Engaging in Conceptual Understanding of Limits in Secondary Calculus

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Abstract

This curriculum project includes conceptual understanding support for teaching limits in high school calculus. The non-sequential lessons focus on polynomial expansion and division, and composite functions as they apply to understanding limits. The definition of the derivative is presented conceptually. These are presented through guided notes, partner practice, and homework. Keys for each of the lessons can be found in the Appendix.

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Introduction

Conceptual understanding in secondary calculus classrooms has been a topic of discussion for many mathematics educators (NCTM, 2000, 2014; NGA Center & CCSSO, 2010; Wade et al., 2017). Cognitive psychology informs that students need to learn the why that concepts present before focusing on teaching procedures and shortcuts (Wade et al., 2023). When students understand the why they are better equipped to transfer their knowledge and skills to other classes and real-world scenarios. Skemp (2006) referred to conceptual understanding in mathematics as relational and instrumental. Relational means students know what to do and why while instrumental refers to rules without reason. As students are pushed to take more high school mathematics it is important that the courses offered are taught to better prepare them for success in college or university level mathematics. This curriculum project provides ways for teachers to build conceptual understanding by connecting prior knowledge with connections to real life and focusing on patterns and relationships within the concepts. The prior knowledge activated within these lessons come from Algebra I, Algebra II, and Precalculus. There has been a push for students to take more math credits in high school, so they are better prepared for postsecondary education.

The first lesson was designed to engage learners in secondary calculus by having students practice expanding and factoring polynomials needed to solve limits at specific x-values. The second lesson provides practice of polynomial division used in long division which is needed to find the limit as x approaches positive or negative infinity. Students will learn that the quotient they get is the slant asymptote equation. The third lesson builds conceptually how to find the average rate of change using equations. This serves as an introductory lesson for the limit definition of the derivative. In this lesson students will recall the differences between a secant line and a tangent line, and how they relate to average rate of change and instantaneous rate of change, respectively. Lastly, lesson four builds on lesson 3 and asks students to apply the limit definition of the derivative with composite functions. It scaffolds student understanding of the derivative by showing students how we apply composite functions to the limit definition. By utilizing these lesson plans, teachers will provide students with a better conceptual understanding of calculus in the secondary classroom.

Literature Review

The importance of developing conceptual understanding in advanced secondary mathematics has been discussed at great lengths (NCTM, 2000, 2014; NGA Center & CCSSO, 2010; Wade et al., 2017). The purpose is to enable the students to grasp a deeper understanding of the relations between concepts, recall them for future use, and apply their mathematical knowledge to new situations and problems. Rather than students memorizing facts it is beneficial when teachers use multiple representations to support student learning and understanding of concepts (Wade et al., 2023). This means teachers abandon tips, tricks, and shortcuts that can lead to students learning rules without reason, which is known as instrumental understanding (Skemp, 2006; Wade et al., 2017). Instead, mathematics teachers should seek for students to learn the why behind their mathematical computations, which has been referred to as relational understanding (Skemp, 2006; Wade et al., 2017).

Conceptual understanding can be built up using proper representations and mathematical literacy, Students who can represent a problem correctly are more likely to solve It correctly. In other words, “forming correct problem representations is one mechanism underlying the influence of conceptual knowledge on improvements in procedural knowledge” (Rittle-Johnson et al., 2001, pp. 358). This is due to students being able to correctly visualize the problem, which allows them to categorize the type of problem and use the procedural knowledge to complete it. Implementing these representations connects concepts such as numeric, algebraic, and geometric (Wade et al., 2017). For mathematical literacy, teachers need to place an increase emphasis on mathematical definitions, vocabulary, and mathematical reasoning. When teachers place importance on mathematical literacy, there has been shown to be a positive impact on *future* learning (Wade et al. 2017). In fact, Wade et al. (2017) determine that mathematical literacy was a significant indicator of collect calculus performance.

It has been noted that "conceptual and procedural knowledge may develop in a hand-over-hand process, rather than one type strictly preceding the other" (Rittle-Johnson et al., 2001, pp. 347). The conflict arises with administration campaigns for procedural knowledge which typically improves standardized test scores (Wade et al., 2017). Although procedural knowledge can assist students in completing assignments and tests, this type of knowledge does not incorporate the principles and relationships needed for conceptual understanding, nor for transfer of learning. Conceptual understanding is thus crucial to higher-order thinking, which results in better problem-solving and critical thinking skills. When students have a strong foundation of conceptual understanding, they can develop procedural knowledge *fluency*. Smith et al. (2018) further corroborates this by suggesting that lessons can focus particularly on "developing fluency based on previously developed conceptual understanding (p. 41). From this, we can propose that both procedural knowledge and conceptual understanding have a place in the classroom. We can interpret this to mean that when students have both conceptual and procedural knowledge before problem solving, they are more likely to succeed. Procedural knowledge and conceptual understanding do not have to be one or the other, in fact, both works together to improve students' overall understanding of mathematics.

Building upon mathematical concepts, this curriculum project presents four calculus lessons on limits, specifically limits that involve polynomial division, polynomial expansion, slope, and composite functions. Although there are no direct standards for Calculus listed in the New York State (NYS) Next Generation Standards, calculus is an important secondary mathematics class. Some schools across the U.S. offer Advanced Placement (AP) calculus, some offer dual enrollment calculus, and others off non-AP calculus. Each of these types of calculus build upon knowledge and skills learned from Algebra I, Algebra II, and Precalculus. Lesson one is on polynomial expansion as it applies to limits; lesson two focuses on polynomial division as it applies to limits; lesson three focuses on slope as it applies to the limit definition of the derivative; lesson four focuses on composite functions as it applies to the limit definition of the derivative. Each lesson begins with a warmup problem that serves as reminder of where the Calculus skill is derived from. All lessons include a formative assessment in the form of an exit ticket that can be used by the teacher to gauge understanding. For most lessons, there is additional practice in the form of homework that students should complete before returning to the next class. This can serve as a summative assessment to evaluate the students' learning each day.

