Extending the Applicability of the Lagrange Multipliers Method

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Thesis Director: Dr. Gabriel Prajitura

Ashley Case
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Abstract

In this work we studied the use of the Lagrange Multipliers Method. We proved that substitutions can result in the ability to use this method when the method had previously failed. We also look at situations where this is not the case, and the method fails to maximize or minimize the function. In such cases, we will discuss what to do from there.

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Chapter 1

The Lagrange Multipliers Method

1.1 Introduction

1.1. Definition. *The Lagrange Multipliers method is a common method of optimization. This method is used to find the maxima and minima of a function subject to a constraint. The constraint will serve as the boundary of the region we are examining.*

1.2. Definition. *The Lagrange Function can be defined as,*

$$L(x, y) = f(x, y) - \lambda \times g(x, y)$$

where λ is the variable known as the Lagrange Multiplier within the function.

1.2 History

The Lagrange Multipliers method is named after Joseph-Louis Lagrange. The original function that we just looked at is the method with one constraint. However, this method is very versatile and can be used for multiple constraints as well.

Chapter 2

Extending the Applicability of the Lagrange Multipliers Method

2.1 Extending the Method Using Substitution

2.1.1 Example 1

Leta, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\sqrt{2} \leq \sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}} \leq \sqrt[4]{27}$$

When do equalities occur?

The inequality is equivalent to

$$2 \leq \sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \leq \sqrt{2}\sqrt[4]{27}$$

Since a, b, c are nonnegative numbers we can use the substitutions $a = x^2$, $b = y^2$ and $c = z^2$, in attempt to make this a Lagrange Multipliers problem. Then we have x, y, z are real numbers subject to

$$x^4 + y^4 + z^4 = 1.$$

We will use Lagrange multipliers to determine the maximum and the minimum of the function

$$f(x, y, x) = \sqrt{x^2 + y^2} + \sqrt{y^2 + z^2} + \sqrt{z^2 + x^2}$$

subject to

$$x^4 + y^4 + z^4 = 1.$$

The multiplier system is

$$\begin{cases} \frac{x}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{z^2+x^2}} = 4\lambda x^3 \\ \frac{y}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{y^2+z^2}} = 4\lambda y^3 \\ \frac{z}{\sqrt{y^2+z^2}} + \frac{z}{\sqrt{z^2+x^2}} = 4\lambda z^3 \\ x^4 + y^4 + z^4 = 1 \end{cases}$$

We will consider four cases:

1. All variables are equal to 0.

Since

$$x^4 + y^4 + z^4 = 1,$$

this case is not possible.

2. Only two of the variables are equal to 0.

Because of the symmetry of the system it suffices to consider only the case $x = y = 0$.

Then $z^4 = 1$ and thus $z = \pm 1$.

This gives us the points $(0, 0, \pm 1)$, and, by symmetry, $(0, \pm 1, 0)$ and $(\pm 1.0, 0)$.

3. Only one of the variables equals 0

Because of the symmetry of the system it suffices to consider only the case $x = 0$.

The system becomes

$$\begin{cases} \frac{y}{\sqrt{y^2}} + \frac{y}{\sqrt{y^2+z^2}} = 4\lambda y^3 \\ \frac{z}{\sqrt{y^2+z^2}} + \frac{z}{\sqrt{z^2}} = 4\lambda z^3 \\ y^4 + z^4 = 1 \end{cases}$$

or

$$\begin{cases} \frac{1}{|y|} + \frac{1}{\sqrt{y^2+z^2}} = 4\lambda y^2 \\ \frac{1}{\sqrt{y^2+z^2}} + \frac{1}{|y|} = 4\lambda z^2 \\ y^4 + z^4 = 1 \end{cases}$$

It is clear from the first equation that $\lambda > 0$.

Subtracting the first two equations we get

$$\frac{1}{|y|} - \frac{1}{|z|} = 4\lambda(y^2 - z^2)$$

If $|y| > |z|$, the left hand side is negative while the right hand side is positive. If $|y| < |z|$ the lefthand side is positive while the right hand side is negative.

Therefore $|y| = |z|$ and thus $y^4 = z^4$. This implies that

$$2y^4 = 1$$

and so

$$y = \pm\frac{1}{\sqrt[4]{2}}$$

We got the points $(0, \pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}})$ and, by symmetry, $(\pm\frac{1}{\sqrt[4]{2}}, 0, \pm\frac{1}{\sqrt[4]{2}})$ and $(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}, 0)$.

4. None of the variables is 0.

The system is equivalent with

$$\begin{cases} \frac{1}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{z^2+x^2}} = 4\lambda x^2 \\ \frac{1}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{y^2+z^2}} = 4\lambda y^2 \\ \frac{1}{\sqrt{y^2+z^2}} + \frac{1}{\sqrt{z^2+x^2}} = 4\lambda z^2 \\ x^4 + y^4 + z^4 = 1 \end{cases}$$

Again λ must be a positive number.

