Mathematical Modeling of Fish Populations in Lake Ontario using Differential Equations

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By

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Introduction

The purpose of this research is to use mathematical models to study the connection between the rainbow trout fish population and the lamprey population in Lake Ontario. These species have a parasite/host relationship. The lamprey, a destructive and invasive species, give the rainbow trout scars and wounds that hinder their life spans. I chose to use models that are traditionally used for predator/prey relationships. It is an acceptable method because by definition predation includes parasitism [8]. Besides, mathematical models will only take the most dominant features into account.

The predator/prey model quantifies what happens when the predators eat their prey. In this case the lamprey are not eating the trout, but they are still causing them harm. Because the harm tends to negatively impact the trout's reproduction rate, the model seems well-suited for this situation. After studying available data I adopted a system of two differential equations that incorporate parameters measuring four factors. These factors include how aggressively the trout are being depleted by the lamprey, how much the lamprey benefit from the trout, and the natural growth of the trout (absent lamprey) and decline of the lamprey (absent trout). By using these equations to quantify the fish population levels as time passes, we find clues to which types of dynamics can be expected.

As an experimental methodology, we modify each parameter to see the effect upon the population dynamics. We find mathematical evidence that the observed increases and decreases in lamprey population in [8] are due not just to human intervention, but arise naturally from the parasite-host relationship. The effects of various methods to help reduce the impact of the hurtful species in the environment will be also discussed.

Section 1: Ecological Relationships

There are various types of relationships that are studied in ecology. The different ways that organisms affect each other can be categorized into groups in which organisms are given symbols based on the nature of their relationship (see "Table 1"). Mutualism occurs when both organisms involved benefit from each other in some way. The symbol for this occurrence is +/+. An example of this is when a bee pollinates a flower. The bee collects nectar from the flower which is necessary for its survival. Meanwhile, the pollen from the flower is spread to other flowers which is necessary for its reproduction. Commensalism takes place when one organism benefits from the relationship while the other experiences neither harm nor benefit from the exchange. The symbol for this situation is +/0. In nature this happens in an instance such as when a sea anemone uses a clown fish to travel from place to place. As the fish travels through the water, the anemone travels with it to help reach its destination. This act does not harm the clownfish in any way, nor does it necessarily help it. Competition happens when the interaction is detrimental to both of the species involved. The symbol for this case is -/-. An example of this is how cheetahs and lions both eat prey that are very alike. This causes them to both hurt from each other since there will be less food available for both of them to eat. Therefore, they will not be able to consume all of the nutrients that they require. Predation comes about when one organism benefits by killing its prey as they rely on them as a source of their nutrients. The symbol for this circumstance is +/-. An example of traditional predation would be a lion catching a deer for its dinner. In the study of ecology, predation shares the same symbol pair as parasitism. They are similar because one organism is harmed while the other one has an advantage. However they slightly differ from one another because predation involves the harm of one organism in which it is killed in order to be used for the other organism's food and survival.

In both cases, the harmed organism has a reduced growth rate. Parasitism arises when one organism causes harm to the other organism without killing it in order to benefit itself. The symbol for this is +/- as well. An instance of this is when tapeworms live inside of and feed off of the digestive systems of animals and humans. Ticks and lice feed off of and live on their hosts which cause them harm to their bodies; this is another example of parasitism [3]. Most importantly for this discussion, this type of relationship is seen between the lamprey and rainbow trout in Lake Ontario.

Type of Interaction	**Organism 1**	**Organism 2**
Mutualism	+	+
Commensalism	+	0
Competition	-	-
Predation (includes Parasitism)	+	-

Table 1

Key:

"+" = benefits organism

"-" = harms organism

"0" = no effect

Section 2: Wounded Trout and Efforts to Reduce Devastation

According to National Geographic [6], the species of rainbow trout, scientifically known as *oncorhynchus mykiss*, are a type of fish that have bodies built similarly to the shape of a torpedo. They are carnivores and have an average life span of around four to six years. Generally they are around 20-30 inches and weigh about 8 pounds, however they can grow up to about 4 feet and weigh 53 pounds. They have a very beautiful appearance and come in all sorts of different colors. These colors range from blue, green or yellow with pink on the side of their bodies and black spots as well as patterns depending on where they live and how old they are. "They prefer cool, clear rivers, streams, and lakes, though some will leave their freshwater homes and follow a river out to the sea. Rainbow trout survive on insects, crustaceans, and small fish. Their populations are healthy worldwide and they have no special status or protections. However, they are now considered a non-native pest species in some areas where they have been introduced".