These lessons encourage students to remember what was taught in previous mathematics courses, and to connect their prior knowledge to what they are currently learning. Teachers should consider prior knowledge and build relational connections into each of the lessons. By reinforcing conceptual connections across prior mathematics courses students can develop a deeper understanding of the limit, which is a crucial topic for understanding Calculus.

Curriculum

Lesson 1: Factoring with Limits

Overview	1. Students will have 5-7 minutes to complete the Warm-up, and then the teacher will go over it with the class. 2. Teacher will model factoring and expanding with the class as students complete the first three problems on the Introduction Practice sheet. 3. The students will pair up and complete the next three problems, practicing their "new" knowledge. The teacher will walk around and monitor. After 10-15 minutes, the class will regroup and discuss the problems. 4. Teacher will model applying the factorization method to Limits as students complete the first three problems on the Application to Limits sheet. 5. The students will pair up and complete the next three problems, applying their knowledge to Limits. The teacher will walk around and monitor. After 10-15 minutes, the class will regroup and discuss the problems. 6. The students will complete an Exit Ticket during the last 5-7 minutes of class and pick up their Homework on their way out of the classroom. The teacher will use this to gauge the students' understanding.
NYS Standards	None
Lesson Objectives	1. I can factor a quadratic equation using the Uncombine Like Terms (ULT) method. 2. I can apply the ULT method to solve a limit.
Assessment	<u>Summative:</u> Formal assessment at the end of the unit <u>Formative:</u> Exit ticket
Materials Needed	1. Warm-up 2. Introduction Practice 3. Application to Limits 4. Exit Ticket 5. Homework 6. Pencil/Pen

Daily Warm-Up

Name: ______________________________ Date: __________

Complete the following matching:

1. $(a+b)(a-b)$ _______
2. $a(b+c)$ _______
3. $(a+b)^2$ _______
4. $(a-b)^2$ _______

A. $a^2+2ab+b^2$

B. $a^2-2ab-b^2$

C. a^2-b^2

D. $ab+ac$

Introduction Practice

Name: ________________________________ Date: __________

Practice expanding and factoring (Uncombine Like Terms Method):
<u>We do together</u>

1. (x+6)(x-2)

2. $2x^2 - 5x - 12$

3. (3x+4)(2x-1)

4. $x^2 + 8x + 16$

You do with a partner

1. (x+1)(x+5)

2. $2x^2 - 11x + 15$

3. (2x+1)(2x-7)

4. $x^2 - 9x + 20$

Application to Limits

Name: ______________________________ Date: __________

Solve the limits using factorization method:

<u>We do together</u>

1. $\lim_{x \to -3} \frac{x^2+9x+18}{x+3}$

2. $\lim_{x \to 2} \frac{2x^2-7x+6}{x-2}$

3. $\lim_{x \to \frac{-2}{3}} \frac{9x^2-6x-8}{3x+2}$

<u>You do with a partner</u>

1. $\lim_{x \to 5} \frac{x^2-8x+15}{x-5}$

2. $\lim_{x \to 1} \frac{2x^2+3x-5}{x-1}$

3. $\lim_{x \to \frac{-2}{3}} \frac{9x^2+12x+4}{3x+2}$

Exit Ticket

Name: ______________________________ Date: __________

Complete the following problems on your own before leaving class:

1. $\lim_{x \to -5} \frac{3x^2+19x+20}{x+5}$

2. $\lim_{x \to 1} \frac{2x^2-9x-7}{x-1}$

Homework

Name: ________________________________ Date: __________

Complete the following problems:

1. $\lim_{x \to -5} \frac{x^2-25}{x+5}$

2. $\lim_{x \to 6} \frac{x^2-2x-24}{x-6}$

3. $\lim_{x \to -3} \frac{2x^2+4x+3}{x+3}$

4. $\lim_{x \to -\frac{5}{2}} \frac{6x^2+29x+35}{2x+5}$

5. $\lim_{x \to \frac{1}{3}} \frac{9x^2-9x+3}{3x-1}$

Lesson 2: Polynomial Division with Limits

Overview	1. Students will have 5-7 minutes to complete the Warm-up, and then the teacher will go over it with the class. 2. Teacher will go over key characteristics of a slant/oblique asymptote with the class. Then, will model polynomial division with square and cube powers. Students will complete the first three problems on the Class Notes sheet as teacher models. 3. The students will pair up and complete the next three problems, implementing their new knowledge. The teacher will walk around and monitor. After 10-15 minutes, the class will regroup and discuss the problems. 4. Teacher will go over why polynomial division and slant/oblique asymptotes are important for limits approaching infinity. Then, will model applying polynomial division to Limits. Students will complete the first three problems on the Application to Limits sheet as teacher models. 5. The students will pair up and complete the next three problems, applying the foundational knowledge to Limits. The teacher will walk around and monitor. After 10-15 minutes, the class will regroup and discuss the problems. 6. The students will complete an Exit Ticket during the last 5-7 minutes of class and pick up their Homework on their way out of the classroom. The teacher will use this to gauge the students' understanding.
NYS Standards	None
Lesson Objectives	1. I understand the purpose of slant/oblique asymptotes. 2. I understand the application of polynomial division to limits.
Assessment	<u>Summative:</u> Formal assessment at the end of the unit <u>Formative:</u> Exit ticket
Materials Needed	1. Warm-up 2. Class Notes 3. Application to Limits 4. Exit Ticket 5. Homework 6. Pencil/Pen

Daily Warm-Up

Name: ______________________________ Date: __________

Complete the following division problems using long division:

1. $96 \div 3$
2. $194 \div 6$
3. $237 \div 14$
4. $629 \div 13$

Class Notes

Name: ______________________________ Date: __________

What is a slant/oblique asymptote? (Think characteristics!)

1. A slant asymptote occurs when the degree of the numerator is exactly _______________ than the degree of the denominator.
2. A slant asymptote is neither _____________ nor ______________.
3. The function ________________ but does not touch the line.
4. The equation of a slant asymptote is always __________.

