

Running Head: Is AP Calculus the best option?

High School Calculus: Is AP Calculus the best option?

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Abstract

In the 2015-2016 academic school year, 3.3 million high school seniors are expected to graduate, and six percent of them are gifted and talented individuals who may be entering an AP Calculus course. Research shows that many students entering AP Calculus end up terminating their mathematical careers before even starting college. This thesis provides an alternative approach to AP Calculus; a non- AP Calculus exemplar unit that provides deeper conceptual knowledge of the slope, connected to the derivative, which may better prepare students for their college mathematics career. High school is a place to encourage students to accomplish their goals and better preparation at the high school level for college calculus may better equip students for future success.

Table of Contents

Abstract	2
CHAPTER 1	5
INTRODUCTION	5
CHAPTER 2	9
LITERATURE REVIEW	9
United States Education	9
Common Core State Standards	10
High School Mathematics Course Requirement	10
Gifted Education	11
AP Calculus	12
Non-AP Calculus Course	13
Purpose of High School	14
CHAPTER 3	15
CURRICULUM PROJECT	15
Lesson 1: The slope of a linear function	17
Lesson 2: What is a tangent line?	19
Lesson 3: Secant Lines	22
Lesson 4: Slope of a tangent line	26
Lesson 5: Extra examples for Lesson 4	30
Lesson 6: Definition of a derivative	33
Lesson 7: Extra examples for Lesson 6	35
Lesson 8: Quiz	38
CHAPTER 4	39
VALIDITY OF CURRICULUM	39
Teacher Review	39
CHAPTER 5	41
CONCLUSION	41
References	43
Lesson 1: The slope of a linear function	45
Lesson 2: What is a tangent line?	46
Lesson 3: Secant Lines	47
Lesson 4: Slope of a tangent line	48
Lesson 5: Extra examples for Lesson 4	49

Lesson 6: Definition of a derivative.....	53
Lesson 7: Extra examples for Lesson 6	54
Lesson 8: Quiz	62

CHAPTER 1

INTRODUCTION

Academically gifted and talented students in the United States make up approximately six to ten percent of the total student population, according to the National Association for Gifted Children (NAGC, 2007). This population accounts for roughly three to five million students. As of August 2013, New York State defined gifted students as:

...those who show evidence of high performances capability and exceptional potential in area such as general intellectual ability, special academic aptitude and outstanding ability in visual and performing arts. Such definition shall include those pupils who require educational programs or services beyond those normally provided by the regular school program in order to realize their full potential (NAGC, 2013).

Even though almost 10% of the student population can be identified as gifted and talented, the NAGC states that these students “spend the majority of their time in the regular education classroom, and are taught by teachers who are not trained to meet their special needs” (2007, p.1). In addition, there is a lack of financial support offered by the federal government for these gifted programs. Monetary support is needed to run these programs and provide trainings for the teachers. “The federal government’s support for gifted children now stands at only 2 cents of every \$100 dollar it invests in K-12 education”(NAGC, 2007, p.1). Each individual state is allowed to decide how to handle their gifted and education programs, and most states allow the individual districts to decide.

A school district is normally the deciding factor on how to challenge gifted students and they usually follow two methods: differentiating instruction in the classroom, or accelerating the students to an advanced pace (Ysseldyke, 2004). Each method has its advantages and

disadvantages. Most schools offer an acceleration program, especially after the No Child Left Behind Act of 2002 which required more accountability and responsibility of schools to provide evidence of each student's heightened education (Klein, 2015).

The first method to enrich education for the gifted and talented students is to differentiate the instruction within a classroom, or enhance the content. This entails the teacher to teach all students then provide extra material to challenge the gifted students as well as supportive materials for struggling students. This is a time consuming task. Due to the time constraints, teachers spend the majority of their time focusing on lower-achieving students to meet the state standards, since most high-achieving students will pass the state exams (Wiesman, 2013). Due to the lack of attention and engagement, "bright students do not find learning difficult; but they do not find school very interesting, either" (Lloyd, 1998).

The second method is to accelerate the students who show mathematical potential in early years and allow them to skip 8th grade mathematics, and instead place them in Algebra 1. This method assumes the students will be challenged because they are taking a mathematics class one year above the requirement. This method tends to be more popular because it is easier for the school to accelerate the students than to require each teacher to differentiate instruction. One reason is that classroom teachers "have difficulty matching instruction to the skill levels in today's diverse classrooms" (Ysseldyke, 2004, p. 295). The second reason is that once students are accelerated, they will continue that path throughout high school and should continue to be challenged by the material. The accelerated path means students take Algebra 1 in the 8th grade, Geometry in 9th grade, Algebra 2 (Algebra 2 Trig) in 10th grade, pre-calculus in 11th grade, and AP Calculus in 12th grade.

Gifted students are given the opportunity to earn college credit through the College Board Accelerated Placement (AP) program their senior year. However, can AP Calculus best prepare them for college level mathematics? The AP Calculus curriculum is considered by many mathematics professors as a “breadth of material to be mastered” and professors believe students often earn a satisfactory grade by focusing on algorithms and procedures instead of understanding (Bressoud, 2010, p. 3 of 4; Wade 2011). Unfortunately, the answer may not matter because many students are not given another option for their senior level mathematics course.

AP Calculus Programs

The AP Program “provides willing and academically prepared students with the opportunity to earn college credit, advanced placement, or both” (College Board, 2012). This program was developed in the mid twentieth century, and over the decades has expanded the courses available and the number of schools enrolled. As of today, there are 34 available courses and over 60% of college policies accepting these credits. According to a College Board survey in 2013, 68% of policies will give students credit for a score of 3 or higher, with another 30% requiring a 4 or higher (Adams, 2014). If students pass the AP Calculus exam, some chose to accept the college credit. However, there are many who either do not pass the exam or chose to not take the college credit. Based on this lost opportunity, are students truly benefiting from taking this class?

Rosenstein, a mathematics professor at Rutgers, did a study on whether or not students were benefiting from AP credit or even using it towards advanced placement. He found “only about 1 out of 16 students who are accelerated into a calculus course in high school earn and make use of advanced placement in mathematics”(Rosenstein, p. 5). He also discovered that

other students used their years of acceleration in high school to slow down or end their math career once entering college. In fact, the results of the study show that “students who took AP Calculus but did not receive advanced placements did not really benefit from taking AP Calculus in high school and would have been better served by a non-AP Calculus course” (Rosenstein, p.5). Should there be another option for seniors besides AP Calculus?

Thus the question is posed: is it the school’s main responsibility to support student’s early college credit, or better prepare them for a college education and the expectations that come with it? The AP Calculus course gives the students the college credit they seek, but are they truly ready to skip Single Variable College Calculus in college? Some students may in fact be ready to take that leap, but the majority of students often are not. This paper provides insight into a non-AP Calculus unit that may offer more support and options to high school seniors as they seek to prepare for success in college calculus.

CHAPTER 2

LITERATURE REVIEW

United States Education

Mathematics Education in the United States (US) went through a paradigm shift with the change from the National Council of Teachers of Mathematics (NCTM) Standards to the Common Core State Standards (CCSS). A major reason behind the shift was due to the US not being able to compete mathematically with other countries. The US based National Assessment of Educational Progress (NAEP), the internationally focused Trends in International Mathematics and Science Study (TIMSS), and the Program for International Student Assessment (PISA) have all verified that students educated in the American School System “lack the ability to understand and apply mathematical principles” (Schimdt, 2012). For example, the TIMSS study in 2011 illustrated that US 4th graders scored a 541 (average is 500) are in the top 15 education systems, while US 8th graders scored a 509 (average is 500) and indicate they are in the top 24 education systems. These declining results are concerning, despite the fact that the US spends more money on education than other countries.

US students are underperforming compared to those of other countries, which is problematic because in our current global economy, US students compete with students from all across the globe (Schimdt, 2012). In order for US students to compete in the international job market, the US set a goal to better prepare students and increase performance by developing national standards, like all other countries that participated in the TIMMS study, and this was the motivation for the CCSS.

Common Core State Standards

The CCSS were developed in 2009, and by December 2013, was adopted by 45 different states. Even though these states adopted the CCSS, for most the plan will not be fully implemented until the 2014-2015 school year (Kober, 2012). The creators of the CCSS saw “the value of consistent, real-world learning goals” to promote higher-level thinking and claimed that upon high school graduation, the standards could “create students who were college and career ready” (Common Core State Standards Development Process, 2010). Porter explained that the CCSS “are explicit in their focus on what students are to learn, what we call here ‘the content of the intended curriculum’” (2011, p. 103). The common core state standards outline the knowledge and skills that students are expected to learn in math throughout each grade level (Kober, 2012).

High School Mathematics Course Requirement

A more rigorous mathematics curriculum created by the CCSS will allow the United States education system to compete on an international level (Common Core State Standards Criteria, 2010). In addition to the new state standards, many states have increased the total required mathematical credits needed to graduate high school. As seen in the Table 1 below, as of June 2013, 18 states require 4 mathematical courses to graduate; a 260% increase from before the adoption of CCSS to now. It would appear that the rest of the states would soon follow as well. The four math credit requirement will allow students to learn more mathematics in their high school career with the goal of preparing them for college level mathematics.

Table 1

Trend of required mathematics courses in the States since the adoption of the CCSS

Number of Mathematics Courses Required for HS Graduation	Before 2010 Adoption of CCSS	As of June 2013	% Change (New-Old)/Old
2	15	5	-67%
3	26	24	-8%
4	5	18	260%
Unknown	4	3	-25%

Source 2010 *Education Commission of the States* as compared to the source listed above

Gifted Education

The trend is that 4 mathematics classes will be required for high school graduation. With six to ten percent of the student population being gifted and talented (three to five million students), they will need a more challenging mathematics class (NAGC, 2007). The most common approach is to accelerate students by moving them directly to Algebra 1 in 8th grade. If the student continues this route, they will take geometry (9th), algebra 2 (10th), and followed by pre-calculus (11th). These account for three mathematics courses, but the trend shows that students may be required to take 4 mathematics courses. There are limited other options available in most schools due to a lack of trained teachers in higher level math, or not enough funds to provide additional courses. Focusing on students who need a course to build upon their pre-calculus education, the author finds that AP Calculus may not be the best designed course in their progression.

AP Calculus

Rosenstein asked “what percentage of students who take AP Calculus are actually advantaged by the acceleration” (2011, p. 1)? He examined how many students used their exam score to advance in the college calculus sequence (Rosenstein, p. 2). He researched freshman cohorts entering Rutgers and found the following results: “Only about 1 out of 16 students who are accelerated into a calculus course in high school earn and make use of advanced placement in mathematics” (2011, p. 2). In addition, a lot of students use “their years of acceleration in order to slow down (or even end) their mathematical studies” (2011, p. 2). The US cannot have students ending their mathematical career in high school if they hope to compare or compete on an international education playing field. Rosenstein’s results indicate that for a lot of students, AP Calculus is too difficult and discourages students from continuing their mathematics education. Bressoud (2009) researched the effects of AP Calculus in high school and found that 31% of high school AP Calculus students decided to take pre-calculus when entering college and another 32% took no calculus in college. If these results stay consistent year over year, more than half of high school AP Calculus students will not benefit from taking the course.