Unfortunately, these gorgeous fish are harmed by another species known as sea lamprey (*petromyzon marinus*). These fish are parasites who live off of the blood and bodily fluids of other fish and have been able to survive multiple near extinctions. They have a distinctive mouth that they use for sucking the blood and body fluids of their prey. As described by the Great Lakes Fishery Commission, they have "a large oral sucking disk filled with sharp, horn-shaped teeth surrounding a razor sharp rasping tongue" [4]. By latching onto and digging into the bodies of rainbow trout, these invasive lamprey inflict wounds which turn into scars, generally without killing their host. Wildlife managers have discovered that this been happening in the Great Lakes, specifically Lake Ontario, for years. For those trout that are fortunate enough to survive, they are left with painful attack wounds and their quality of life is hindered. This ultimately hinders the span of their lives as well which decreases their population levels and the overall flourishment of fisheries in general. "In 2016, 27% of fish had lamprey marks (wound or scar), representing a 7% increase from 2015" [9].

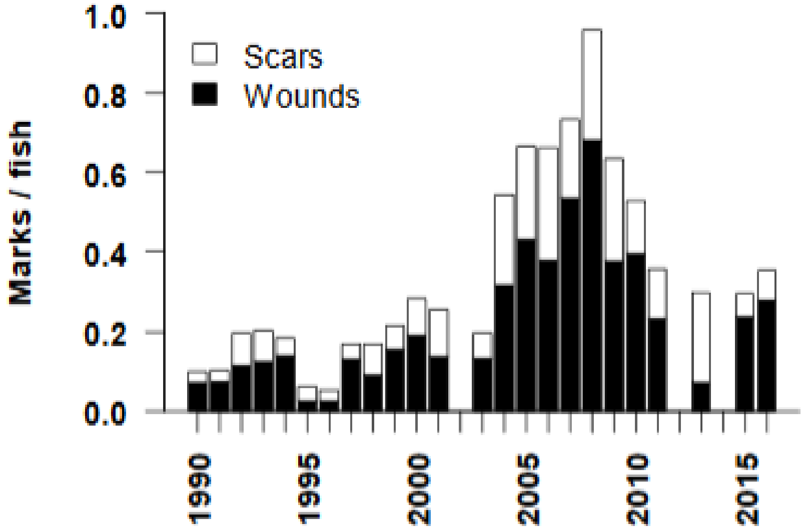


Figure 1: A plot of the increases and decreases of wounds found on the trout from [9]

Year	Wounds/ fish	Scars/ fish	Marks/ fish	% with wounds	% with scars	% with marks	Sample Size
1974	0.083	0.676	0.759	7.0	33.2	37	527
1975	0.095	0.725	0.820	8.0	37.2	40	599
1976	0.090	0.355	0.445	6.6	23.3	28	1280
1977	0.076	0.178	0.254	6.4	13.5	18	2242
1978	0.097	0.380	0.476	8.1	28.4	34	2722
1979	0.122	0.312	0.434	10.3	22.8	30	3926
1981			0.516			36	5489
1983	0.113	0.456	0.569	9.7	33.4	39	833
1985	0.040	0.154	0.193	3.7	11.5	14	1256
1990	0.030	0.071	0.101	2.8	5.8	8	466
1991	0.026	0.076	0.103	2.4	6.4	8	419
1992	0.079	0.117	0.197	6.3	11.1	17	315
1993	0.077	0.126	0.203	6.9	11.5	17	261
1994	0.044	0.141	0.185	4.0	12.4	15	298
1995	0.036	0.026	0.063	3.6	2.6	6	303
1996	0.028	0.025	0.053	2.8	2.5	5	396
1997	0.035	0.132	0.167	3.5	10.3	13	311
1998	0.075	0.092	0.168	6.8	8.5	13	400
1999	0.057	0.157	0.214	5.5	12.4	16	477
2000	0.091	0.191	0.283	8.0	16.9	24	361
2001	0.118	0.138	0.257	10.0	12.5	19	608
2003	0.063	0.134	0.197	5.9	10.9	16	238
2004	0.227	0.316	0.543	17.6	25.0	38	392
2005	0.231	0.433	0.664	17.1	33.6	41	321
2006	0.282	0.379	0.661	22.6	30.1	45	319
2007	0.199	0.534	0.733	15.5	39.3	49	206
2008	0.274	0.682	0.956	18.6	43.8	51	274
2009	0.256	0.377	0.633	20.4	29.8	42	289
2010	0.134	0.394	0.528	10.4	31.2	38	231
2011	0.124	0.235	0.359	10.7	21.8	30	298
2013	0.229	0.071	0.300	17.4	6.8	22	380
2015	0.058	0.238	0.296	4.9	16.5	20	206
2016	0.075	0.280	0.356	7.5	21.8	27	239