Subtracting the first two equations, we get

$$\frac{1}{\sqrt{z^2+x^2}} - \frac{1}{\sqrt{y^2+z^2}} = 4\lambda(x^2 - y^2)$$

If $x^2 > y^2$ then the left hand side is negative while the right hand side is positive. If $x^2 < y^2$ then the left hand side is positive while the right hand side is negative. Therefore $x^2 = y^2$.

In the same way, subtracting the third equation from the second we get that $y^2 = z^2$. Thus

$$3x^4 = 1$$

and $x = \pm\frac{1}{\sqrt[4]{3}}$. This gives us the points $(\pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[4]{3}})$.

For all points in case 2 the value of the function is 2. For all points in case 3 the value of the function is $\sqrt[4]{8} + \sqrt[4]{2}$. For all the points in case 4 the value of the function is $\sqrt{2}\sqrt[4]{27}$.

Comparing the three values we conclude that the first is the minimum and the last is the maximum of the function within the constraints.

In terms of a, b, c, the minimum is attained when two of the variables are 0 and the third equals 1 while the maximum attained when all variables are equal and equal to $\frac{1}{\sqrt{3}}$.

2.1.2 Example 2

Leta, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$a\sqrt{a+3} + b\sqrt{b+3} + c\sqrt{c+3} \geq 6$$

Since a, b, c are nonnegative numbers we can use the substitutions $a = x^2$, $b = y^2$ and $c = z^2$, where x, y, z are real numbers subject to

$$x^2 + y^2 + z^2 = 3.$$

We will use Lagrange multipliers to determine the minimum of the function

$$f(x, y, z) = x^2\sqrt{x^2+3} + y^2\sqrt{y^2+3} + z^2\sqrt{z^2+3}$$

subject to

$$x^2 + y^2 + z^2 = 3.$$

We will also find the maximum.

The multiplier system is

$$\begin{cases} 2x\sqrt{x^2+3} + \frac{x^3}{\sqrt{x^2+3}} = 2\lambda x \\ 2y\sqrt{y^2+3} + \frac{y^3}{\sqrt{y^2+3}} = 2\lambda y \\ 2z\sqrt{z^2+3} + \frac{z^3}{\sqrt{z^2+3}} = 2\lambda z \\ x^2 + y^2 + z^2 = 3 \end{cases}$$

We will consider four cases

1. All variables are equal to 0.

Since

$$x^2 + y^2 + z^2 = 3,$$

this case is not possible.

2. Only two of the variables are equal to 0.

Because of the symmetry of the system it suffices to consider only the case $x = y = 0$.

Then $z^2 = 3$ and thus $z = \pm\sqrt{3}$.

This gives us the points $(0, 0, \pm\sqrt{3})$, and, by symmetry, $(0, \pm\sqrt{3}, 0)$ and $(\pm\sqrt{3}.0, 0)$.

3. Only one of the variables equals 0

Because of the symmetry of the system it suffices to consider only the case $x = 0$.

The system becomes

$$\begin{cases} 2\sqrt{y^2+3} + \frac{y^2}{\sqrt{y^2+3}} = 2\lambda \\ 2\sqrt{z^2+3} + \frac{z^2}{\sqrt{z^2+3}} = 2\lambda \\ y^2 + z^2 = 3 \end{cases}$$

Let
$$g(y) = 2\sqrt{y^2+3} + \frac{y^2}{\sqrt{y^2+3}}$$
Then
$$g'(y) = \frac{2y}{\sqrt{y^2+3}} + \frac{y^3+6y}{\sqrt{y^2+3}^3} > 0.$$
Therefore the function g is strictly increasing and thus one to one.
Since $g(y) = gzy)$ we conclude that
$$y = z = \pm\sqrt{3}2.$$
We got the points $(0, \pm\sqrt{3}2., \pm\sqrt{3}2.)$ and, by symmetry, $(\pm\sqrt{3}2., 0, \pm\sqrt{3}2.)$ and $(\pm\sqrt{3}2., \pm\sqrt{3}2., 0)$.

4. None of the variables is 0.

The system is equivalent to
$$\begin{cases} 2\sqrt{x^2+3} + \frac{x^2}{\sqrt{x^2+3}} = 2\lambda \\ 2\sqrt{y^2+3} + \frac{y^2}{\sqrt{y^2+3}} = 2\lambda \\ 2\sqrt{z^2+3} + \frac{z^2}{\sqrt{z^2+3}} = 2\lambda \\ x^2+y^2+z^2 = 3 \end{cases}$$
This time we have $g(x) = g(y) = g(z) = \pm 1$.
This gives us the points $(\pm 1, \pm 1, \pm 1)$.
For all points in case 2 the value of the function is $3\sqrt{6}$. For all points in case 3 the value of the function is $3\sqrt{2}$. For all the points in case 4 the value of the function is 6.