Bressoud discovered the Chrisman Morgan study that researched students who used or intended to use the Advanced Placement option upon entering college. The number of students who passed the exam and intended the use the credit was 80% (Bressoud, p. 2). Unfortunately, this study does not tell us the percentage of students who took the exam, and therefore cannot tell us how many students failed the exam. David Lutzer conducted a similar study and found that “students who took Calculus 1 at William and Mary (College) were more likely to take the next mathematics class than those who arrived with AP credit for Calculus” (Bressoud, p. 3). These two results combined show that AP Calculus may not have been beneficial for the group of

students who used the credit for college. Calculus is still a relevant mathematics subject though, so we must explore alternate ways to teach this material.

There are gifted students who benefit from AP Calculus, but this is only a select number. Many others are not ready for the fast-paced course. Rosenstein discovered that “students who took AP Calculus but did not receive advanced placements did not really benefit from taking AP Calculus in high school and would have been better served by a non-AP Calculus course” (2011, p. 5). Bressoud came to the same conclusion; “there is no evidence that taking calculus in high school is of any benefit unless a student learns it well enough to earn college credit for it, and there is some evidence that an introduction to calculus that builds on an inadequate foundation can be detrimental” (2009, p. 6). Thus, the students who do not earn college credit did not benefit from this fast-paced course; another option needs to be available for their fourth required math course.

Non-AP Calculus Course

These findings cannot exclusively tell us that AP Calculus is the best choice for gifted seniors, but it can tell us that a majority of the students did not benefit from taking an AP Calculus course. How can we decide what is the best choice? Should there be more options available to these students even with the limited funds? The new standards are given as a curriculum guide that students are expected to learn, and if mastered will prepare them for college (Kober, 2012). The CCSS does not expect students to have earned college credit upon graduation, but does expect them to be equipped with the ability for deeper level thinking. Therefore, the schools should provide a course that examines the underlying concepts of calculus

curriculum, and connects this new conceptual understanding to previously learned material in order to give the students a full grasp on high school mathematics.

Purpose of High School

One of the main goals for gifted and talented students is it to be adequately prepared for college; therefore high schools should set this goal as one of their top priorities. In a study done by Camacho and Cook, they found that many high schools across the nation are adopting the goal of preparing students for college. This was heavily due in part to the low level of preparedness in high school students seeking a college education (2007, p.9). This is especially true in mathematics; “some researchers claim that students are underprepared for college calculus because teachers tend to focus on procedural instruction instead of conceptual understanding” (Wade, p. 2). The most important objective is to have students be prepared; “Solid mathematical preparation is far more important than exposure to calculus” (Bressoud, p.1). A non-AP Calculus course can provide this exposure and prepare students for college level mathematics. The course would delve into the major topics of calculus by relating them back to Algebra, Geometry, and Trigonometry. This is also supported by the CCSS; the standards “represent a modest shift toward higher levels of cognitive demand than are currently represented in state standards” (Porter, p. 106). While AP Calculus may provide the same material, a non-AP Calculus course has two extra months for there would be no exam or exam prep time, and allow the teacher to dive into the material and teach the concepts rather than procedures. This higher-level thinking can be provided and allows students more time to gain a deeper conceptual understanding of Calculus which also better prepare them for college level Calculus.

CHAPTER 3

CURRICULUM PROJECT

This curriculum project is focused on a non-AP Calculus derivative unit, which may better prepare students for college mathematics. This unit takes a closer look at the definition of a derivative. The goal of a non-AP calculus course should be to provide an in depth look at crucial components of Calculus and this unit is an example. It defines the derivative and scaffolds instruction to provide greater conceptual understanding. The first lesson recalls information that the student learned in Algebra and builds upon that knowledge to lead to the definition of the derivative. The unit takes information from Larson, R., & Hostetler, R.'s (2006) Calculus of a single variable (8th ed.) book as well as other online resources that have been cited. How the unit plan is put together and presented to the students is what makes it unique, as well as the time-line.

The main difference between this unit and the similar unit provided in AP Calculus is the time-line. This unit allows an extra five days of instruction, and that extra time allows the students to dive into the definition of the derivative and understand the underlying concepts. The unit starts with a refresher on slope, a concept learned in Algebra, and connects it to the definition of the derivative. Extra days are provided for more examples that an AP Calculus fast-paced course would not allow. The connection made between Algebra and Calculus is what allows students to better grasp the material.

Unit plan: The definition of a derivative

Lesson 1: Slope of a linear function

Lesson 2: What is a tangent line?

Lesson 3: Secant lines

Lesson 4: Slope of a tangent line

Lesson 5: Extra practice for Lesson 4

Lesson 6: Definition of the derivative

Lesson 7: Extra practice for Lesson 6

Lesson 8: Quiz

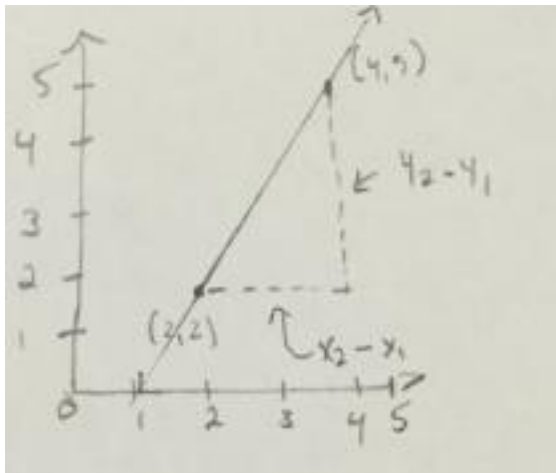
Lesson 1: The slope of a linear function

Objective: Refresh the student's prior knowledge of the slope of a linear function, and function notation.

Reason: The goal is to connect the concept of slope in algebra to the concept of the derivative in calculus. To do so, students need to be reminded of their prior knowledge.

Notes:

Graph the points (2,2) and (4,5)



This picture gets drawn as the teacher goes through the notes. Each student will draw their own graph in their notes.

What is the slope?

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{5 - 2}{4 - 2} = \frac{3}{2}$$

What does $y_2 - y_1$ actually mean?

The change in y . (How much the y -coordinate changed from one point to the next.)

Notation: Δy

What does $x_2 - x_1$ actually mean?

The change in x . (How much the x -coordinate changed from one point to the next)

Notation: Δx

$$\therefore m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

Example 1: Given the points $A(-1,3)$ and $B(-5,7)$, what is the change of y ? What is the change of x ? What is the slope?

Example 2: Given the following information, find the slope of the function.

$$a = 2 \quad f(a) = 5$$

$$b = 6 \quad f(b) = -1$$

These examples will allow the students to refresh their function notation.

Example 3: Given the function $f(x) = 3x - 5$, find the slope of the function for $a = -2$, and $b = 4$.

The purpose of these examples is to review the process of calculation the slope of a linear function, as well as reviewing function notation. There is no homework assigned after this lesson. I would assign an exit ticket that is similar to example 3 in order to find out which students are still struggling with algebraic concepts.

Lesson 2: What is a tangent line?

Objective: Define the word tangent, as well as visualize and draw tangent lines. Introduce the tangent line problem.

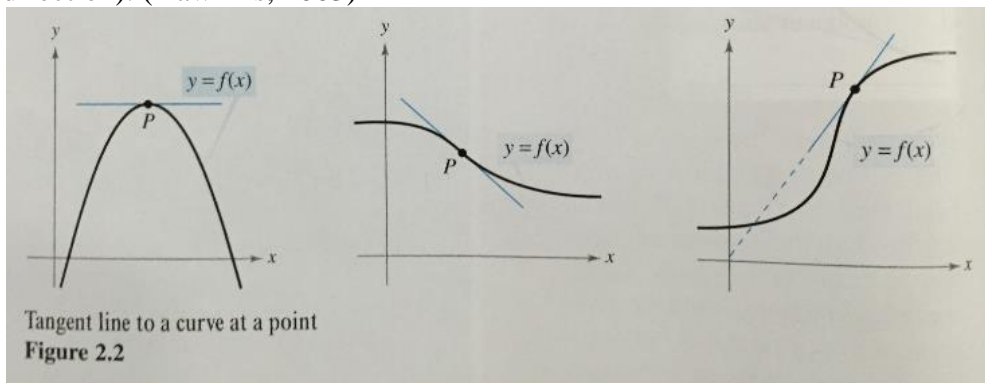
Reason: The students need a clear definition of a tangent line. A visual picture will help students understand the definition. In the future, it will allow them to better understand the concept of a derivative (the ultimate goal). Outlining the tangent problem allows the student to start thinking of solutions, and the possibilities instead of being told the outcome.

Warm up:

1. Given $f(1) = 8$ and $f(5) = -3$, find the slope.
2. Find the slope if two points on the line are $(2,3)$ and $(-5,6)$

Notes:

Definition of a tangent line – A line that touches a curve at one point without crossing over, almost as if it is parallel to the graph **at that point** (by parallel we mean move in the same direction). (Dawkins, 2003)

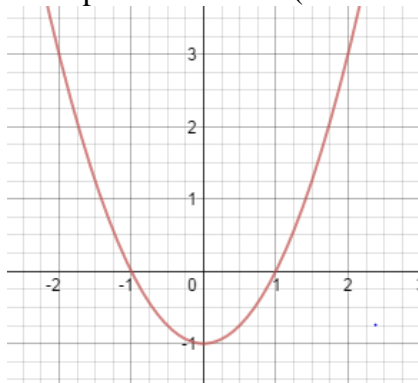


Emphasize at that point, because the line itself can cross over the curve at some point. (As seen in example 3).

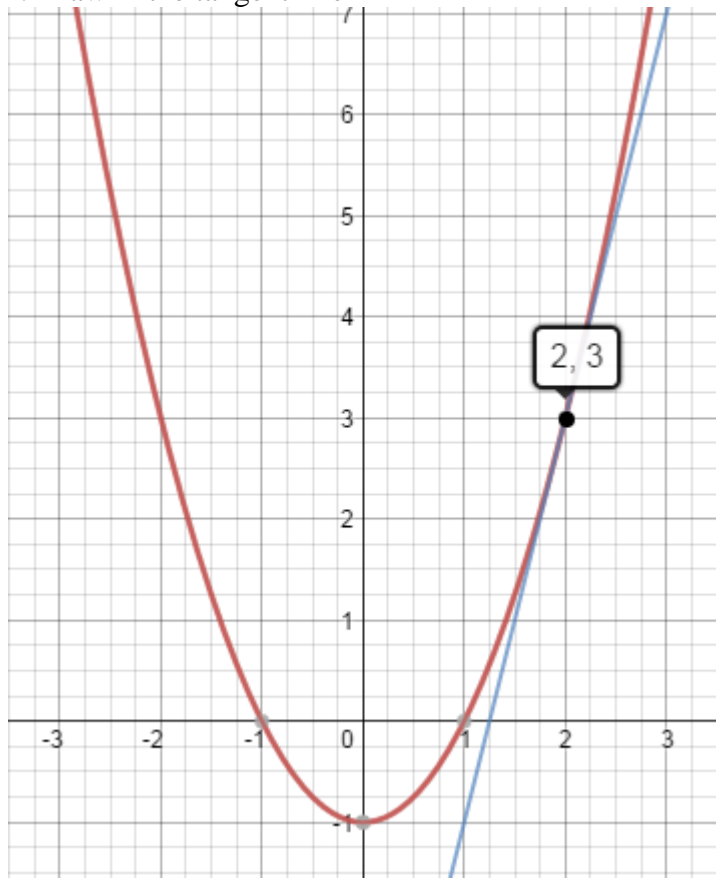
Tangent line problem

Question: Find the tangent line to the function $f(x) = x^2 - 1$ at $x = 2$.