Table 2: A data table of the increases and decreases of wounds found on the trout from [9]

Year	Observed	Estimated
1974	527	527
1975	591	591
1976	1,281	1,281
1977	2,237	2,237
1978	2,724	2,724
1979	4,004	4,004
1980		5,817
1981	7,306	7,306
1982		10,127
1983	7,907	7,907
1984		8,277
1985	14,188	14,188
1986		12,785
1987	10,603	13,144
1988	10,983	15,154
1989	13,121	18,169
1990	10,184	14,888
1991	9,366	13,804
1992		12,905
1993	7,233	8,860
1994	6,249	7,749
1995	7,859	9,262
1996	8,084	9,454
1997	7,696	8,768
1998	3,808	5,288
1999	5,706	6,442
2000	3,382	4,050
2001	5,365	6,527
2002		5,652
2003	3,897	4,494
2004	4,452	5,308
2005	4,417	5,055
2006	5,171	5,877
2007	3,641	4,057
2008	3,963	4,713
2009	3,290	4,502
2010	4,705	6,923
2011	6,313	9,058
2012	7,256	8,486
2013	8,761	12,021
2014	8,218	9,611
2015	5,890	6,669
2016	4,225	4,987

Table 3: A data table of the recorded trout moving upstream from [9]

"Figure 1" shows the trend in lamprey marks on rainbow trout and "Table 2" also shows the lamprey marks on rainbow trout. The fluctuations in the number of wounds on the trout in both of these data segments represent a measure of the lamprey. "Table 3" shows the trend in trout population over time. It is evident that the trout population has specific lows/highs soon after the lamprey population has highs/lows. This is evidence that the upcoming predator-prey equations that will be used are appropriate. These equations are especially suitable for our situation because they predicted the fluctuation patterns that are reflected in the data. For example in 2008 we see a significantly low number in trout population and a significantly high amount of wounds and scars of the trout. This high number of marks on the trout simultaneously represents a high lamprey population.

Various control techniques are used to reduce the devastation of the sea lamprey in their environments. The UMESC (Upper Midwest Environmental Sciences Center) has made efforts to manipulate the lamprey by making them more vulnerable to the harmful actions that are taken against them during the beginning or middle stages of their cycle of life. Lampricides are successfully able to kill lamprey larvae that live in the bottom of bodies of water. These pesticides are cleverly designed to only harm lamprey while not posing any danger to other organisms. The formula of these chemicals have been improved by the Environmental Protection Agency by decreasing the amount of chemicals required while still being effective. In addition, their ability to travel upstream and reproduce are diminished through the use of barricades and traps that are in place [7].

Section 3: The Mathematical Model

Many effects are revealed in the varying math models with which we have experimented. We have used the program Maple to input mathematical equations that model the lamprey and trout populations. From this, various graphs have been created which help visualize the cyclic population patterns of the two species who coexist with one another. We constructed a system of differential equations in order to model the changes in population growth/decay. These equations incorporate components from a simple form of natural growth, which are seen in the parts of the equations that have either an x or a y term next to the constant. Natural growth represents only the growth of one given species by itself without any influence from the other species. This portion of the equation is then added to a constant that has an xy term attached to it. This portion of the equation represents the result of how the two species effect each other. We have slightly changed the different parameters of each equation one at a time in order to show the different effects of each part of the predator/prey cycle. This is applicable since each one of these parameters means something different relative to the cycle.

All of the parameters are changed separately while keeping everything else constant and they can all be explained by a natural phenomenon. For all of the following graphs we use the same time domain $0 \leq t \leq 6$ to allow for comparison. We also select the same four initial conditions ($x_0 = 3$ and $y_0 = 0.5, 1.0, 1.5,$ or 2.0) for each plot. However, we vary the range for x and y (the "Plot Window") so as to best display each family of solutions.