Comparing the three values we conclude that theminimum is 6 and the maximum is $3\sqrt{6}$.

2.1.3 Example 3

Let$a$ and b be non negative real numbers such that $a+b=1$. Prove that

$$\sqrt{289}256 \le (1+a^4)(1+b^4) \le 2.$$

Since a, b are nonnegative numbers we can use the substitutions $a = x^2$ and $b = y^2$ where x, y are real numbers subject to

$$x^2 + y^2 = 1.$$

We will use Lagrange multipliers to determine the minimum and the maximum of the function

$$f(x,y) = (1+x^8)(1+y^8) = 1 + x^8 + y^8 + x^8y^8$$

subject to

$$x^2 + y^2 = 1.$$

The multiplier system is

$$\begin{cases} 8x^7 + 8x^7y^8 = 2\lambda x \\ 8y^7 + 8x^8y^7 = 2\lambda y \\ x^2 + y^2 = 1 \end{cases}$$

We will consider three cases

1. All variables are equal to 0.

Since

$$x^2 + y^2 = 1,$$

this case is not possible.

2. Only one of the variables is equal to 0.

Because of the symmetry of the system it suffices to consider only the case $x = 0$.

Then $y^2 = 1$ and thus $y = \pm 1$.

This gives us the points $(0, \pm 1$, and, by symmetry, $(\pm 1, 0)$.

3. None of the variables is 0.

The system is equivalent to

$$\begin{cases} 4x^6 + 4x^6y^8 = \lambda \\ 4y^6 + 4x^8y^6 = \lambda \\ x^2 + y^2 = 1 \end{cases}$$

If we subtract the first 2 equations we get

$$4(x^6 - y^6) + 4x^6y^6(y^2 - x^2) = 0 \iff (x^2 - y^2)(x^4 + x^2y^2 + y^4) - (x^2 - y^2)x^6y^6 = 0$$

$$\iff (x^2 - y^2)(x^4 + x^2y^2 + y^4 - x^6y^6) = 0.$$

The equation $x^2 - y^2 = 0$ implies $|x| = |y|$ and since $x^2 + y^2 = 1$ we get the solutions $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$.

$$x^4 + x^2y^2 + y^4 - x^6y^6 = 0 \iff (x^2 + y^2)^2 - x^2y^2 - x^6y^6 = 0$$
$$\iff 1 = x^2y^2 + x^6y^6.$$

We will show that this equation does not have solutions.

By AM - GM inequality,

$$|x \cdot y| \leq \frac{x^2 + y^2}{2} = \frac{1}{2}.$$

Therefore

$$x^2y^2 \leq \frac{1}{4}$$

and

$$x^6y^6 \leq \frac{1}{64}.$$

This implies that

$$x^2y^2 + x^6y^6 \leq \frac{17}{64} < 1.$$

For all points in case 2 the value of the function is 2. For all points in case 3 the value of the function is $\frac{289}{256}$.

Comparing the two values we conclude that the minimum is $\frac{289}{256}$ and the maximum is 2.

2.1.4 Example 4

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$ and let k be a nonnegative real number. Prove that

$$a + b + c + \sqrt{k\left(k - 1 + \frac{a^2 + b^2 + c^2}{3}\right)} \leq k + 3.$$

Let $a + b + c = s$ and $abc = p$.

The inequality is equivalent to

$$\sqrt{k\left(k - 1 + \frac{4 - p}{3}\right)} \leq k + 3 - s$$

$$\iff \sqrt{k\left(k + \frac{1 - p}{3}\right)} \leq k + 3 - s$$

Since

$$s^2 \leq 2(a^2 + b^2 + c^2) < 8 \iff s < 2\sqrt{2} < 3,$$

the right hand side of the inequality is positive so

$$\iff k\left(k + \frac{1 - p}{3}\right) \leq (k + 3 - s)^2$$

$$\iff k^2 + \frac{1 - p}{3}k \leq k^2 + 2(3 - s)k + (3 - s)^2$$

$$\iff 0 \leq k\left(6 - 2s - \frac{1 - p}{3}\right) + (3 - s)^2$$

If

$$6 - 2s - \frac{1 - p}{3} \geq 0$$

the previous inequality is true.

We will show that this is always the case

$$\iff 18 - 6s - 1 + p \geq 0 \iff 17 - 6s + p \geq 0$$

which is true because $p > 0$ and $6sleq12\sqrt{2} < 17$.