1. Graph the function. (Label the point $(2,3)$)



2. Draw in the tangent line



The students would be drawing this graph in their own notes. The answer would only be shown after enough time is given to the students to draw their graph.

In order to write the equation of a line we need either two points on the line or a single point and the slope of the line.

Problem: How can we find that second point or the slope?

This is an easier example, and the students could find the slope based on looking at the graph. The goal of this lesson is to have them start thinking about how can we find slopes of lines or curves that are not linear, or we can't discover on graph paper.

“And I dare say that this is not only the most useful and general problem in Geometry that I know, but even that I ever desire to know” – 1637, René Descartes. Interesting fact to share with students, the geometry problem they are examining today, was also a problem in 1637.

Homework (Lesson 2)

1. Find the slope for the following situations

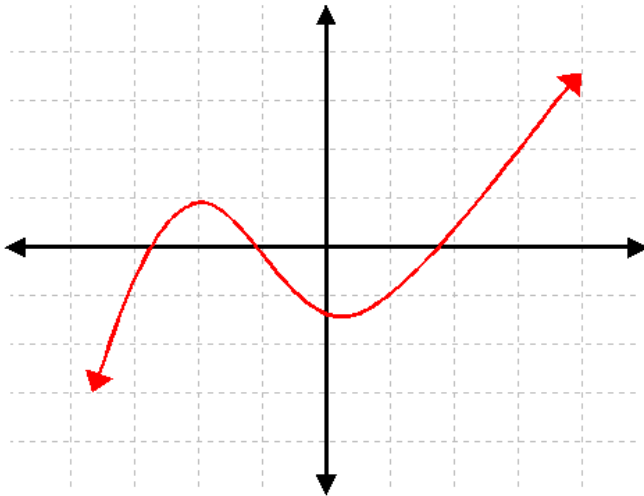
a. Two points of the linear function are (3,5) and (7,2).

b. $f(2) = 6$ and $f(5) = -3$

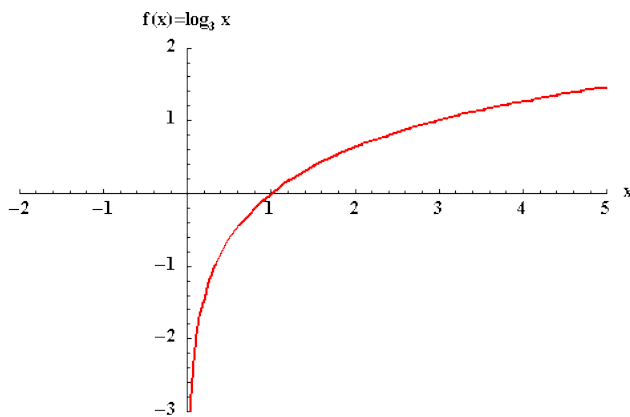
Additional practice on slope and function notation will allow them to become experts. Now we can start expanding their knowledge.

2. Draw a tangent line to the following curves at two different points.

a.



b.



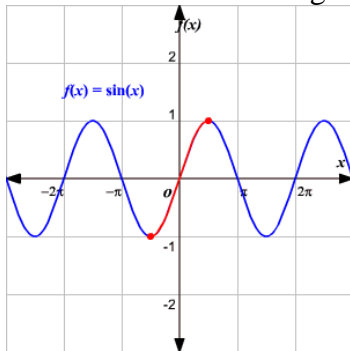
Lesson 3: Secant Lines

Objective: Students will learn about secant lines and that its slope is: $m_{sec} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$

Reason: This will allow the student's to move closer to the definition of a tangent line. Approximating the secant line is necessary to find the tangent line; therefore students need a clear understanding of what a secant line is, and its slope.

Warm up:

Draw three different tangent lines to the function below:



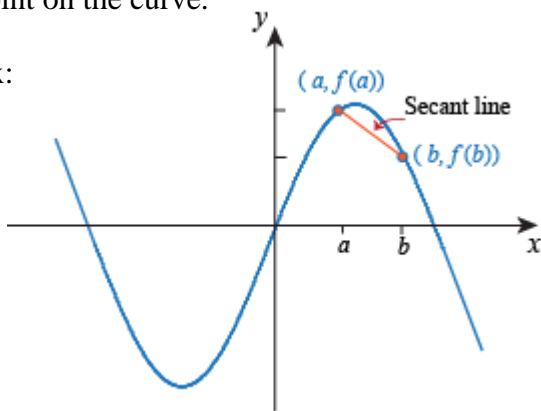
Review Homework:

- Answer any questions the students might have.
- A formative assessment that will allow you to assess student knowledge.

Notes:

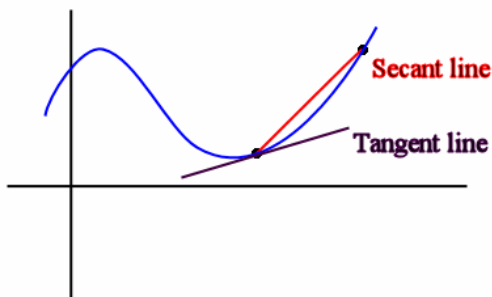
Definition of a secant line – It is a line that goes through the point of tangency and a second point on the curve.

Ex:



Make sure the students understand the notation.

Secant lines and Tangent lines - The secant line and tangent line all meet at one point on the curve.

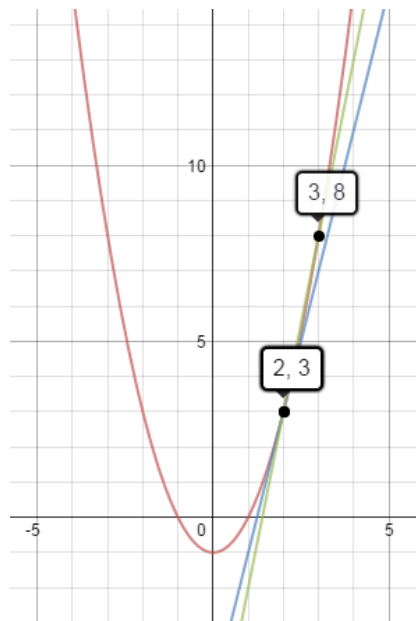


How does the secant line help?

A secant line is used to approximate the slope of a curve and helps us solve our tangent line problem.

Question: Find the tangent line to the function $f(x) = x^2 - 1$ at $x = 2$.

Graph the function. Then graph its tangent line at $x=2$. Then graph the secant line. Let it go through the points $(2,3)$ and $(3,8)$.



Do one step at a time. Given all the directions at once may lead to confusion.

Red = $f(x)$
Blue = tangent line
Green = secant line

Again, the student will be graphing this in their notes, with the solution given after the students attempted the problem.

The secant line's slope is an approximation of the tangent line's slope. Find the slope of the secant line.

$$m = \frac{8-3}{3-2} = \frac{5}{1} = 5$$

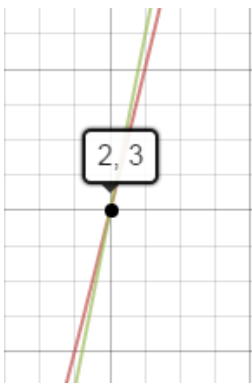
Then let's assume 5 is also the slope of the tangent line. Let's create the equation of the tangent line. (We want to use slope 5, and point $(2,3)$)

$$y - 3 = 5(x - 2)$$

$$y - 3 = 5x - 10$$

$$y = 5x - 7$$

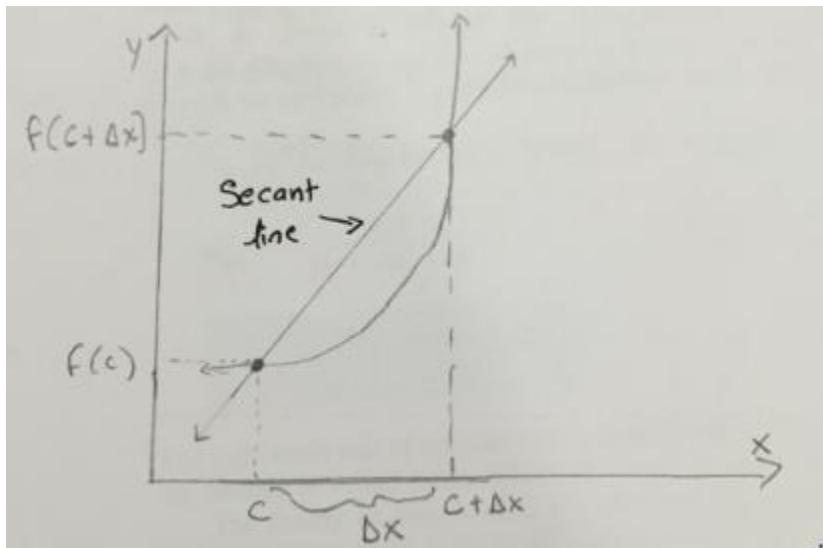
Graph this tangent line with our curve. How accurate does it look?



It is not 100% accurate, because the line crosses our curve.
How can we make this more accurate?