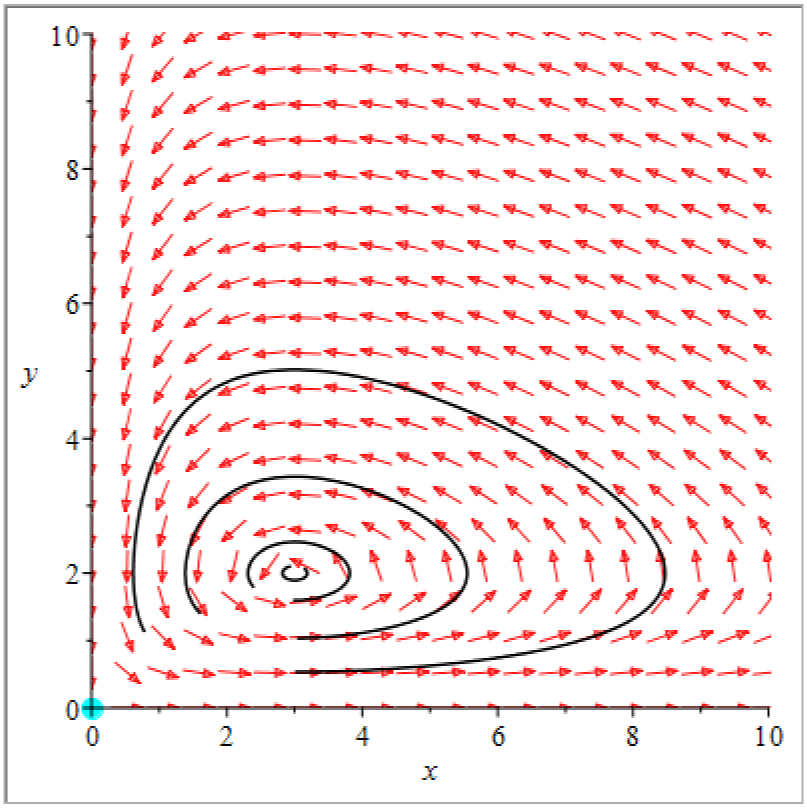
For the base case, **Plot Window:** $0 \leq x \leq 10$ and $0 \leq y \leq 10$

Differential Equations:

$$\frac{dx}{dt} = F(x, y) = x - .5xy$$

$$\frac{dy}{dt} = G(x, y) = -.75y + .25xy$$

The above equations were adopted directly from [1]. The first choice of parameters is somewhat arbitrary. Changes to each of the parameters will be explored in the next section.



At first we can see that both the predator and prey start at a relatively small state. Then, the prey are first to increase since there is little predation occurring. Next, the predators increase in population as well since there is now abundant food. This causes heavier predation, and the prey tend to decrease. Lastly, the predator population also decreases due to a diminishing food supply and the system returns to the original state.

Section 4: Changes in Parameters

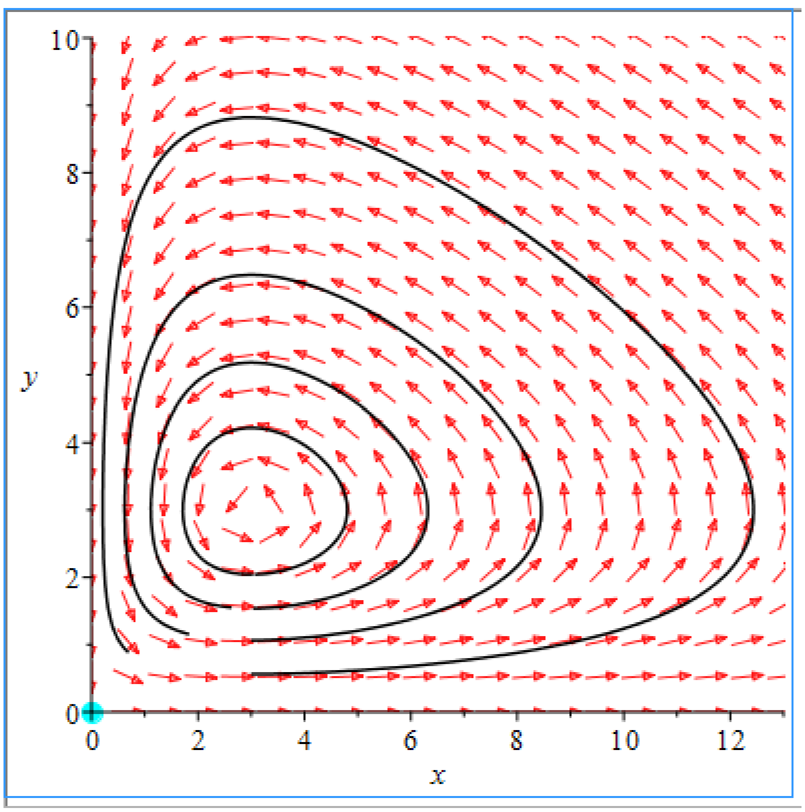
We will use the above graph from Section 4 with the accompany equations as the base case. These are intended to be used as the standard scenario that will be used for comparison purpose to make various changes. The variable x stands for the number of trout and the variable y stands for the number of lamprey. The variable t stands for time. By altering the parameters, different effects can be observed. These factors include the overall shape of the graph, the change in range (maximum/minimum) for each population, and perceivable changes in cycle time.