Practice polynomial division with square and cube powers:

We do together

1. $(x^2 + 5x + 6) \div (x + 2)$
2. $(x^2 - 25) \div (x + 5)$
3. $(x^3 + 2x^2 - 16x + 4) \div (x^2 - 3x - 1)$

You do with a partner

1. $(x^2 - 4) \div (2x - 3)$
2. $(x^2 + 6x + 9) \div (x + 2)$

3. $(x^3 + 3\mathrm{x}^2 - 4x - 5) \div (x^2 - 4)$

Application to Limits

Name: ________________________________ Date: __________

Solve the limits using polynomial division:

Why do we use polynomial division in limit problems?

1. Finding the slant asymptote is useful when evaluating a limit that approaches ____.

Important Note: we ignore the remainder of the division problem when describing the slant asymptote.

<u>We do together</u>

1. $\lim\limits_{x \to -\infty} \frac{x^2+10x+29}{x+3}$

2. $\lim\limits_{x \to \infty} \frac{2x^2+3x-1}{2x+1}$

3. $\lim\limits_{x \to -\infty} \frac{6x^3-37x^2+57x-11}{2x^2-9x+4}$

<u>You do with a partner</u>

1. $\lim\limits_{x \to -\infty} \frac{5x^2-3x-36}{x-3}$

2. $\lim\limits_{x \to \infty} \frac{2x^2}{3x+1}$

3. $\lim_{x\to\infty} \frac{x^3+4x^2-5x-14}{x^2+6x+7}$

Exit Ticket

Name: ______________________________ Date: __________

Complete the following problems on your own before leaving class:

1. In your own words, jot down what a slant asymptote is. (Think characteristics!)

2. $\lim_{x \to -\infty} \frac{5x^3+26x^2-25x+13}{5x^2-4x-1}$

Homework

Name: ______________________________ Date: __________

Complete the following problems using polynomial division:

1. $(x^3 + 8) \div (x^2 - 2x + 4)$
2. $(2x^2 + 7x - 4) \div (2x + 13)$

Solve the following limits using polynomial division:

1. $\lim_{x \to -\infty} \frac{2x^2+3x+1}{x+2}$
2. $\lim_{x \to \infty} \frac{3x^3+11x^2-14x}{x^2+5x+2}$

Lesson 3: Limit Definition of the Derivative Introduction Using Slope

Overview	1. Students will have 5-7 minutes to complete the Warm-up, and then the teacher will go over it with the class. 2. Teacher will begin with the Class Notes. The Class Notes review previous knowledge (average slope). Teacher will bridge the previous knowledge to the new knowledge (instantaneous slope) through secant and tangent lines. Allow 15-20 minutes for this. 3. Students will pair up with a peer to work through the Partner Practice. This sheet reinforces the bridge between previous knowledge and new knowledge form the Class Notes. Teacher will walk around and monitor understanding. Allow 15-20 minutes for this. 4. The students will complete an Exit Ticket during the last 5-7 minutes of class. The teacher will use this to gauge the students' understanding. There will be no Homework since this is an introductory lesson to the unit, there has been no application of the knowledge yet.
NYS Standards	None
Lesson Objectives	1. I understand what the purpose of the limit definition is. 2. I understand the relationship between the average slope and the instantaneous slope. 3. I understand the purpose between the secant line and the tangent line.
Assessment	<u>Summative:</u> Formal assessment at the end of the unit <u>Formative:</u> Exit ticket
Materials Needed	1. Warm-up 2. Class Notes 3. Partner Practice 4. Exit Ticket 5. Pencil/Pen

Teachers Notes: while walking the students through the Class Notes, make sure to draw connections between average rate of change and secant lines, and instantaneous rate of change and tangent lines. For example, after identifying the rate of change as average in question 3, ask students if they think this would produce a secant line or a tangent line. In question 4, pose the same question regarding instantaneous rate of change. Allow time for discussion if any students are confused or need clarification. (If I need to move this, I will, and all graphs were created by me).

Daily Warm-Up

Name: ______________________________ Date: __________

Complete the following:

1. Jot down what the slope is, including the definition and the equation.

2. Find the slope of the line that connects the two points R (2, 6) and S (-2, -1).

3. Find the slope of the line that connects the two points A (1, -4) and B (-4, 5).

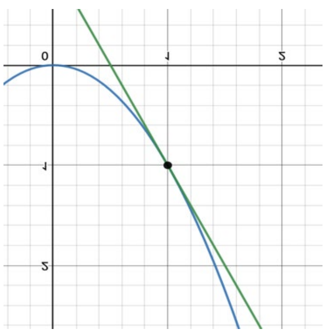
Class Notes

Name: ______________________________ Date: __________

Important definitions:

1. A secant line is a line passing through _______________ of a curve.
2. A tangent line is a line that touches a curve at ______________, matching the slopes curve there.
3. Label the graphs based on the line:

____



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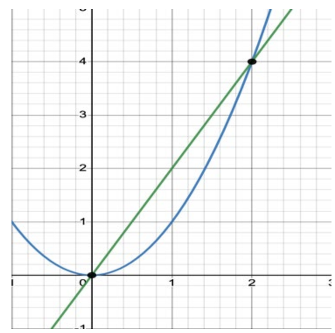


Consider an object is moving and its position s(t) is measured in feet and depends on t in seconds.

$$s(t) = t^2 + 3$$

1. Where is the object at the 1st second?
2. Where is the object at the 2nd second?
3. What is the rate of change between the 1st and the 2nd second?

This is the _________________________!

4. What is the instantaneous rate of change at exactly the 1st second?

These steps allow us to get infinitely closer to 1 but does not give us our exact answer. To find the instantaneous rate of change at the 1st second, we would need to use the limit definition!

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Label the parts of the limit definition:

1. $x \to a$: ______________
2. $f(x)$: ______________
3. $f(a)$: ______________
4. x : ______________
5. a : ______________

Partner Practice

Name: ______________________________ Date: __________

Complete the following questions with a partner:

1. Match the following terms to their corresponding picture or definition:

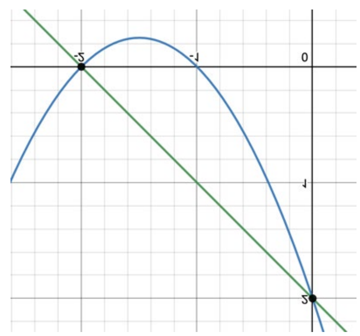
   a. Secant line

   b. Tangent line

   c. Average rate of change

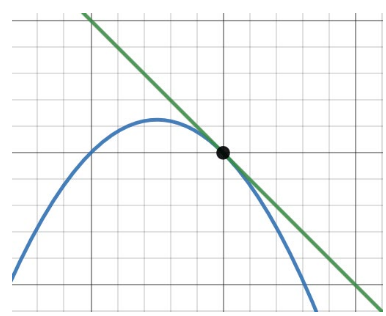
   d. Instantaneous rate of change

   i.



   ii. The change in rate at a particular instant.

   iii.



   iv. A measure of how much the function changed per unit over an interval.