Take a closer look at the slope of a secant line:

1. Point on the curve: $(c, f(c))$
2. We need another point on the curve in order to draw the secant line.
This will be some distance away from c . Let's call this distance Δx .
3. Second point will be: $(c+\Delta x, f(c+\Delta x))$
4. Now we need to find the slope:



Two ways (think about it):

First way

$$m = \frac{\text{change of } y}{\text{change of } x}$$

$$\text{Change of } y = f(c + \Delta x) - f(c)$$

$$\text{Change of } x = c + \Delta x - c = \Delta x$$

$$\text{Thus } m_{\text{sec}} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

Second way

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y_2 = f(c + \Delta x) \text{ and } y_1 = f(c)$$

$$x_2 = c + \Delta x \text{ and } x_1 = c$$

$$\text{Thus } m = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

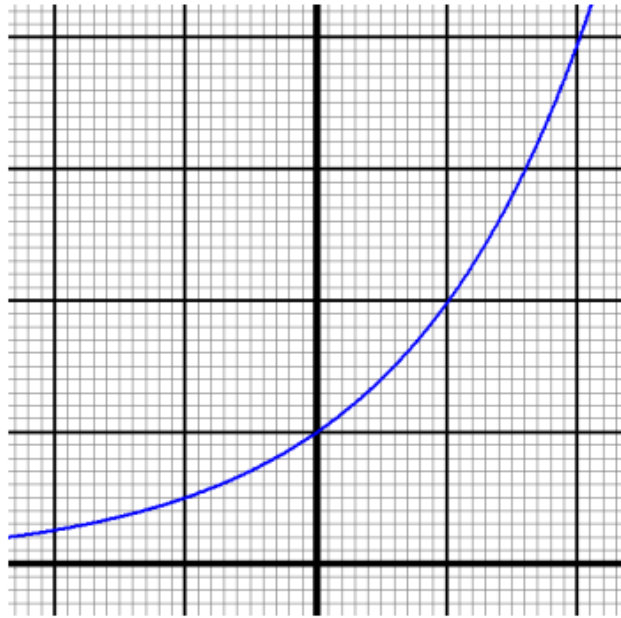
Each student learns differently, and therefore multiple ways are provided to benefit more students.

This lesson may take two days. Each graph that is shown, students would have drawn by hand on a piece of graph paper. The worksheet that goes along with this lesson starts the process of thinking about minimizing Δx . This worksheet comes from:

<http://www1.american.edu/cas/mathstat/People/kalman/calc1/day1/worksheet1.pdf>. It is provided below for easier viewing, and the author's additional notes to justify the worksheet.

Worksheet:**Approximating slopes of tangent lines**

The figure at right shows the graph of the function $f(x) = 2^x$. Using a straightedge, draw a tangent line as accurately as possible at the point $(1,2)$. Then draw a secant line through the points $(1,2)$ and $(2,4)$. We will refer to that as the *right* secant line. Draw another secant line, this time through the points $(-1, .5)$ and $(1,2)$. This is the *left* secant line. Answer the following questions.



A new function, the students have not seen thus far in the unit.

1. What is the slope of the left secant line?
2. Based on the graph, is the slope of the left secant line greater than, equal to, or less than the slope of the tangent line?
3. What is the slope of the right secant line?
4. Based on the graph, is the slope of the right secant line greater than, equal to, or less than the slope of the tangent line?
5. Using the results above, fill in the blanks with numerical values:
 _____ < slope of tangent line < _____
6. Based on the foregoing, what is your best numerical estimate of the tangent line, and how far off might this estimate be, at worst?

These questions allow the students not only to practice drawing tangent and secant lines, but allow them to start thinking about the equation of a tangent line, and the importance of a secant line. In addition it gives more practice on the definition of slope.

Lesson 4: Slope of a tangent line

Objective: Connect the previous three lessons, as well as the previous unit of limits, and define the slope of a tangent line. This brings us one step closer to the definition of a derivative.

Reason: The definition of a derivative is based on the slope of the tangent line, thus it is crucial that students can conceptually understand a tangent line, and its slope.

Warm up:

Find the slope of the secant line that goes through the function at (-2,3) and (5,4).

Review Homework:

- Answer any questions the students might have.
- A formative assessment that will allow you to assess student knowledge.

Notes:

Recall: $m_{sec} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$

The slope found using the two points of (2,3) and (3,8) was not 100% accurate.

In order to approximate the slope of the tangent line as accurately as possible, is to decrease the distance between the two points. In other words, we need to minimize Δx .

Example: $f(x) = x^2 - 1$ at $x = 2$.

Second Point	f(second point)	Slope
3	8	5
2.75	6.5625	4.75
2.5	5.25	4.5
2.2	3.84	4.2
2.05	3.2025	4.05
2.01	3.0401	4.01
2.0001	3.00040001	4.0001

This chart would be created by the students. The teacher would walk through the first couple together, and the students would fill out the rest on their own with the first column given to them. It is additional practice on computing the slope.

What do you discover about the slope?

The slope is moving closer and closer to 4.

This chart allows them to compute the slope, and visually see how the slope is narrowing down to 4.

Minimize Δx : To minimize Δx and obtain a more accurate approximation of the slope of the tangent line, we need to take the limit of the secant slope as Δx approaches 0. We are minimizing the distance between the two points, and that means Δx goes to 0.

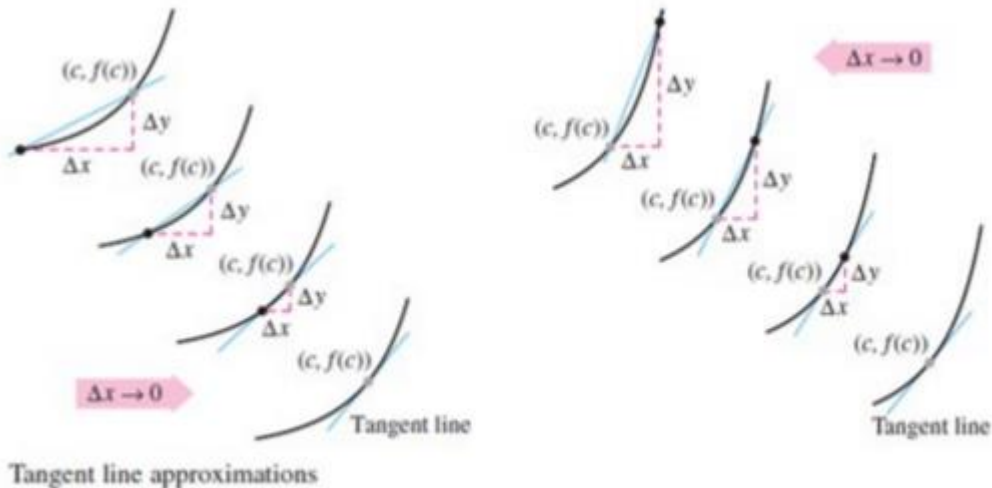


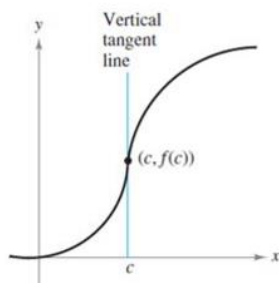
Figure 2.4

Definition of a tangent line at point c with slope m :

$$\lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

The equation of the tangent line: $y - f(c) = m(x - c)$

(This definition does not cover vertical tangent lines. If a tangent is vertical then the slope is either ∞ or $-\infty$.)



The graph of f has a vertical tangent line at $(c, f(c))$.

Figure 2.7

Before continuing on to an easy example to finish out the lesson, the teacher should take a few minutes and explain how to calculate $f(c+\Delta x)$. Students get confused on this concept.

Side bar:

$f(c) \rightarrow$ this means for every x in the equation, you would replace with c .

example: $f(x) = x^2 - 1 \rightarrow f(c) = c^2 - 1$

$f(c+\Delta x) \rightarrow$ this means for every x in the equation, you would replace with $c+\Delta x$

example: $f(x) = x^2 - 1 \rightarrow f(c + \Delta x) = (c + \Delta x)^2 - 1$

Example 1: Find the slope of the tangent line of $f(x) = 4x - 9$ at $x=1$.

$$\begin{aligned} & \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} \\ & \lim_{\Delta x \rightarrow 0} \frac{f(1 + \Delta x) - f(1)}{\Delta x} \\ & \lim_{\Delta x \rightarrow 0} \frac{4(1 + \Delta x) - 9 - [4(1) - 9]}{\Delta x} \\ & \lim_{\Delta x \rightarrow 0} \frac{4 + 4\Delta x - 9 - (-5)}{\Delta x} \\ & \lim_{\Delta x \rightarrow 0} \frac{4\Delta x}{\Delta x} \\ & \lim_{\Delta x \rightarrow 0} 4 = 4 \end{aligned}$$

Therefore $m = 4$.

The point of choosing an easy linear function is to allow the students to practice with the notation. The worksheet that follows this just has a couple examples for students to get more practice. The last question has a quadratic function to see if students can apply their knowledge.

Worksheet:

Directions: Find the slope of the tangent line at the given points.

1. $f(x) = 6x + 3, x = 2$

2. $f(x) = \frac{1}{2}x - 4, x = -4$

3. $f(x) = x^2 - 3, x = 5$

Lesson 5: Extra examples for Lesson 4

Objective: Give the students extra examples on tangent lines.

Reason: Students need to be able to conceptually understand the slope of tangent lines as well as calculate them.

Example 1 (this can be the warm up)

Find the slope of the tangent line of $f(x) = -\frac{3}{2}x + 6$ at $x = 2$.

Example 2: Find the slope of the tangent line of $f(x) = x^2 - 5x + 3$ at $x = 3$.

Model this example.

Example 3: Find the slope of the tangent line of $f(x) = x^3 - 1$ at $x = -2$.

Students try this on their own.

Example 4: Find the slope of the tangent line of $f(x) = x^2 - 2x$ at $x=c$.

Model this example. This is a generic example, and can then be applied to many different values. This also leads one step closer to the definition of the derivative.

$$\begin{aligned} & \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} \\ & \lim_{\Delta x \rightarrow 0} \frac{(c + \Delta x)^2 - 2(c + \Delta x) - [c^2 - 2c]}{\Delta x} \\ & \lim_{\Delta x \rightarrow 0} \frac{c^2 + 2c\Delta x + \Delta x^2 - 2c - 2\Delta x - c^2 + 2c}{\Delta x} \\ & \lim_{\Delta x \rightarrow 0} \frac{2c\Delta x + \Delta x^2 - 2\Delta x}{\Delta x} \\ & \lim_{\Delta x \rightarrow 0} 2c + \Delta x - 2 = 2c - 2 \end{aligned}$$

Let $c = 2 \rightarrow m = 2(2) - 2 = 2$

We can now apply it for any c .

Let $c = 4 \rightarrow m = 2(4) - 2 = 6$

Now write the tangent line for $c = -1$. Graph the function and its tangent line.