2.1.5 Example 5

Let x, y, z be non negative real numbers such that $x + y + z = 1$ and let $1 \leq \lambda \leq \sqrt{3}$. Determine the minimum and the maximum of the function

$$f(x, y, z) = \lambda(xy + yz + zx) + \sqrt{x^2 + y^2 + z^2}$$

in terms of λ.

Our goal is to find the maximum and the minimum of

$$f(x, y, z) = \lambda(xy + yz + zx) + \sqrt{x^2 + y^2 + z^2}$$

subject to

$$x + y + z = 1$$

with $x, y, z \geq 0$.

The Lagrange Multiplier system is

$$\begin{cases} \lambda(y+z) + \frac{x}{\sqrt{x^2+y^2+z^2}} = \alpha \\ \lambda(x+z) + \frac{y}{\sqrt{x^2+y^2+z^2}} = \alpha \\ \lambda(x+y) + \frac{z}{\sqrt{x^2+y^2+z^2}} = \alpha \end{cases}$$

Subtracting the first two equations we get

$$\lambda(y - x) + \frac{x - y}{\sqrt{x^2 + y^2 + z^2}} = 0 \iff (y - x)\left(\lambda - \frac{1}{\sqrt{x^2 + y^2 + z^2}}\right) = 0$$

and, in the same way, subtracting the last two equations, we get

$$(z - y)\left(\lambda - \frac{1}{\sqrt{x^2 + y^2 + z^2}}\right) = 0$$

We will consider two cases:

I

$$\frac{1}{\sqrt{x^2 + y^2 + z^2}} \neq \lambda$$

Then $y - x = z - y = 0$ which implies that $x = y = z = 1/3$. The value of the function is, in this case,

$$\frac{\lambda}{3} + \frac{1}{\sqrt{3}}.$$

II

$$\frac{1}{\sqrt{x^2 + y^2 + z^2}} = \lambda$$

In this case the function becomes

$$f(x, y, z) = \lambda(xy + yz + zx) + \frac{1}{\lambda}$$

Since

$$xy + yz + zx = \frac{1}{2}((x + y + z)^2 - (x^2 + y^2 + z^2)) = \frac{1}{2}(1 - \frac{1}{\lambda^2})$$

we get that the value of the function is

$$\frac{\lambda}{2}(1 - \frac{1}{\lambda^2}) + \frac{1}{\lambda} = \frac{1}{2}(\lambda + \frac{1}{\lambda})$$

We will show that

$$\frac{1}{2}(\lambda + \frac{1}{\lambda}) \geq \frac{\lambda}{3} + \frac{1}{\sqrt{3}}$$

which will imply that the first is the maximum and the second is the minimum.

The inequality

$$\iff \frac{\lambda}{6} + \frac{1}{2\lambda} \geq \frac{1}{\sqrt{3}}$$

which follows from the AM - GM inequality.

Chapter 3

When the Lagrange Multipliers Method Fails

3.1 When the method fails due to a sharp domain

3.1.1 Example 1

We will consider the problem; The function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = |x|$$

has a minimum at $x = 0$. However, the derivative of f is never equal to 0. To find it we need to look at the points where the derivative does not exist. To find it we need to look at the points where the derivative does not exist.

3.1.2 Example 2

Something similar can happen in the case of functions of several variables. Except that in this case we need to pay attention not only to the function itself but to the constraint function as well. Let us consider the following;

Find the maximum and the minimum of the function

$$f(x, y) = x$$

subject to

$$x^4 - x^3 + y^2 = 0.$$

Therefore,

$$g(x, y) = x^4 - x^3 + y^2$$

Here we have;

$$f_x = 1 \qquad \text{and} \qquad f_y = 0$$

and

$$g_x = 4x^3 - 3x^2 \qquad \text{and} \qquad g_y = 2y$$

Therefore the multiplier system is;

$$\begin{cases} 1 = \lambda(4x^3 - 3x^2) \\ 0 = \lambda(2y) \\ x^4 - x^3 + y^2 = 0 \end{cases}$$

The first equation implies that $\lambda \neq 0$ and so, from the second equation we get $y = 0$. If we plug in the third equation we get

$$x^4 - x^3 = 0$$

$$\iff x^3(x-1) = 0$$

so $x = 0$ or $x = 1$.

The solution $x = 0$ turns the first equation into $1 = 0$, so we conclude that $x = 1$ is the only solution. It turns out this is the maximum of the function.

However, since the constraint

$$y^2 = x^4 - x^3$$

is a compact set the function f has both a maximum and a minimum and the method failed to identify one of them.

Notice that both $f(x, y) = x$ and $g(x, y) = x^4 - x^3 + y^2$ have partial derivatives everywhere, so the minimum does not occur at a point where the derivative does not exist, as in the one dimensional case.

Here the graph of

$$x^4 - x^3 + y^2 = 0$$

has a sharp point at $x = 0$, which is the source of the failure.