2. Consider an object moving and its position s(t) is measured in yards and depends on t in minutes.

$$s(t) = t^3 + 2t^2 - 5t - 6$$

a. Find the objects position at t=1.

b. Find the objects position at t=3.

c. What is the average rate of change from t=1 to t=3?

d. List the first four "steps" of finding the instantaneous rate of change. (Hint: look at Class Notes question 4).

    i. What are the steps of finding the instantaneous rate of change doing at t=3?

Exit Ticket

Name: ______________________________ Date: __________

Answer the following question before leaving class:

1. Explain the differences between a tangent line and a secant line. Be sure to include which line corresponds to which rate of change.

Lesson 4: Limit Definition of the Derivative Application with Composite Functions

Overview	1. Students will have 5-7 minutes to complete the Warm-up, and then the teacher will go over it with the class. 2. Teacher will begin with the Class Notes. The Class Notes review previous knowledge (composite functions). Teacher will bridge the previous knowledge to the new knowledge (limit definition of the derivative) through instantaneous rate of change/slope of the tangent line. Allow for 15-20 minutes for this. 3. Students will pair up with a peer to work through the Partner Practice. This sheet reinforces the bridge between previous knowledge and new knowledge form the Class Notes. Teacher will walk around and monitor understanding. Allow for 15-20 minutes for this. 4. The students will complete an Exit Ticket during the last 5-7 minutes of class and pick up their Homework on their way out of the classroom. The teacher will use this to gauge the students' understanding.
NYS Standards	None
Lesson Objectives	1. I understand what a composite function is. 2. I understand how comparative functions relate to the limit definition of derivatives.
Assessment	<u>Summative:</u> Formal assessment at the end of the unit <u>Formative:</u> Exit ticket
Materials Needed	1. Warm-up 2. Class Notes 3. Partner Practice 4. Exit Ticket 5. Homework 6. Pencil/Pen

Daily Warm-Up

Name: ______________________________ Date: __________

Complete the following:

1. Write down the limit definition of derivatives.

2. Why do we find the derivative of a function at a particular x-value?

Class Notes

Name: ______________________________ Date: __________

Important Notes:

1. What is a composite function?

2. What is the notation of function composition?

Evaluate the following composite functions:

1. f(x) = 3x+7 g(x) = x-4
   (f∘g)(x) = __________

2. f(x) = x^2+1 g(x) = 2x+6
   g(f(2)) = ________

3. f(x) = x^2+4 g(x) = 2x^2-3
   g(f(x)) = __________

4. f(x) = 3x+11 g(x) = x^3+2x^2+x-5
   (f∘g)(2) = ________

Let's apply our knowledge of composite functions to the limit definition of the derivative:

1. f(x) = 2x-5 ; x = 4

2. f(x) = -3x-4 ; x = -1

3. $g(x) = x^2-4$; $x = 3$

4. $g(x) = x^2-4x+2$; $x = 3$

5. $h(x) = x^3-8x^2-2x+1$; $x = 2$

6. $h(x) = -2x^2-3x-1$; $x = -5$

Partner Practice

Name: ______________________________ Date: __________

Complete the following questions using function composition and the limit definition with a partner:

1. $f(x) = 6x-5$; $x = -2$

   a. What does the answer mean?

2. $g(x) = 2x+9$; $x = 5$

3. $h(x) = x^2-9$; $x = -4$

4. $g(x) = x^3-2x^2-x+4$; $x = -3$

5. $h(x) = 2x^3-4x+3$; $x = 1$

6. f(x) - $3x^2$+2x-5 ; x = -6

Exit Ticket

Name: ______________________________ Date: __________

Answer the following question before leaving class:

1. Given f(x) = x^2-3x+2 and g(x) = 2x-9, find the instantaneous rate of change of h(x) = (g∘f) at x = 1.

Homework

Name: ______________________________ Date: __________

Complete the following problems using function composition:

1. $f(x) = 3x-11$ $g(x) = x^2+4x+1$
   $(f(g(2)) =$ _______

2. $f(x) = 4x-1$ $g(x) = 3x+7$
   $(g \circ f)(3) =$ _______

Complete the following problems using function composition and the limit definition of derivatives:

1. $g(x) = 2x^2+3x+8$; $x = -2$

2. $f(x) = x^3-2x^2+4x+1$; $x = 1$

3. Given $f(x) = 3x+8$ and $g(x) = x^2-4x+5$, find the slope of the tangent line for $f(g(x))$ at $x = -2$.

Conclusion

This curriculum project offers students the opportunity to see how mathematical ideas evolve and interconnect, rather than appear as disjointed topics. Students who make these connections can grow in their mathematical confidence. The secondary calculus content chosen for this curriculum project was selected based on the conceptual connections between algebra and calculus. The topics chosen not only represent foundational skills in the study of calculus, but also the powerful bridge that connects calculus ideas to algebraic reasoning. Students in high school (secondary) calculus must recall algebraic topics such as factoring and polynomial division to understand limits. When teachers connect core algebraic concepts that are required to build understanding in calculus then connections can develop in meaningful ways. Such learning opportunities allow students to build relational understanding (knowing what to do and why) by anchoring the rise-over-run structure to foundational calculus concepts. It is my hope that other mathematics teachers can use these lessons in their classroom to scaffold student understanding of resolving indeterminate forms of limits.