$c = -1 \rightarrow m = 2(-1) - 2 = -4$

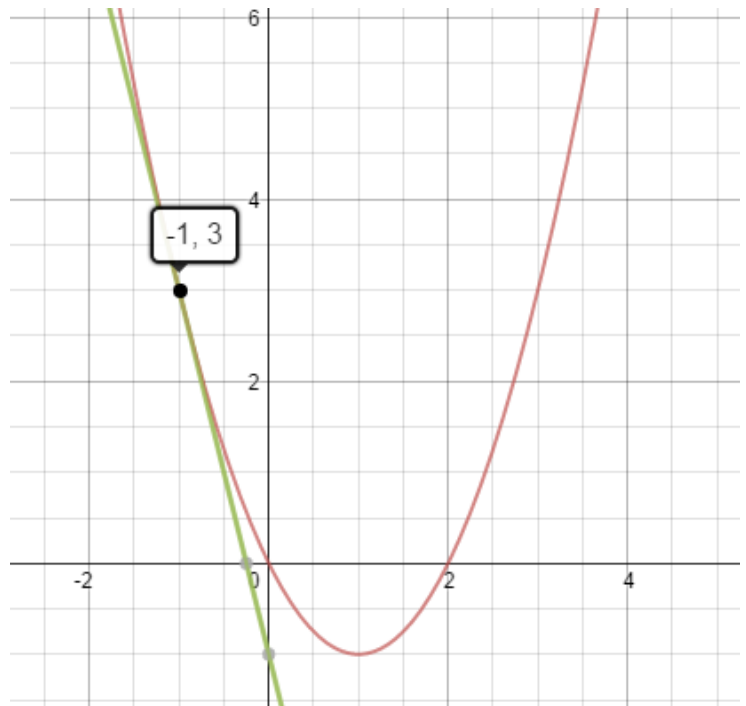
$f(-1) = (-1)^2 - 2(-1) = 1 + 2 = 3$

Equation:

$y - 3 = -4(x - -1)$

$y - 3 = -4x - 4$

$y = -4x - 1$



Worksheet: This worksheet allows the students to practice finding the slope of the tangent line as well as writing the equation. It was taken from the site: www.uplifteducation.org and slightly modified to fit the format and formula of this unit plan. The worksheet should become homework if the students do not finish in the time allotted.

Calculus 1
Worksheet – Tangent Line Slope

Name _____
Date _____

Use the limit definition to find the equation for the slope of any tangent line, the slope of the tangent line at the given value, and the equation of the tangent line at the given value of x .

1. $f(x) = x^2 + 1$ at $x = 2$

$m_{\text{tan}} =$ _____ $m =$ _____ point = _____ equation = _____

2. $f(x) = x^3 - 2$ at $x = 1$

$m_{\text{tan}} =$ _____ $m =$ _____ point = _____ equation = _____

3. $f(x) = 3x^2 - 2x + 5$ at $x = 4$

$m_{\text{tan}} =$ _____ $m =$ _____ point = _____ equation = _____

4. $f(x) = \frac{3}{x}$ at $x = 3$

$m_{\text{tan}} =$ _____ $m =$ _____ point = _____ equation = _____

5. $f(x) = \frac{x}{x-2}$ at $x = 3$

$m_{\text{tan}} =$ _____ $m =$ _____ point = _____ equation = _____

Lesson 6: Definition of a derivative

Objective: Learn the definition of the derivative and how it relates to the slope of the tangent line. All the lessons lead up to this, and this is the crucial point of the unit. This is a fundamental concept of calculus.

Reason: Since this is one of two main fundamental concepts of Calculus, it is important that students understand the concept behind the derivative, and how it relates to all previous lessons.

Warm up:

Find the slope of the tangent line of $f(x) = x^2 - 5x + 3$ at $x = c$.

This will lead into the discussion of the notes.

Review homework:

- Answer any questions the students might have.
- A formative assessment that will allow you to assess student knowledge.

Notes:

The slope of a tangent line at point c is:

$$\lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

The derivative of a function is defined as:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Note:

- Assuming the limit exists
- The derivative itself is also a function
- instantaneous rate of change or the rate of change of the function

Notation:

Derivative: $f'(x)$, $\frac{dy}{dx}$, y' , $\frac{d}{dx}[f(x)]$

Vocabulary:

Differentiation – the process of finding the derivative of a function

Differentiable – when it is possible to take a derivative, then function is differentiable

Example 1: Find the derivative of $f(x) = 5x - 6$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{5(x + \Delta x) - 6 - [5x - 6]}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{5x + 5\Delta x - 6 - 5x + 6}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{5\Delta x}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} 5 = 5$$

Again, start with an easier example. This process should be getting simpler now as students have been practicing this for a few days now.

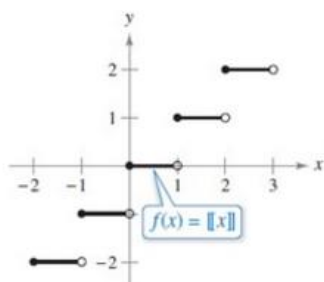
Example 2: Find the derivative of $f(x) = x^2$

Have the students try this example on their own, and then review it as a class. If the student is done early, have them create the tangent line at the point (2,4).

Differentiability and Continuity:

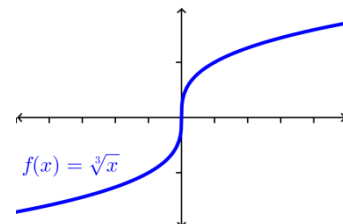
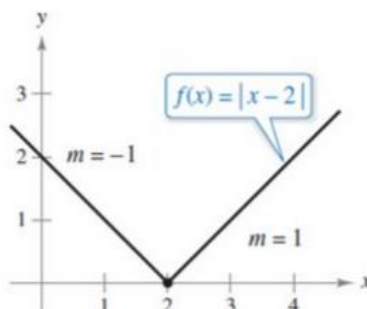
The derivative is also a function, and based on a limit. Thus the limit from the left and the right need to be the same. There are several functions, where the derivative does not exist because the limit does not exist.

Examples: The step function, a graph with a sharp turn (absolute value function), or a graph with a vertical tangent line.



The greatest integer function is not differentiable at $x = 0$ because it is not continuous at $x = 0$.

Figure 2.11



Theorem 2.1: If f is differentiable at $x=c$, then it is continuous at $x=c$.

Lesson 7: Extra examples for Lesson 6

Objective: Give students extra practice on finding the derivative of a function using the definition of a function.

Reason: The students need to understand this concept because it is a fundamental part of calculus.

Examples:

Example 1: Find the derivative of the function $f(x) = x^3 - 2x^2 + 9$. What is $f'(1)$?
This can be the warm up.

Example 2: Find the derivative of the function $f(x) = \sqrt{5x - 3}$. Write the equation of the tangent line at the point $(1, \sqrt{2})$.

Give the students a few minutes to try and find the derivative on their own or with a partner. Make sure to review this with the class because some students will struggle with the square root sign.

Worksheets: There are two worksheets below that were taken from their respective sites. The first worksheet allows the student to work through examples and get more comfortable with the procedure. All problems should not be assigned. The last question makes the students think about the concept of the derivative and is a great question. The second worksheet gives an additional practice, but goes above and beyond and has the students apply the derivatives to create tangent lines. This is also a great tool to provide differentiation to the students. Below are snapshots of both worksheets for easier viewing, along with my notes.

Worksheets:

1. <http://cdn.kutasoftware.com/Worksheets/Calc/03%20-%20Definition%20of%20the%20Derivative.pdf>

Kuta Software - Infinite Calculus

Name _____

Definition of the Derivative

Date _____ Period _____

Use the definition of the derivative to find the derivative of each function with respect to x .

1) $y = -2x + 5$

2) $f(x) = -4x - 2$

3) $y = 4x^2 + 1$

4) $f(x) = -3x^2 + 4$

5) $y = -4x^2 - 5x - 2$

6) $y = 3x^2 + 3x + 3$

7) $y = \sqrt{-3x - 5}$

8) $f(x) = \sqrt{4x - 5}$

The square root function ups the level of difficulty and will challenge the top students.

9) $y = \frac{1}{x+2}$

10) $f(x) = -\frac{2}{2x-1}$

The fraction functions will also challenge the students.

Critical thinking question:

11) Use the definition of the derivative to show that $f'(0)$ does not exist where $f(x) = |x|$.

This is a great concept question because the students need to rely on their knowledge of limits in order to answer this.

2. http://www2.johnabbott.qc.ca/~olivier.dubois/103/103_worksheet_defn_derivative.pdf

201-103-RE - Calculus 1

WORKSHEET: DEFINITION OF THE DERIVATIVE

1. For each function given below, calculate the **derivative at a point** $f'(a)$ using the limit definition.

(a) $f(x) = 2x^2 - 3x$ $f'(0) = ?$

(b) $f(x) = \sqrt{2x+1}$ $f'(4) = ?$

(c) $f(x) = \frac{1}{x-2}$ $f'(3) = ?$

2. For each function $f(x)$ given below, find the **general derivative** $f'(x)$ as a new function by using the limit definition.

(a) $f(x) = \sqrt{x-4}$ $f'(x) = ?$

(b) $f(x) = -x^3$ $f'(x) = ?$

(c) $f(x) = \frac{x}{x+1}$ $f'(x) = ?$

(d) $f(x) = \frac{1}{\sqrt{x}}$ $f'(x) = ?$

The variety of questions here will allow a student to practice all types of questions, and challenge them as well.

3. For each function $f(x)$ given below, find the **equation of the tangent line** at the indicated point.

(a) $f(x) = x - x^2$ at $(2, -2)$

(b) $f(x) = 1 - 3x^2$ at $(0, 1)$

(c) $f(x) = \frac{1}{2x}$ at $x = 1$

(d) $f(x) = x + \sqrt{x}$ at $x = 1$

ANSWERS:

1. (a) $f'(0) = -3$ (b) $f'(4) = 1/3$ (c) $f'(3) = -1$

2. (a) $f'(x) = \frac{1}{2\sqrt{x-4}}$ (b) $f'(x) = -3x^2$ (c) $f'(x) = \frac{1}{(x+1)^2}$ (d) $f'(x) = \frac{-1}{2x^{3/2}}$

3. (a) $y = -3x + 4$ (b) $y = 1$ (c) $y = -\frac{1}{2}x + 1$ (d) $y = \frac{3}{2}x + \frac{1}{2}$

Lesson 8: Quiz

Objective: It is always important to test the student's knowledge to gain insight onto what they have actually learned.

Reason: Formative assessments are a necessity to see what concepts were learned and what might need to be re-taught.

Quiz:

Directions: For the following functions, find the derivative of the function, and create the equation of its tangent line at the given point.

1. $f(x) = 5x + 3, x = 4$

2. $f(x) = x^2 - 3x + 5, x = 1$

3. $f(x) = \sqrt{x - 5}, x = 9$

CHAPTER 4

VALIDITY OF CURRICULUM

This curriculum project was submitted to a Teacher who has instructed mathematics for 20 years, taught AP Calculus for 14 of those years, whose also won a NYS Teaching Excellence award. This Teacher's expertise and knowledge on the topic provides validity to the curriculum.

Teacher Review

The Teacher reviewed the curriculum and discussed her thoughts with the author. The following summarizes the feedback provided by the Teacher during this discussion. Lesson 1 is very close to what the Teacher instructs in her class, and is a good introduction to this topic.

She tends to do a blend of lesson 2 and lesson 3. The Teacher introduces secant lines before tangent lines, and tries to have the students to discover the slope of the tangent line. She believes this curriculum may be better because her students end up making parallel lines using the tangent and secant lines, where students using the methods in this curriculum would instead create the tangent and secant lines the teacher intended for them to make. The Teacher stressed that she does teach the AP course and therefore wants her students to discover on their own, but this often results in re-teaching the same material over again.