For the first graph, **Plot Window:** $0 \leq x \leq 13$ and $0 \leq y \leq 10$

Differential Equations:

$$\frac{dx}{dt} = F(x, y) = 1.5x - .5xy$$

$$\frac{dy}{dt} = G(x, y) = -.75y + .25xy$$



The F equation has been altered by increasing the coefficient next to the x variable. This component displays the natural growth rate of just the trout on their own which shows how fast their population is increasing. This could be achieved by restricting harvesting, the removal of the trout, which would help increase the population of the trout as seen in the graph. If more harvesting were to occur then the coefficient next to the x variable would be smaller and we would see a lower population of trout.

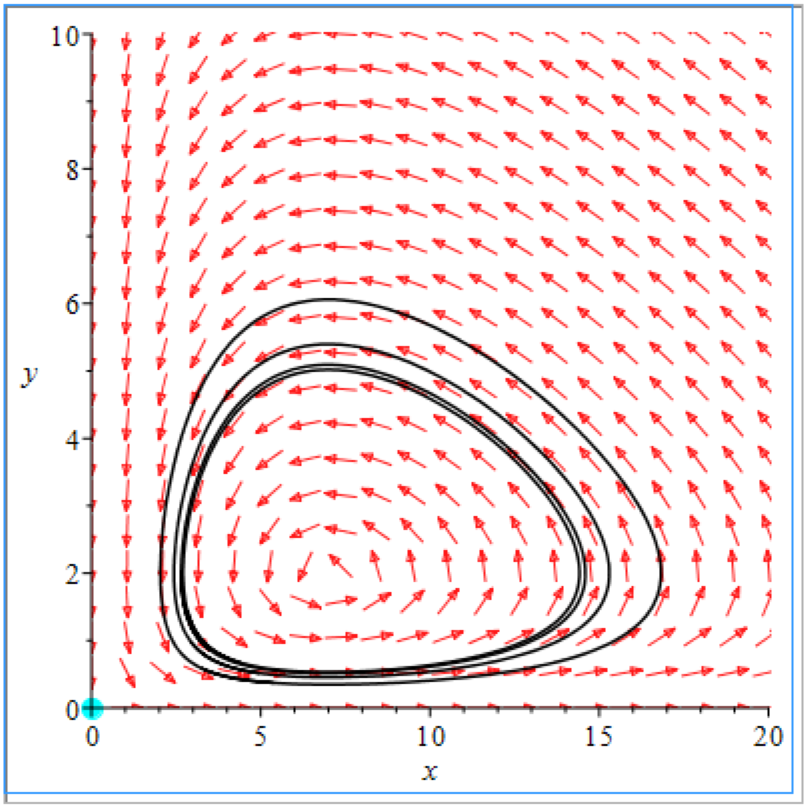
In comparison to the original graph that we are using as a starting point, it is clear that this graph has a larger maximum trout population. In addition, its extremes are much more extreme than those of the original graph.

For the second graph, **Plot Window:** $0 \leq x \leq 20$ and $0 \leq y \leq 10$

Differential Equations:

$$\frac{dx}{dt} = F(x, y) = x - .5xy$$

$$\frac{dy}{dt} = G(x, y) = -1.75y + .25xy$$



The G equation has been changed by making the coefficient next the y variable greater which in this case is making it more negative. This part of the equation represents the natural growth rate of just the lamprey which shows how well the population of lamprey are doing on their down. When this value decreases, that could mean that the pesticides are being sprayed which kills off the lamprey.