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Appendix

Lesson 1: Factoring with Limits

Daily Warm-Up

Name: ______________________________ Date: __________

Complete the following matching:

1. $(a+b)(a-b)$ ___C___ A. $a^2+2ab+b^2$

a. $a(b+c)$ ___D___ B. $a^2-2ab-b^2$

b. $(a+b)^2$ ___A___ C. a^2-b^2

c. $(a-b)^2$ ___B___ D. $ab+ac$

Introduction Practice

Name: ______________________________ Date: __________

Practice expanding and factoring (Uncombine Like Terms Method):

<u>We do together</u>

1. (x+6)(x-2)

$x^2 + 4x - 12$

2. $2x^2 - 5x - 12$

(2x+3)(x-4)

3. (3x+4)(2x-1)

$6x^2 + 5x - 4$

4. $x^2 + 8x + 16$

(x+4)(x+4)

<u>You do with a partner</u>

1. (x+1)(x+5)

$x^2 + 6x + 5$

2. $2x^2 - 11x+15$

(2x-5)(x-3)

3. (2x+1)(2x-7)

$4x^2 - 12x - 7$

4. $x^2 - 9x + 20$

(x-5)(x-4)

Application to Limits

Name: ________________________________ Date: __________

Solve the limits using factorization method:

<u>We do together</u>

1. $\lim_{x \to -3} \frac{x^2+9x+18}{x+3}$

$\lim_{x \to -3} \frac{x^2+9x+18}{x+3} = 3$

2. $\lim_{x \to 2} \frac{2x^2-7x+6}{x-2}$

$\lim_{x \to 2} \frac{2x^2-7x+6}{x-2} = 1$

3. $\lim_{x \to \frac{-2}{3}} \frac{9x^2-6x-8}{3x+2}$

$\lim_{x \to \frac{-2}{3}} \frac{9x^2-6x-8}{3x+2} = -6$

<u>You do with a partner</u>

4. $\lim_{x \to 5} \frac{x^2-8x+15}{x-5}$

$\lim_{x \to 5} \frac{x^2-8x+15}{x-5} = 2$

5. $\lim_{x \to 1} \frac{2x^2+3x-5}{x-1}$

$\lim_{x \to 1} \frac{2x^2+3x-5}{x-1} = 7$

6. $\lim_{x \to \frac{-2}{3}} \frac{9x^2+12x+4}{3x+2}$

$\lim_{x \to \frac{-2}{3}} \frac{9x^2+12x+4}{3x+2} = 0$

Exit Ticket

Name: ______________________________ Date: __________

Complete the following problems on your own before leaving class:

1. $\lim_{x \to -5} \frac{3x^2+19x+20}{x+5}$

$\lim_{x \to -5} \frac{3x^2+19x+20}{x+5}$ = -11

2. $\lim_{x \to 1} \frac{2x^2-9x-7}{x-1}$

$\lim_{x \to 1} \frac{2x^2-9x-7}{x-1}$ = -5

Homework

Name: ________________________________ Date: __________

Complete the following problems:

1. $\lim_{x \to -5} \frac{x^2-25}{x+5}$

$\lim_{x \to -5} \frac{x^2-25}{x+5}$ = -10

2. $\lim_{x \to 6} \frac{x^2-2x-24}{x-6}$

$\lim_{x \to 6} \frac{x^2-2x-24}{x-6}$ = 10

3. $\lim_{x \to -3} \frac{2x^2+4x+3}{x+3}$

$\lim_{x \to -3} \frac{2x^2+4x+3}{x+3}$ = -5

4. $\lim_{x \to -\frac{5}{2}} \frac{6x^2+29x+35}{2x+5}$

$\lim_{x \to -\frac{5}{2}} \frac{6x^2+29x+35}{2x+5}$ = - $\frac{1}{2}$

5. $\lim_{x \to \frac{1}{3}} \frac{9x^2-9x+3}{3x-1}$

$\lim_{x \to \frac{1}{3}} \frac{9x^2-9x+3}{3x-1}$ = -1

Lesson 2: Polynomial Division with Limits

Daily Warm-Up

Name: ______________________________ Date: __________

Complete the following division problems using long division:

1. $96 \div 3$

$96 \div 3 = 32\ R\ 0$

2. $194 \div 6$

$194 \div 6 = 32\ R\ 2$

3. $237 \div 14$

$237 \div 14 = 16\ R\ 13$

4. $629 \div 13$

$629 \div 13 = 48\ R\ 5$

Class Notes

Name: ______________________________ Date: __________

What is a slant/oblique asymptote? (Think characteristics!)

1. A slant asymptote occurs when the degree of the numerator is exactly one more than the degree of the denominator.
2. A slant asymptote is neither horizontal nor vertical.
3. The function approaches but does not touch the line.
4. The equation of a slant asymptote is always linear.

Practice polynomial division with square and cube powers:

We do together

1. $(x^2 + 5x + 6) \div (x + 2)$

$(x^2 + 5x + 6) \div (x + 2) = x + 3\ R\ 0$

2. $(x^2 - 25) \div (x + 5)$

$(x^2 - 25) \div (x + 5)$ = x+5 R 0

3. $(x^3 + 2x^2 - 16x + 4) \div (x^2 - 3x - 1)$

$(x^3 + 2x^2 - 16 + 4) \div (x^2 - 3x - 1)$ = x+5 R 9

You do with a partner

4. $(x^2 - 4) \div (2x - 3)$

$(x^2 - 4) \div (2x - 3)$= $\frac{1}{2}x + 1\ R$ 1

5. $(x^2 + 6x + 9) \div (x + 2)$

$(x^2 + 6x + 9) \div (x + 2)$= x+4 R 1

6. $(x^3 + 3\text{x}^2 - 4x - 5) \div (x^2 - 4)$

$(x^3 + 3\text{x}^2 - 4x - 5) \div (x^2 - 4)$= x+3 R 7

Application to Limits

Name: ________________________________ Date: __________

Solve the limits using polynomial division:

Why do we use polynomial division in limit problems?

1. Finding the slant asymptote is useful when evaluating a limit that approaches positive or negative infinity.

Important Note: we ignore the remainder of the division problem when describing the slant asymptote.