In lesson 4, the Teacher has never displayed slopes in a table due to time constraints. She feels that this is in an impactful way to convey the idea as well as key differences between the two courses. The side bar that is mentioned in the lesson 4 will also be beneficial because this is a topic she often re-teaches. The Teacher stated that since lesson 4 covers a crucial topic, it will be beneficial to the extra practice mentioned in lesson 5. She feels that the last example shown in lesson 5 is critical because it displays the concept graphically, algebraically, numerically, and

verbally. The solution displayed in various formats will help reach more students. This is something she cannot do due to time constraints.

Lesson 6 has a good introduction to the definition of a derivative, and the Teacher particularly likes the explanations of the vocabulary words. She commented on the smooth transition from the slope of the tangent line to the definition of a derivative. Lesson 7 has good examples because it increases the difficulty, particularly through the square root problem. The Teacher likes the differentiated worksheets because they will create opportunities for group work.

Overall, the Teacher feels that this unit plan takes a more in-depth look at the definition of a derivative but it does take more time to teach than the AP course layout. She does not have the extra time in her curriculum map to go into the in-depth knowledge, but some of her students are missing that conceptual understanding. Many of her students would benefit from this curriculum for it would give them a deeper understanding of the course content and Calculus in general.

CHAPTER 5

CONCLUSION

As previously presented, US students are lagging behind other countries in mathematical performance. In an attempt to allow students to compete on a more international level, the US shifted from the NCTM standards to the Common Core State Standards. Mathematics is a core subject in the US, and the current trend is to require four mathematics courses in high school. Therefore, there is a need for more choices of mathematics for all students, other than just an AP course.

Part of the student population is the gifted and talented students; they consist of six to ten percent of the population. The identified group of students took Algebra 1 in 8th grade, Geometry in 9th grade, Algebra II in 10th grade, and Pre-Calculus in 11th grade. Their senior year, most have the option of AP Calculus and some schools also offer AP Statistics. However, research shows that 1 out of 16 AP Calculus students do not benefit from taking AP Calculus, and would benefit more from a non-AP Calculus course (Rosenstein, 2011). This thesis provides a unit of the curriculum that would be offered in such a course.

This new mathematics course provides students with an in-depth understanding of the key components of calculus, as well as strengthening of previously learned material from prior courses in this high school sequence. The goal of a non AP Calculus course, and this unit, would be to give students the conceptual knowledge of Calculus, improve and build upon concepts learned in prior high school mathematics courses, and provide a basis for higher level mathematics education.

The curriculum was reviewed by a current AP Calculus teacher, she believes this course delivers more conceptual knowledge of Calculus due to the pace and the focus on previously learned material, something she cannot offer her students due to the time restraints she faces.

With a focus on the core concepts and an understanding of the students who are eligible, this course can be an important part of high school curriculum throughout the country. It would fulfill the extra course requirement that is rolling out across the states in a way that is meaningful to the students and beneficial as a capstone to their mathematics education. This course would solidify the methods learned in previous classes, lay groundwork for a pursuit of higher education, and provide the critical thinking skills required for US students to excel on a global platform.

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APPENDIX**Solutions to Unit Plan****Lesson 1: The slope of a linear function****Lesson 1: The slope of a linear function**

Example 1: Given the points A(-1,3) and B(-5,7), what is the change of y? What is the change of x? What is the slope?

$$\text{change of } y = 7 - 3 = 4$$

$$\text{change of } x = -5 - (-1) = -4$$

$$m = \frac{4}{-4} = \boxed{-1}$$

Example 2: Given the following information, find the slope of the function.

$$a = 2 \quad f(a) = 5$$

$$b = 6 \quad f(b) = -1$$

$$m = \frac{-1 - 5}{6 - 2} = \boxed{\frac{-6}{4}} = \boxed{-\frac{3}{2}}$$

Example 3: Given the function $f(x) = 3x - 5$, find the slope of the function for $a = -2$, and $b = 4$.

$$f(-2) = 3(-2) - 5 = -11$$

$$f(4) = 3(4) - 5 = 7$$

$$m = \frac{7 - (-11)}{4 - (-2)} = \frac{18}{6}$$

$$\boxed{m = 3}$$

Lesson 2: What is a tangent line?Warm up:1. Given $f(1) = 8$ and $f(5) = -3$, find the slope.

$$m = \frac{-3-8}{5-1} = \boxed{\frac{-11}{4}}$$

2. Find the slope if two points on the line are (2,3) and (-5,6)

$$m = \frac{6-3}{-5-2} = \boxed{\frac{3}{-7}}$$

Homework (Lesson 2)

1. Find the slope for the following situations

a. Two points of the linear function are (3,5) and (7,2).

$$m = \frac{2-5}{7-3} = \boxed{\frac{-3}{4}}$$

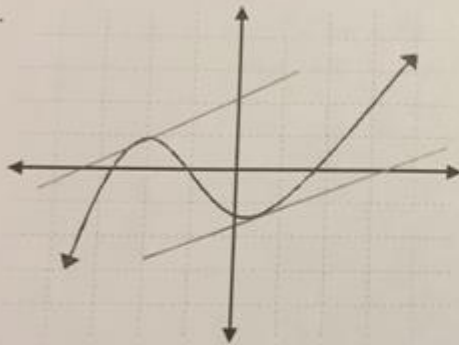
b. $f(2) = 6$ and $f(5) = -3$

$$m = \frac{-3-6}{5-2} = \frac{-9}{3}$$

$$\boxed{m = -3}$$

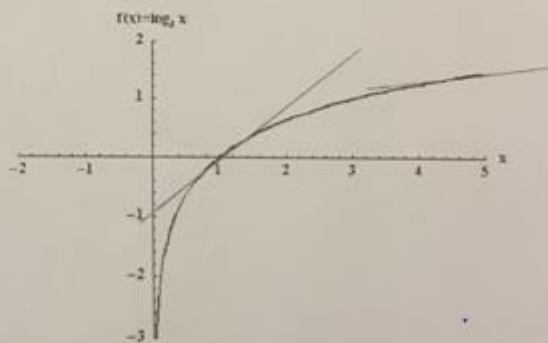
2. Draw a tangent line to the following curves at two different points.

a.



many solutions

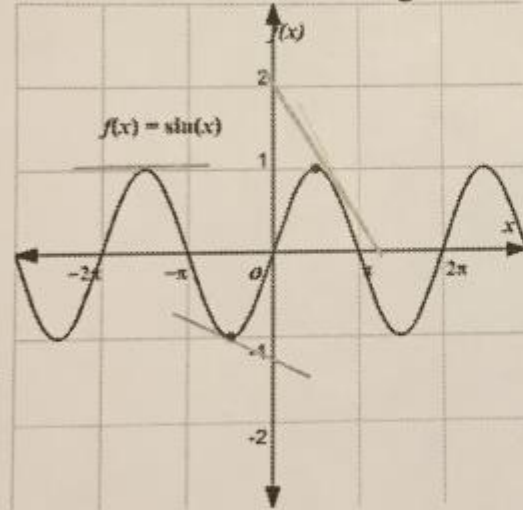
b.



Lesson 3: Secant Lines

Warm up:

Draw three different tangent lines to the function below:

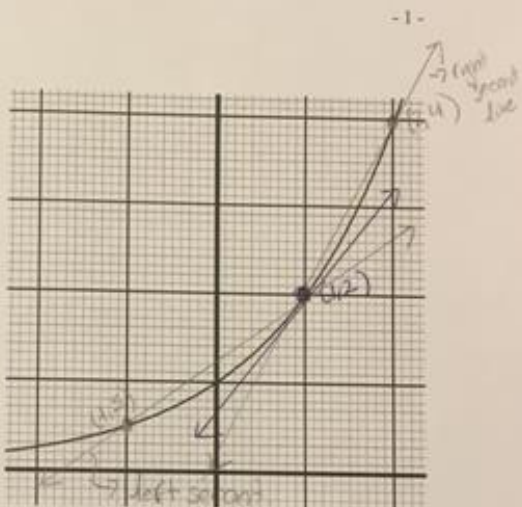


many solutions

Worksheet:

Calc 1 Worksheet 1

Approximating slopes of tangent lines
 The figure at right shows the graph of the function $f(x) = 2^x$. Using a straightedge, draw a tangent line as accurately as possible at the point $(1, 2)$. Then draw a secant line through the points $(1, 2)$ and $(2, 4)$. We will refer to that as the *right* secant line. Draw another secant line, this time through the points $(-1, .5)$ and $(1, 2)$. This is the *left* secant line. Answer the following questions.



1. What is the slope of the left secant line?

$$m_L = \frac{2 - .5}{1 - (-1)} = \frac{1.5}{2} = .75$$

2. Based on the graph, is the slope of the left secant line greater than, equal to, or less than the slope of the tangent line?

The slope of the left secant line is less than the slope of the tangent line.

3. What is the slope of the right secant line?

$$m_R = \frac{4 - 2}{2 - 1} = \frac{2}{1} = 2$$

4. Based on the graph, is the slope of the right secant line greater than, equal to, or less than the slope of the tangent line?

The slope of the right secant line is greater than the slope of the tangent line.

5. Using the results above, fill in the blanks with numerical values:

$$m_L < \text{slope of tangent line} < m_R$$

6. Based on the foregoing, what is your best numerical estimate of the tangent line, and how far off might this estimate be, at worst?

Students should give an answer between .75 and 2.

Lesson 4: Slope of a tangent line

Warm up:

Find the slope of the secant line that goes through the function at (-2,3) and (5,4).

$$m = \frac{4-3}{5-(-2)} = \frac{1}{7}$$

Worksheet:

Directions: Find the slope of the tangent line at the given points.