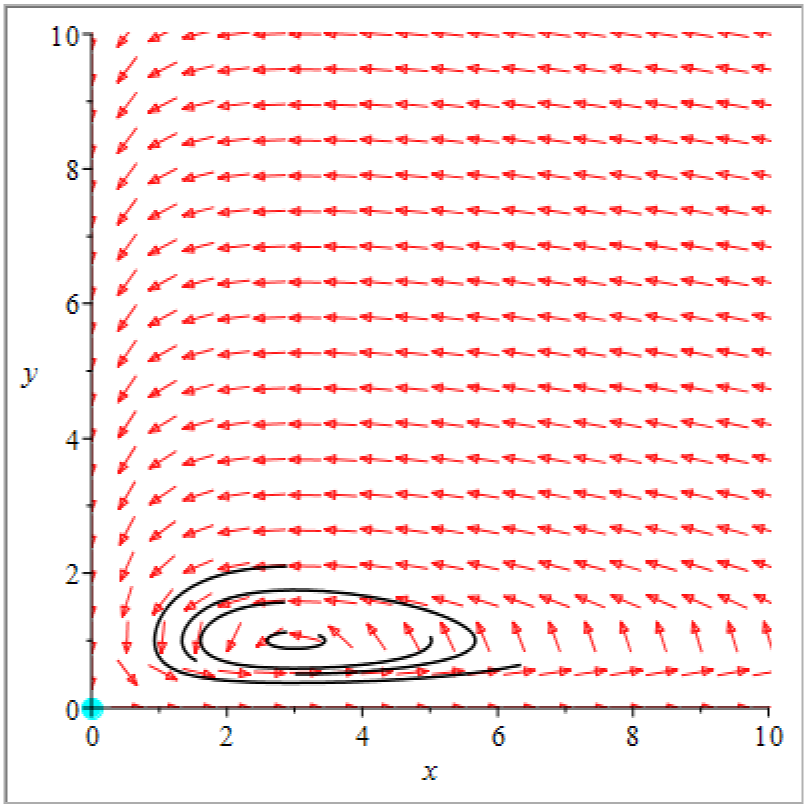
In contrast to the original graph, is it very noticeable that the cycles of this graph are a lot closer together. This means that there is not much of an overall impact of this specific parameter change. It is also apparent that the rate of the lamprey population is decreasing more quickly since the initial values are very close together this time.

For the third graph, **Plot Window:** $0 \leq x \leq 10$ and $0 \leq y \leq 10$

Differential Equations:

$$\frac{dx}{dt} = F(x, y) = x - 1xy$$

$$\frac{dy}{dt} = G(x, y) = -.75 + .25xy$$



The F equation has been altered by making the coefficient next to the xy component more negative. This means that the prey is more harmed. The smaller that this number is, the less they are being harmed. This could have been caused by a change in water temperature which could potentially introduce more bacteria into the open wounds of the trout inflicted by the lamprey.

Another possible explanation is that there could be more predators around. An increased presence of lamprey would make it easier for them to catch the wounded fish since they would now be weaker. Lastly, it is possible that an antibiotic could have been given to the trout in effort to help heal their wounds. This medication would improve their overall health and give them a better chance at survival, which would move the parameter the other way. However this could actually end up harming them if they had developed a resistance to their antibiotics which is very probable.

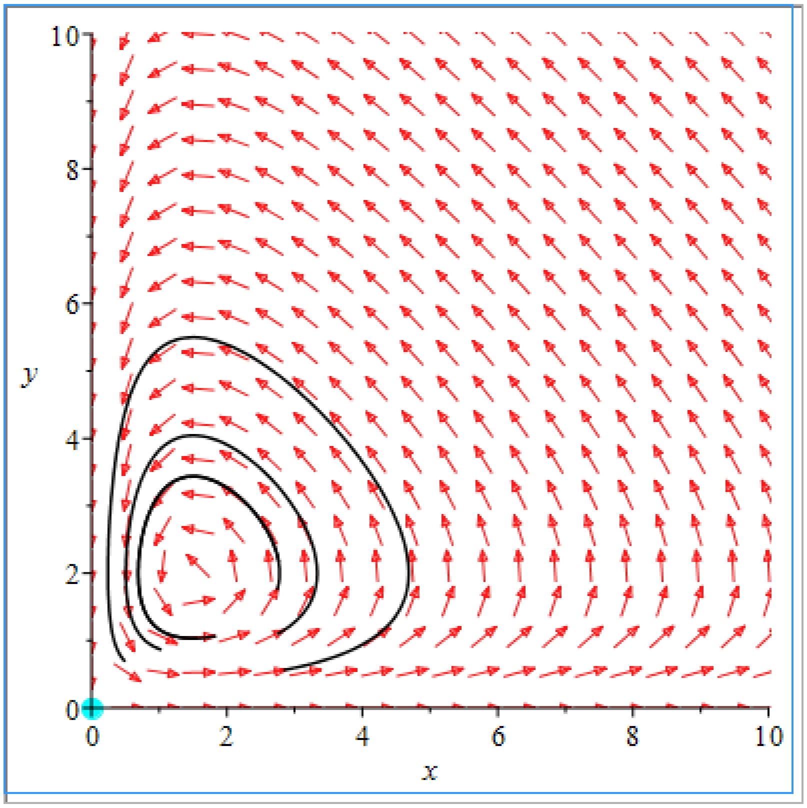
In comparison to the original graph, this graph shows a lower equilibrium point. It is also interesting to notice that the trout population is starting off in a decreased state.