<u>We do together</u>

1. $\lim\limits_{x \to -\infty} \frac{x^2+10x+29}{x+3}$

$\frac{x^2+10x+29}{x+3}$ = x+7 R 8

$\lim\limits_{x \to -\infty} x + 7 = -\infty$

2. $\lim\limits_{x \to \infty} \frac{2x^2+3x-1}{2x+1}$

$\frac{2x^2+3x-1}{2x+1} = x + 1\ R - 2$

$\lim\limits_{x \to \infty} x + 1 = \infty$

3. $\lim\limits_{\to -\infty} \frac{6x^3-37x^2+57x-11}{2x^2-9x+4}$

$\frac{6x^3-37x^2+57x-11}{2x^2-9x+4}$= 3x-5 R 9

$\lim\limits_{x \to -\infty} 3x - 5$= -∞

<u>You do with a partner</u>

4. $\lim\limits_{x \to -\infty} \frac{5x^2-3x-36}{x-3}$

5. $\lim\limits_{x \to \infty} \frac{2x^2}{3x+1}$

$\frac{5x^2-3x-36}{x-3}$= 3x+12 R 0

$\lim_{x\to-\infty} 3x + 12$= -∞

$\frac{2x^2}{3x+1}$= $\frac{2}{3}x - \frac{2}{9}$ R $\frac{2}{9}$

$\lim_{x\to\infty} \frac{2}{3}x - \frac{2}{9}$= ∞

6. $\lim_{x\to\infty} \frac{x^3+4x^2-5x-14}{x^2+6x+7}$

$\frac{x^3+4x^2-5x-14}{x^2+6x+7}$= x-2 R 0

$\lim_{x\to\infty} x - 2$= ∞

Exit Ticket

Name: ______________________________ Date: __________

Complete the following problems on your own before leaving class:

1. In your own words, jot down what a slant asymptote is. (Think characteristics!)

- Numerator is one more than denominator
- Always linear
- Approaches but does not touch line
- Not horizontal or vertical

2. $\lim\limits_{x \to -\infty} \frac{5x^3+26x^2-25x+13}{5x^2-4x-1}$

$\frac{5x^3+26x^2-25x+13}{5x^2-4x-1}$ = x+6 R 19

$\lim\limits_{x \to -\infty} x + 6$ = -∞

Homework

Name: ______________________________ Date: __________

Complete the following problems using polynomial division:

1. $(x^3+8) \div (x^2-2x+4)$
2. $(2x^2+7x-4) \div (2x+13)$

$(x^3+8) \div (x^2-2x+4)$ = x+2 R 0

$(2x^2+7x-4) \div (2x+13)$ = x-3 R 35

Solve the following limits using polynomial division:

1. $\lim\limits_{x\to-\infty} \frac{2x^2+3x+1}{x+2}$
2. $\lim\limits_{x\to\infty} \frac{3x^3+11x^2-14x}{x^2+5x+2}$

$\frac{2x^2+3x+1}{x+2}$ = 3x +1 R 1

$\lim\limits_{x\to\infty} 3x+1 = \infty$

$\frac{3x^3+11x^2-14x}{x^2+5x+2} = 3x-4\ R\ 8$

$\lim\limits_{x\to-\infty} 3x-4 = -\infty$

Lesson 3: Limit Definition of the Derivative Introduction Using Slope

Daily Warm-Up

Name: ________________________________ Date: __________

Complete the following:

1. Jot down what the slope is, including the definition and the equation.

- Measures the steepness of a function from one coordinate point to another
- $m = \frac{y_2 - y_1}{x_2 - x_1}$

2. Find the slope of the line that connects the two points R (2, 6) and S (-2, -1).

$$m = \frac{\Delta y}{\Delta x} = \frac{6-(-1)}{2-(-2)} = \frac{7}{4}$$

3. Find the slope of the line that connects the two points A (1, -4) and B (-4, 5).

$$m = \frac{\Delta y}{\Delta x} = \frac{-4-5}{1-(-4)} = \frac{-9}{5}$$

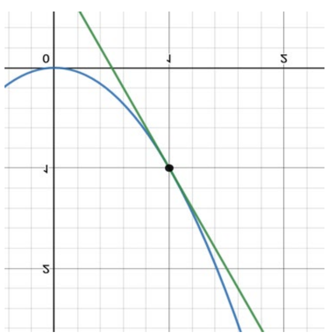
Class Notes

Name: ________________________________ Date: __________

Important definitions:

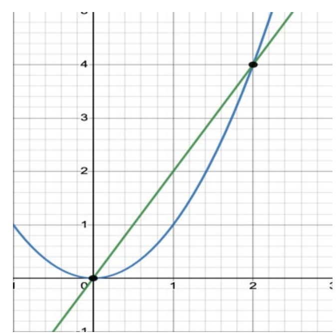
1. A secant line is a line passing through two points of a curve.

2. A tangent line is a line that touches a curve at only one point, matching the slopes curve there.



3. Label the graphs based on the line:

Tangent line



Secant line

Consider an object is moving and its position s(t) is measured in feet and depends on t in seconds.

$$s(t) = t^2 + 3$$

1. Where is the object at the 1st second?

$s(1) = (1)^2+3 = 4$ feet

2. Where us the object at the 2nd second?

$s(2) = (2)^2+3 = 7$ feet

3. What is the rate of change between the 1st and the 2nd second?

m = $\frac{s(t_2)-s(t_1)}{t_2-t_1} = \frac{7-4}{2-1}$ = 3 ft/sec

This is the average rate of change!

4. What is the instantaneous rate of change at exactly the 1st second?

m = $\frac{s(1.1)-s(1)}{1.1-1}$

m = $\frac{s(1.01)-s(1)}{1.01-1}$

m = $\frac{s(1.001)-s(1)}{1.001-1}$

and so on...

These steps allow us to get infinitely closer to 1 but does not give us our exact answer. To find the instantaneous rate of change at the 1st second, we would need to use the limit definition!