1. $f(x) = 6x + 3, x = 2$

$$\lim_{\Delta x \rightarrow 0} \frac{6(2+\Delta x) + 3 - [6(2) + 3]}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{12 + 6\Delta x + 3 - 15}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{15 + 6\Delta x - 15}{\Delta x}$$

$$\lim_{\Delta x \rightarrow 0} \frac{6\Delta x}{\Delta x} = 6$$

$$\boxed{m = 6}$$

2. $f(x) = \frac{1}{2}x - 4, x = -4$

$$\lim_{\Delta x \rightarrow 0} \frac{\frac{1}{2}(-4 + \Delta x) - 4 - (\frac{1}{2}(-4) - 4)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-2 + \frac{1}{2}\Delta x - 4 - (-2 - 4)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-6 + \frac{1}{2}\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{2}\Delta x}{\Delta x} = \frac{1}{2}$$

$m = \frac{1}{2}$

3. $f(x) = x^2 - 3, x = 5$

$$\lim_{\Delta x \rightarrow 0} \frac{(5 + \Delta x)^2 - 3 - (5^2 - 3)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{25 + 10\Delta x + \Delta x^2 - 3 - (25 - 3)}{\Delta x}$$

$$\lim_{\Delta x \rightarrow 0} \frac{25 + 10\Delta x + \Delta x^2 - 3 - 25 + 3}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{10\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} 10 + \Delta x$$

$$= 10 + 0 = 10$$

$m = 10$

Lesson 5: Extra examples for Lesson 4Example 1Find the slope of the tangent line of $f(x) = -\frac{3}{2}x + 6$ at $x = 2$.

$$m = \lim_{\Delta x \rightarrow 0} \frac{-\frac{3}{2}(2 + \Delta x) + 6 - (-\frac{3}{2}(2) + 6)}{\Delta x}$$

$$m = \lim_{\Delta x \rightarrow 0} \frac{-3 - \frac{3}{2}\Delta x + 6 - (-3 + 6)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3 - \frac{3}{2}\Delta x - 3}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-\frac{3}{2}\Delta x}{\Delta x} = -\frac{3}{2}$$

$$m = -\frac{3}{2}$$

Example 2: Find the slope of the tangent line of $f(x) = x^2 - 5x + 3$ at $x = 3$.

$$m = \lim_{\Delta x \rightarrow 0} \frac{(3 + \Delta x)^2 - 5(3 + \Delta x) + 3 - (3^2 - 5(3) + 3)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{9 + 6\Delta x + \Delta x^2 - 15 - 5\Delta x + 3 - 9 + 15 - 3}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} 1 + \Delta x = 1 + 0$$

$$m = 1$$

Example 3: Find the slope of the tangent line of $f(x) = x^3 - 1$ at $x = -2$.

$$\begin{aligned}
 m &= \lim_{\Delta x \rightarrow 0} \frac{(-2 + \Delta x)^3 - 1 - ((-2)^3 - 1)}{\Delta x} \\
 m &= \lim_{\Delta x \rightarrow 0} \frac{-8 + 12\Delta x - 6\Delta x^2 + \Delta x^3 - 1 - (-9)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\cancel{-8} + 12\Delta x - 6\Delta x^2 + \Delta x^3 \cancel{-1}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{12\Delta x - 6\Delta x^2 + \Delta x^3}{\Delta x} = \lim_{\Delta x \rightarrow 0} 12 - 6\Delta x + \Delta x^2 \\
 &= 12 - 6(0) + 0^2 = 12
 \end{aligned}$$

$m = 12$

Worksheet

Calculus I

Worksheet - Tangent Line Slope

Name Answer Key

Date _____

Use the limit definition to find the equation for the slope of any tangent line, the slope of the tangent line at the given value, and the equation of the tangent line at the given value of x .

1. $f(x) = x^2 + 1$ at $x = 2$

$m_{\text{tan}} = 2x$ $m = 4$ point = 5 equation = $y = 4x - 3$

$$m_{\text{tan}} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 + 1 - (x^2 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 \cancel{+ 1} - \cancel{x^2} - \cancel{1}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2x + \Delta x$$

$$m_{\text{tan}} = 2x$$

$$f(2) = 2^2 + 1 = 5$$

$$M_{\text{at } 2} = 2(2) = 4$$

$$y - y_1 = m(x - x_1)$$

$$y - 5 = 4(x - 2)$$

$$\begin{array}{r} y - 5 = 4x - 8 \\ \hline y = 4x - 3 \end{array}$$

$$y = 4x - 3$$

2. $f(x) = x^3 - 2$ at $x = 1$

$m_{\text{tan}} = 3x^2$ $m = 3$ point = $(1, -1)$ equation = $y = 3x - 4$

$$m_{\text{tan}} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - 2 - (x^3 - 2)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - \cancel{x^3} - \cancel{-2} + \cancel{2}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x} = \lim_{\Delta x \rightarrow 0} 3x^2 + 3x\Delta x + \Delta x^2$$

$$= 3x^2 + 3x(0) + (0)^2 = 3x^2$$

$m_{\text{at } 1} = 3(1)^2 = 3$

$f(1) = 1^3 - 2 = 1 - 2 = -1$

equation = $y - (-1) = 3(x - 1)$ \rightarrow $y = 3x - 4$
 $y + 1 = 3x - 3$

3. $f(x) = 3x^2 - 2x + 5$ at $x = 4$

$m_{\text{tan}} = 6x - 2$ $m = 22$ point = $(4, 45)$ equation = $y = 22x - 43$

$$m_{\text{tan}} = \lim_{\Delta x \rightarrow 0} \frac{3(x + \Delta x)^2 - 2(x + \Delta x) + 5 - (3x^2 - 2x + 5)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\cancel{3x^2} + 6x\Delta x + 3\Delta x^2 - \cancel{2x} - 2\Delta x + \cancel{5} - \cancel{3x^2} + \cancel{2x} - \cancel{5}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{6x\Delta x + 3\Delta x^2 - 2\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 6x + 3\Delta x - 2 = 6x - 2$$

$m_{\text{at } 4} = 6(4) - 2 = 24 - 2 = 22$

$f(4) = 3(4)^2 - 2(4) + 5 = 3(16) - 8 + 5 = 45$

eg = $y - 45 = 22(x - 4)$ \rightarrow $y = 22x - 43$
 $\frac{y - 45}{+45} = \frac{22x - 88}{+45}$

4. $f(x) = \frac{3}{x}$ at $x=3$

$m_{\text{tan}} = \frac{-3}{x^2} = m = \frac{-1}{3}$ point = $(3, 1)$ equation = $y = -\frac{1}{3}x + 2$

$$m_{\text{tan}} = \lim_{\Delta x \rightarrow 0} \frac{\frac{3}{x+\Delta x} - \frac{3}{x}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{3x - 3x - 3\Delta x}{x(x+\Delta x)}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-3\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-3}{x(x+\Delta x)} = \frac{-3}{x(x+0)} = \frac{-3}{x^2}$$

$$m_{\text{at } 3} = \frac{-3}{3^2} = \frac{-3}{9} = -\frac{1}{3}$$

$$f(3) = \frac{3}{3} = 1$$

Eq = $y - 1 = -\frac{1}{3}(x - 3)$

$$y - 1 = -\frac{1}{3}x + 1$$

$$\begin{array}{r} y - 1 \\ +1 \\ \hline y = -\frac{1}{3}x + 2 \end{array}$$

5. $f(x) = \frac{x}{x-2}$ at $x=3$

$m_{\text{tan}} = \frac{-2}{(x-2)^2} = m = \frac{-2}{1}$ point = $(3, 3)$ equation = $y = -2x + 9$

$$m_{\text{tan}} = \lim_{\Delta x \rightarrow 0} \frac{\frac{x+\Delta x}{x+\Delta x-2} - \frac{x}{x-2}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{x^2 - 2x + x\Delta x - 2\Delta x - (x^2 + x\Delta x - 2x - 2\Delta x)}{(x-2)(x+\Delta x-2)}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\cancel{x^2} - 2x + \cancel{x\Delta x} - 2\Delta x - \cancel{x^2} - \cancel{x\Delta x} + 2x + 2\Delta x}{(x-2)(x+\Delta x-2)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-2\Delta x}{(x-2)(x+\Delta x-2)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-2}{(x-2)(x+\Delta x-2)} = \frac{-2}{(x-2)(x+0-2)} = \frac{-2}{(x-2)^2}$$

$$m_{\text{at } 3} = \frac{-2}{(3-2)^2} = \frac{-2}{1} = -2$$

$$f(3) = \frac{3}{3-2} = \frac{3}{1} = 3$$

Eq = $y - 3 = -2(x - 3)$

$$y - 3 = -2x + 6$$

$$\begin{array}{r} y - 3 \\ +3 \\ \hline y = -2x + 9 \end{array}$$

Lesson 6: Definition of a derivativeWarm up:Find the slope of the tangent line of $f(x) = x^2 - 5x + 3$ at $x = c$.

$$\begin{aligned}
 m &= \lim_{\Delta x \rightarrow 0} \frac{(c+\Delta x)^2 - 5(c+\Delta x) + 3 - (c^2 - 5c + 3)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\cancel{c^2} + 2c\Delta x + \cancel{\Delta x^2} - 5c - 5\Delta x + 3 - \cancel{c^2} + 5c - 3}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2c\Delta x + \cancel{\Delta x^2} - 5\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2c + \Delta x - 5 \\
 &= \boxed{2c - 5}
 \end{aligned}$$

Notes:Example 2: Find the derivative of $f(x) = x^2$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\cancel{x^2} + 2x\Delta x + \cancel{\Delta x^2} - \cancel{x^2}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + \cancel{\Delta x^2}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} 2x + \Delta x
 \end{aligned}$$

$$\boxed{f'(x) = 2x}$$

Lesson 7: Extra examples for Lesson 6Examples:**Example 1:** Find the derivative of the function $f(x) = x^3 - 2x^2 + 9$. What is $f'(1)$?

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^3 - 2(x+\Delta x)^2 + 9 - (x^3 - 2x^2 + 9)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 2x^2 - 4x\Delta x - 2\Delta x^2 + 9 - x^3 + 2x^2 - 9}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 4x\Delta x - 2\Delta x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} 3x^2 + 3x\Delta x + \Delta x^2 - 4x
 \end{aligned}$$

$$f'(x) = 3x^2 - 4x \quad f'(1) = 3(1)^2 - 4(1) = 3 - 4 = \boxed{-1}$$

Example 2: Find the derivative of the function $f(x) = \sqrt{5x-3}$. Write the equation of the tangent line at the point $(1, \sqrt{2})$.

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{5(x+\Delta x)-3} - \sqrt{5x-3}}{\Delta x} \cdot \frac{\sqrt{5(x+\Delta x)-3} + \sqrt{5x-3}}{\sqrt{5(x+\Delta x)-3} + \sqrt{5x-3}} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{5(x+\Delta x)-3 - (5x-3)}{\Delta x (\sqrt{5(x+\Delta x)-3} + \sqrt{5x-3})} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{5x + 5\Delta x - 3 - 5x + 3}{\Delta x (\sqrt{5(x+\Delta x)-3} + \sqrt{5x-3})} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{5}{\sqrt{5x+5\Delta x-3} + \sqrt{5x-3}}
 \end{aligned}$$

$$f'(x) = \frac{5}{2\sqrt{5x-3}}$$

$$f'(1) = \frac{5}{2\sqrt{5(1)-3}} = \frac{5}{2\sqrt{2}}$$

side bar

$$-\frac{5}{2\sqrt{2}} + \frac{\sqrt{2} \cdot 2\sqrt{2}}{1 \cdot 2\sqrt{2}}$$

$$\frac{-5}{2\sqrt{2}} + \frac{4}{2\sqrt{2}} = \frac{-1}{2\sqrt{2}}$$

$$\begin{aligned}
 \text{Eq} &= y - y_1 = f'(x_1)(x - x_1) \\
 y - \sqrt{2} &= \frac{5}{2\sqrt{2}}(x - 1) \\
 y - \sqrt{2} &= \frac{5}{2\sqrt{2}}x - \frac{5}{2\sqrt{2}} \\
 &\quad +\sqrt{2} \qquad \qquad +\sqrt{2} \\
 \hline
 \boxed{y} &= \frac{5}{2\sqrt{2}}x - \frac{1}{2\sqrt{2}}
 \end{aligned}$$

Worksheets:

1. <http://cdn.kutasoftware.com/Worksheets/Calc/03%20-%20Definition%20of%20the%20Derivative.pdf>

Kuta Software - Infinite Calculus

Name _____

Definition of the Derivative

Date _____ Period _____

Use the definition of the derivative to find the derivative of each function with respect to x .

