardian. Feel free to ask Mr. Hammond if you have any questions or concerns about the project.

If you choose to participate, please sign and return the consent form as soon as possible.

Student Consent Form

SUNY Fredonia

Thank you for being a part of this project. Please print and sign your name in the space provided to show that you agree to participate. Remember that signing the form below allows Mr. Hammond to use your data for this research project. Also remember that if you choose not to participate in this study, you must still participate in class as usual.

Voluntary Consent: I have read this memo and am fully aware of the study and all that it involves. My signature below shows that I freely agree to participate in this study. I understand that there will be no penalty if I choose not to participate, and that I may withdraw from the study at any time without penalty. I understand that my name and other personal information will be kept out of the study. I understand that if I have any questions about the study, I may contact Mr. Hammond at 716-962-8581 or rhammond@cvcs.wnyric.org

Please return this consent form as soon as possible. Thank you very much for your cooperation.

Student Name (please print):_____

Student Signature:_____

Date:_____

Systems of Equations Quiz **Name_____**

For 1 & 2, find the value of *x* and the value of *y* that make both equations work.

1) $3x + y = 14$
$y = 2 + 3x$

2) $2x + 4y = 2$
$4x + 6y = 12$

3) One ball and 1 cup weigh the same as 12 pennies. Four balls and 2 pennies weigh the same as 1 cup. So, how many pennies weigh the same as 1 cup?

4) When Thor and Hulk team up together, they have the combined strength of 35 Iron Men. But when Thor teams up with 6 Hulks, they have the combined strength of 65 Iron Men. So, how many Iron Men would it take to match the strength of just 1 Hulk?

Please answer the following questions about the quiz you just took…

_____1) Which sentence best describes how you feel about the problems you just worked on?

A) The word problems were easier than the problems on the front.
B) The word problems were more difficult than the problems on the front.
C) All of the problems were pretty easy.
D) All of the problems were pretty difficult.

_____2) Which sentence best describes how you feel about the problems you just worked on?

A) I liked the word problems better than the problems on the front.
B) I liked the problems on the front better than the word problems.
C) I liked all of the problems.
D) I didn't like any of the problems.

_____3) Which problem seemed the <u>easiest</u> to solve?

A) Problem # 1
B) Problem # 2
C) Problem # 3
D) Problem # 4

_____4) Which problem seemed the <u>most difficult</u> to solve?

A) Problem # 1
B) Problem # 2
C) Problem # 3
D) Problem # 4

_____5) Which method for solving these problems do you like the most?

A) Substitution
B) Elimination
C) Guess and check
D) Some other method