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Label the parts of the limit definition:

1. $x \to a$: as x approaches the point "a"
2. $f(x)$: the original function f(x)
3. $f(a)$: the function at point "a"
4. x : the variable "x"
5. a : the point "a"

Partner Practice

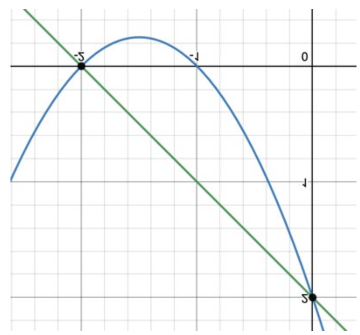
Name: ________________________________ Date: __________

Complete the following questions with a partner:

1. Match the following terms to their corresponding picture or definition:

   a. Secant line

   i.

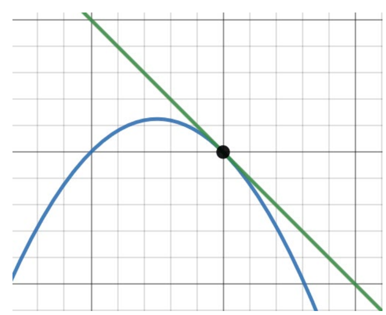


   b. Tangent line

   ii. The change in rate at a particular instant.

   c. Average rate of change

   iii.



   d. Instantaneous rate of change

   v. A measure of how much the function changed per unit over an interval.

2. Consider an object moving and its position s(t) is measured in yards and depends on t in minutes.

$$s(t) = t^3 + 2t^2 - 5t - 6$$

a. Find the objects position at t=1.

$s(1) = (1)^3 + 2(1)^2 - 5(1) - 6 = -8$

b. Find the objects position at t=3.

$s(3) = (3)^3 + 2(3)^2 - 5(3) - 6 = 24$

c. What is the average rate of change from t=1 to t=3?

$m = \frac{24-(-8)}{3-1} = \frac{32}{2} = 16$

d. List the first four “steps” of finding the instantaneous rate of change. (Hint: look at Class Notes question 4).

$m = \frac{s(3.1)-s(3)}{3.1-3}$

$m = \frac{s(3.01)-s(3)}{3.01-3}$

$m = \frac{s(3.001)-s(3)}{3.001-3}$

$m = \frac{s(3.0001)-s(3)}{3.0001-3}$

i. What are the steps of finding the instantaneous rate of change doing at t=3?

Approaching t=3.

Exit Ticket

Name: ______________________________ Date: __________

Answer the following question before leaving class:

1. Explain the differences between a tangent line and a secant line. Be sure to include which line corresponds to which rate of change.

A tangent line touches the function/curve only once. It corresponds to the instantaneous rate of change.

A secant line goes through the function/curve at two points. It corresponds to the average rate of change.

Lesson 4: Limit Definition of the Derivative Application with Composite Functions

Daily Warm-Up

Name: ______________________________ Date: __________

Complete the following:

1. Write down the limit definition of derivatives.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

2. Why do we find the derivative of a function at a particular x-value?

- Find the instantaneous rate of change
- Find the slope of the tangent line at that point

Class Notes

Name: ________________________________ Date: __________

Important Notes:

1. What is a composite function?

An operation "∘" that takes two functions f and g, and produces a function h. in this operation, the function g is applied to the result of applying the function f to x.

2. What is the notation of function composition?

h(x) = g(f(x)) H(x) = (g∘f)(x)

$$\boxed{x} \xrightarrow{g} \boxed{g(x)} \xrightarrow{f} \boxed{f(g(x))}$$

Evaluate the following composite functions:

1. f(x) = 3x+7 g(x) = x-4
   (f∘g)(x) = 3x-5
2. f(x) = x^2+1 g(x) = 2x+6
   g(f(2)) = 16
3. f(x) = x^2+4 g(x) = 2x^2-3
   g(f(x)) = 2x^4+16x^2+29
4. f(x) = 3x+11 g(x) = x^3+2x^2+x-5
   (f∘g)(2) = 2

Let's apply our knowledge of composite functions to the limit definition of the derivative:

1. f(x) = 2x-5 ; x = 4

a = 4 ; f(a=4) = 2(4)-5 = 3

$$\lim_{x\to 4}\frac{2x-5-(3)}{x-4} = \lim_{x\to 4}\frac{2x-8}{x-4} = \lim_{x\to 4}\frac{2(x-4)}{x-4} = \lim_{x\to 4} 2 = 2$$

2. f(x) = -3x-4 ; x = -1

a = -1 ; f(a=-1) = -3(-1)-4 = -1

$\lim_{x \to -1} \frac{-3x-4-(-1)}{x-(-1)} = \lim_{x \to -1} \frac{-3x-3}{x+1} = \lim_{x \to -1} \frac{-3(x+1)}{x+1} = \lim_{x \to -1} -3$ = -3

3. g(x) = x^2-4 ; x = 3

a = 3 ; g(a=3) = $(3)^2$-4 = 5

$\lim_{x \to 3} \frac{x^2-4-(5)}{x-3} = \lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x \to 3} x + 3$ = 3+3 = 6

4. g(x) = x^2-4x+2 ; x = -2

a = -2 ; g(a=-2) = $(-2)^2$-4(-2)+2 = 4+8+2 = 14

$\lim_{x \to -2} \frac{x^2-4x+2-(14)}{x-(-2)} = \lim_{x \to -2} \frac{x^2-4x-12}{x+2} = \lim_{x \to -2} \frac{(x-6)(x+2)}{x+2} = \lim_{x \to -2} x - 6$ = -2-6 = -8

5. h(x) = x^3-8x^2-2x+1 ; x = 2

a = 2 ; g(a=2) = $(2)^3$-8$(2)^2$-2(2)+ = 8-32-4+1 = -27

$\lim_{x \to 2} \frac{x^3-8x^2-2x+1-(-27)}{x-(2)} = \lim_{x \to 2} \frac{x^2-8x^2-2x+28}{x-2} = \lim_{x \to 2} \frac{(x^2-6x-14)(x-2)}{x-2} = \lim_{x \to 2} x^2 - 6x - 14$ = $(2)^2$-6(2)-14 = 4-12-14 = -22

6. h(x) = -2x^2-3x-1 ; x = -5

a = -5 ; g(a=15) = -2$(-5)^2$-3(-5)-1 = -50+15-1 = -36

$\lim_{x \to -5} \frac{-2x^2-3x-1-(-36)}{x-(-5)} = \lim_{x \to -5} \frac{-2x^2-3x+35}{x+5} = \lim_{x \to -5} \frac{(-2x+7)(x+5)}{x+5} = \lim_{x \to -5} -2x + 7 =$ -2(-5)+7 = 17

Partner Practice

Name: ________________________________ Date: __________

Complete the following questions using function composition and the limit definition with a partner:

1. f(x) = 6x-5 ; x = -2

a = -2 ; f(a=-2) = 6(-2)-5 = -12-5 = -17

$\lim_{x \to -2} \frac{6x-5-(-17)}{x-(-2)} = \lim_{x \to -2} \frac{6x+12}{x+2} = \lim_{x \to -2} \frac{6(x+2)}{x+2} = \lim_{x \to -2} 6 = 6$

a. What does the answer mean?
   The slope of the tangent line at x = -2 is 6.