For the fourth graph, **Plot Window:** $0 \leq x \leq 10$ and $0 \leq y \leq 10$

Differential Equations:

$$\frac{dx}{dt} = F(x,y) = x - .5xy$$

$$\frac{dy}{dt} = G(x,y) = -.75 + .5xy$$



The G equation has been changed to make the coefficient next to the xy term greater. This means that the lamprey benefit more from the parasitism and the there is no difference in impact on the trout. A possible explanation is that there could be something in the environment that makes the trout more nutritious for the lamprey. If this coefficient were to be changed to make the xy term smaller, a likely explanation could be that the lamprey had decided to latch onto the trout for a shorter time. Their decision to stay away from the trout more could be because they may not enjoy their taste as much as they used to; they might be poisonous to them or not as nutritious. Or, they might start leaving the trout alone because there could just be another fish population living in the same environment that they like better and prefer to eat more.

In contrast to the original graph, it is evident that this cycle does not reach out as far on the x axis of the plot. It also has a shorter cycle length, meaning that it takes less time to complete a full cycle in this case.

Section 5: Observed Conclusions in Mathematical Models

There are different kinds of long term effects that can be predicted from the predator/prey relationship between the trout and the lamprey. It is important to notice that no matter what actions are taken in regards to either of the species, there always seems to be some type of cycling. These actions include either trying to reduce the population of the parasite (lamprey) or increasing the population of the host (trout). This creates some ups and downs in the cycles related to the population dynamics of the two species interacting with one another. Some instances are closer to equilibrium than others which would seem to be the preferred situation. This shows that their dramatic fluctuations in populations have settled down and they can peacefully coexist in the same ecosystem. This also means that in this case we are not so worried about the trout dying out. Secondary effects demonstrate the concept of cycling in which conditions may improve and then get worse again (though other factors could cause the trout to die out). Just from looking at the data, one might be tempted to think that something changed, or that was done had suddenly stopped working. However this is just the expectation of the natural cycle. Instead of preventing the parasites from living, it may be a better idea to just try to maintain the cycle overall. This can be accomplished by making improvements to have more favorable levels of each type of fish. It is important to not completely eliminate the “bad” part of the cycle which are the lamprey (parasites/predators) so that only the “good” part of the cycle is left which are the lake trout (prey). It is acceptable to actually expect the predators to make some type of recovery once the lamprey do because there is now more for them to eat, hence, there is this natural cycle that we would like to achieve.

All of the models show diverse results in population levels of both the prey and predators that are expected from different types of interventions. Keeping in mind that all of the graphs cycle, it is important to notice that they cycle in different ways. For example, the changes that were made to the second graph led to the largest maximum in trout population. This intervention method included the spraying of pesticides on the lamprey, which makes logical sense because it was actually conducted and scientifically proven to be highly effective. The changes that were made to the first graph led to the largest maximum in lamprey population. This intervention method included restricting harvesting. By limiting the removal of the trout from the lake, they are able to reproduce more. In exchange, a higher trout population therefore results in a higher lamprey population since they have more fish to feed off of.