2. g(x) = 2x+9 ; x = 5

a = 5 ; g(a=5) = 2(5)+9 = 10+9 = 19

$\lim\limits_{x \to 5} \frac{2x+9-(19)}{x-(5)} = \lim\limits_{x \to 5} \frac{2x-10}{x-5} = \lim\limits_{x \to 5} \frac{2(x-5)}{x-5} = \lim\limits_{x \to 5} 2 = 2$

3. h(x) = x^2-9 ; x = -4

a = -4 ; h(a=-4) = $(-4)^2$-9 = 16-9 = 7

$\lim\limits_{x \to -4} \frac{x^2-9-(7)}{x-(-4)} = \lim\limits_{x \to -4} \frac{x^2-16}{x+4} = \lim\limits_{x \to -4} \frac{(x-4)(x+4)}{x+4} = \lim\limits_{x \to -4} x-4$ = -4-4 = -8

4. g(x) = x^3-2x^2-x+4 ; x = -3

a = -3 ; g(a=-3) = $(-3)^3$-2$(-3)^2$-(-3)+4 = -27-28+3+4 = -38

$\lim\limits_{x \to -3} \frac{x^3-2x^2-x+4-(-38)}{x-(-3)} = \lim\limits_{x \to -3} \frac{x^3-2x^2-x+42}{x+3} = \lim\limits_{x \to -3} \frac{(x^2-5x+14)(x+3)}{x+3} =$

$\lim\limits_{x \to -3} x^2-5x+14 = (-3)^2-5(-3)+14 =$ 38

5. h(x) = 2x^3-4x+3 ; x = 1

a = 1 ; h(a=1) = 2$(1)^3$-4(1)+3 = 2-4+3 = 1

$\lim\limits_{x \to 1} \frac{2x^3-4x+3-(1)}{x-(1)} = \lim\limits_{x \to 1} \frac{2x^3-4x+2}{x-1} = \lim\limits_{x \to 1} \frac{(2x^2+2x-2)(x-1)}{x-1} = \lim\limits_{x \to 1} 2x^2+2x-2 =$

$2(1)^2+2(1)-2 =$ 2

6. f(x) = 3x^2+2x-5 ; x = -3

a = -3 ; f(a=-3) = -3$(-3)^2$+2(-3)-5 = 27-6-5 = 16

$\lim\limits_{x \to -3} \frac{3x^2+2x-5-(16)}{x-(-3)} = \lim\limits_{x \to -3} \frac{3x^2+2x-21}{x+3} = \lim\limits_{x \to -3} \frac{(3x-7)(x+3)}{x+3} = \lim\limits_{x \to -3} 3x-7$ = 3(-3)-7

= -16

Exit Ticket

Name: _______________________________ Date: __________

Answer the following question before leaving class:

1. Given f(x) = x^2-3x+2 and g(x) = 2x-9, find the instantaneous rate of change of h(x) = (g∘f) at x = 1.

h(x) = (g∘f)(x) = 2(x^2-3x+2)-9 = 2x^2-6x+4-9 = 2x^2-6x-5

a = 1 ; h(a=1) = 2$(1)^2$-6(1)-5 = 2-6-5 = -9

$\lim_{x\to 1}\frac{2x^2-6x-5-(-9)}{x-1} = \lim_{x\to 1}\frac{2x^2-6x+4}{x-1} = \lim_{x\to 1}\frac{(2x-4)(x-1)}{x-1} = \lim_{x\to 1} 2x-4$ = 2(1)-4 = -2

Homework

Name: ________________________________ Date: __________

Complete the following problems using function composition:

1. f(x) = 3x-11 g(x) = x^2+4x+1
   (f(g(2)) = 28
2. f(x) = 4x-1 g(x) = 3x+7
   (g∘f)(3) = 63

Complete the following problems using function composition and the limit definition of derivatives:

1. g(x) = 2x^2+3x+8 ; x = -2

a = -2 ; g(a=2) = 10

$$\lim_{x\to -2}\frac{2x^2+3x+8-(10)}{x-(-2)} = \lim_{x\to -2}\frac{2x^2+3x-2}{x+2} =$$

$$\lim_{x\to -2}\frac{(2x-1)(x+2)}{x+2} = \lim_{x\to -2} 2x-1 = -5$$

2. f(x) = x^3-2x^2+4x+1 ; x = 1

a = 1 ; f(a=1) = 4

$$\lim_{x\to 1}\frac{x^3-2x^2+4x+1-(4)}{x-(1)} = \lim_{x\to 1}\frac{x^3-2x^2+4x-3}{x-1}$$

$$= \lim_{x\to 1}\frac{(x^2-x+3)(x-1)}{x-1} = \lim_{x\to 1} x^2-x+3$$

= 3

1. Given f(x) = 3x+8 and g(x) = x^2-4x+5, find the slope of the tangent line for f(g(x)) at x = -2.

f(g(x)) = 3x^2-12x+23

a = -2 ; f(g(-2)) = 59

$$\lim_{x\to -2}\frac{3x^2-12x+23-(59)}{x-(-2)} = \lim_{x\to -2}\frac{3x^2-12x-36}{x+2} = \lim_{x\to -2}\frac{(3x-18)(x+2)}{x+2} = \lim_{x\to -2} 3x-18 =$$

-24