Section 6: Carrying Capacity

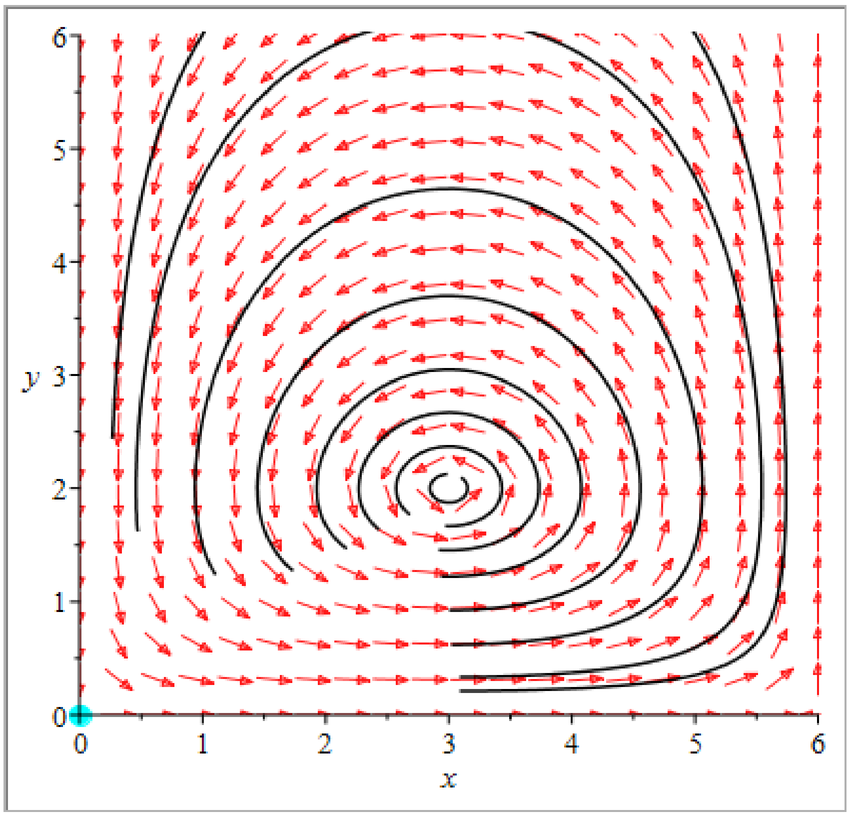
One of the most common natural population phenomena is that of a carrying capacity, which we could introduce to the mathematical model for the trout. "Biologists define carrying capacity as the maximum population of a given species that can survive indefinitely in a given environment" [2,5]. Simply put, a carrying capacity is the largest population that could survive, absent the effects of the parasites. The models used in sections 3 and 4 used the unreasonable simplifying assumption that in the absence of parasites, the trout population would increase with no bound. Note that the data suggest that the trout are most likely not currently close to their carrying capacity. This results in a more precise but more complicated set of equations.

For the graph that exemplifies carrying capacity, the following equations will be used –

Differential Equations:

$$\frac{dx}{dt} = F(x, y) = (1 - .5y)\, x \cdot (1 - \frac{x}{6})$$

$$\frac{dy}{dt} = G(x, y) = -.75 + \ .25xy$$



In this equation we can see that there is now a threshold of $(1 - \frac{x}{6})$ which has been used to model a carrying capacity of $x = 6$. From this graph we can observe a different type of cycle relative to the ones we previously observed. This is exemplified by the fact that it is clear that the fish population is not able to grow past a certain level.

Section 7: Conclusion

Having gained mathematical insight from modelling types of population dynamics among species in their environments and a basic understanding of the biology that accompanies such dynamics, there is more work that can potentially be continued in both fields and the connection between them. In the future, it would be a great idea to propose additional data be collected to see which of our models is best. To do this, it would be optimal to figure out the correct units that are displayed on the graph and to pick parameters so that the numbers match up exactly with the observations. We could also try to take different measures into account to see how they impact the cycles of the fish involved. For instance, harvesting or restocking the trout at a fixed yearly rate can be incorporated into the equations that we were working with in the previous sections. There would be a plus/minus "C" factor introduced into the equation (for example: $\frac{dx}{dt} = (1 - 0.5y)x + 2$). Restocking is when fish are raised in a hatchery and then released into a body of water to enhance current populations or to generate a new population if there isn't already one. During harvesting, people involved in wildlife have control over a definite amount of a certain species of fish being captured from their environment each year.

In conclusion, we have demonstrated that the use of mathematical modeling can provide potential explanations and insights into population dynamics of different species who live and interact amongst each other in a shared environment. This could help aid wildlife managers in making more informed decisions. By integrating aspects of mathematics, biology, and ecology wildlife managers could better understand the results of various interventions in the ecosystems that they work with.

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