1) $y = -2x + 5$

$$\frac{dy}{dx} = -2$$

2) $f(x) = -4x - 2$

$$f'(x) = -4$$

3) $y = 4x^2 + 1$

$$\frac{dy}{dx} = 8x$$

4) $f(x) = -3x^2 + 4$

$$f'(x) = -6x$$

5) $y = -4x^2 - 5x - 2$

$$\frac{dy}{dx} = -8x - 5$$

6) $y = 3x^2 + 3x + 3$

$$\frac{dy}{dx} = 6x + 3$$

7) $y = \sqrt{-3x - 5}$

$$\frac{dy}{dx} = -\frac{3}{2\sqrt{-3x - 5}}$$

8) $f(x) = \sqrt{4x - 5}$

$$f'(x) = \frac{2}{\sqrt{4x - 5}}$$

9) $y = \frac{1}{x + 2}$

$$\frac{dy}{dx} = -\frac{1}{x^2 + 4x + 4}$$

10) $f(x) = -\frac{2}{2x - 1}$

$$f'(x) = \frac{4}{4x^2 - 4x + 1}$$

Critical thinking question:

11) Use the definition of the derivative to show that $f'(0)$ does not exist where $f(x) = |x|$.

Using 0 in the definition, we have $\lim_{h \rightarrow 0} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h}$ which does not exist because the left-handed and right-handed limits are different.

Odd Numbered Solutions to worksheet 1

① $y = -2x + 5$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{-2(x+\Delta x) + 5 - (-2x + 5)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-2x - 2\Delta x + 5 + 2x - 5}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-2\Delta x}{\Delta x} = -2$$

$$\boxed{\frac{dy}{dx} = -2}$$

③ $y = 4x^2 + 1$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{4(x+\Delta x)^2 + 1 - (4x^2 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{4x^2 + 8x\Delta x + 4\Delta x^2 + 1 - 4x^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{8x\Delta x + 4\Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} 8x + 4\Delta x = 8x + 4(0)$$

$$\boxed{\frac{dy}{dx} = 8x}$$

⑤ $y = -4x^2 - 5x - 2$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{-4(x+\Delta x)^2 - 5(x+\Delta x) - 2 - (-4x^2 - 5x - 2)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-4x^2 - 8x\Delta x - 4\Delta x^2 - 5x - 5\Delta x - 2 + 4x^2 + 5x + 2}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-8x\Delta x - 4\Delta x^2 - 5\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} -8x - 4\Delta x - 5$$

$$\boxed{\frac{dy}{dx} = -8x - 5}$$

⑦ $y = \sqrt{-3x - 5}$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{-3(x+\Delta x) - 5} - \sqrt{-3x - 5}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{-3x - 3\Delta x - 5} - \sqrt{-3x - 5}}{\Delta x} \cdot \frac{\sqrt{-3x - 3\Delta x - 5} + \sqrt{-3x - 5}}{\sqrt{-3x - 3\Delta x - 5} + \sqrt{-3x - 5}}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-3\Delta x - (\cancel{-3x - 5})}{\Delta x (\sqrt{-3x - 3\Delta x - 5} + \sqrt{-3x - 5})}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-3}{\sqrt{-3x - 3\Delta x - 5} + \sqrt{-3x - 5}}$$

$$\boxed{\frac{dy}{dx} = \frac{-3}{2\sqrt{-3x - 5}}}$$

⑨ $y = \frac{1}{x+2}$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x+\Delta x+2} - \frac{1}{x+2}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{x+2 - (x+\Delta x+2)}{(x+2)(x+\Delta x+2)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{x+2 - x - \Delta x - 2}{(x+2)(x+\Delta x+2)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-1}{(x+2)(x+\Delta x+2)}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{(x+2)^2}}$$

⑪ $\lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h}$

$$= \lim_{h \rightarrow 0} \frac{|h|}{h}$$

This does not exist because the left-handed and right-handed limits are different.

2. http://www2.johnabbott.qc.ca/~olivier.dubois/103/103_worksheet_defn_derivative.pdf

1. For each function given below, calculate the derivative at a point $f'(a)$ using the limit definition.

(a) $f(x) = 2x^2 - 3x$ $f'(0) = ?$

(b) $f(x) = \sqrt{2x+1}$ $f'(4) = ?$

(c) $f(x) = \frac{1}{x-2}$ $f'(3) = ?$

1.
 (a) $f(x) = 2x^2 - 3x$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{2(x+\Delta x)^2 - 3(x+\Delta x) - (2x^2 - 3x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2(x^2 + 2x\Delta x + \Delta x^2) - 3(x+\Delta x) - (2x^2 - 3x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\cancel{2x^2} + 4x\Delta x + 2\Delta x^2 - \cancel{3x} - 3\Delta x - \cancel{2x^2} + \cancel{3x}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{4x\Delta x + 2\Delta x^2 - 3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} 4x + 2\Delta x - 3 = 4x + 2(0) - 3$$

$$f'(x) = 4x - 3$$

$$\therefore f'(0) = 4(0) - 3 = \boxed{-3}$$

(b) $f(x) = \sqrt{2x+1}$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{2(x+\Delta x)+1} - \sqrt{2x+1}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{2x+2\Delta x+1} - \sqrt{2x+1}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{2x+2\Delta x+1} - \sqrt{2x+1}}{\Delta x} \cdot \frac{\sqrt{2x+2\Delta x+1} + \sqrt{2x+1}}{\sqrt{2x+2\Delta x+1} + \sqrt{2x+1}} = \lim_{\Delta x \rightarrow 0} \frac{2x+2\Delta x+1 - 2x-1}{\Delta x (\sqrt{2x+2\Delta x+1} + \sqrt{2x+1})}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2\Delta x}{\Delta x (\sqrt{2x+2\Delta x+1} + \sqrt{2x+1})} = \frac{2}{\sqrt{2x+1} + \sqrt{2x+1}} = \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

$$f'(4) = \frac{1}{\sqrt{2(4)+1}} = \frac{1}{\sqrt{9}} = \boxed{\frac{1}{3}}$$

(c) $f(x) = \frac{1}{x-2}$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x+\Delta x)-2} - \frac{1}{x-2}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x+\Delta x-2} - \frac{1}{x-2}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\frac{x-2 - (x+\Delta x-2)}{(x+\Delta x-2)(x-2)}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x-2 - x-\Delta x+2}{(x+\Delta x-2)(x-2)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-\Delta x}{(x+\Delta x-2)(x-2)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-1}{(x+\Delta x-2)(x-2)}$$

$$f'(x) = \frac{-1}{(x-2)^2}$$

$$f'(3) = \frac{-1}{(3-2)^2} = \frac{-1}{1} = \boxed{-1}$$

2. For each function $f(x)$ given below, find the **general derivative** $f'(x)$ as a new function by using the limit definition.

(a) $f(x) = \sqrt{x-4}$ $f'(x) = ?$

(b) $f(x) = -x^3$ $f'(x) = ?$

(c) $f(x) = \frac{x}{x+1}$ $f'(x) = ?$

(d) $f(x) = \frac{1}{\sqrt{x}}$ $f'(x) = ?$

2.
a) $f(x) = \sqrt{x-4}$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x+\Delta x-4} - \sqrt{x-4}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x-4} + \sqrt{x-4}}{\sqrt{x+\Delta x-4} + \sqrt{x-4}}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\cancel{x} + \cancel{\Delta x} - \cancel{x} - \cancel{\Delta x}}{\Delta x (\sqrt{x+\Delta x-4} + \sqrt{x-4})} = \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x+\Delta x-4} + \sqrt{x-4}}$$

$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

b) $f(x) = -x^3$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{-(x+\Delta x)^3 - (-x^3)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-\cancel{x^3} - 3x^2\Delta x - 3x(\Delta x)^2 - (\Delta x)^3 + \cancel{x^3}}{\Delta x}$$

$$\lim_{\Delta x \rightarrow 0} \frac{-3x^2 - 3x\Delta x - (\Delta x)^2}{1} = -3x^2 - 3x(0) - (0)^2$$

$$f'(x) = -3x^2$$

$$c) f(x) = \frac{x}{x+1}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{x+\Delta x}{x+\Delta x+1} - \frac{x}{x+1}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{x+\Delta x}{x+\Delta x+1} - \frac{x}{x+1}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta x}{(x+\Delta x+1)(x+1)}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{1}{(x+\Delta x+1)(x+1)}$$

$$f'(x) = \frac{1}{(x+1)^2}$$

$$d) f(x) = \frac{1}{\sqrt{x}}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{\sqrt{x+\Delta x}} - \frac{1}{\sqrt{x}}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{\sqrt{x} - \sqrt{x+\Delta x}}{\sqrt{x}\sqrt{x+\Delta x}}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x} - \sqrt{x+\Delta x}}{\Delta x \sqrt{x}\sqrt{x+\Delta x}}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{x - x - \Delta x}{\Delta x (\sqrt{x}\sqrt{x+\Delta x})(\sqrt{x} + \sqrt{x+\Delta x})} = \lim_{\Delta x \rightarrow 0} \frac{-1}{(\sqrt{x}\sqrt{x+\Delta x})(\sqrt{x} + \sqrt{x+\Delta x})}$$

$$= \frac{-1}{(\sqrt{x}\sqrt{x+0})(\sqrt{x} + \sqrt{x+0})} = \frac{-1}{(\sqrt{x}\sqrt{x})(2\sqrt{x})} = \frac{-1}{x(2\sqrt{x})} \quad f'(x) = \frac{-1}{2x^{3/2}}$$

3. For each function $f(x)$ given below, find the equation of the tangent line at the indicated point.

(a) $f(x) = x - x^2$ at $(2, -2)$

(b) $f(x) = 1 - 3x^2$ at $(0, 1)$

(c) $f(x) = \frac{1}{2x}$ at $x = 1$

(d) $f(x) = x + \sqrt{x}$ at $x = 1$

3.

a) $f(x) = x - x^2, (2, -2)$

$$m(2) = \lim_{\Delta x \rightarrow 0} \frac{(2+\Delta x) - (2+\Delta x)^2 - f(2)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2+\Delta x - (4+4\Delta x + \Delta x^2) - (2-2^2)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2+\Delta x - 4 - 4\Delta x + \Delta x^2 - 2 + 4}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-3\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} -3 + \Delta x$$

$$m = -3$$

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = -3(x - 2)$$

$$y + 2 = -3x + 6$$

$$y = -3x + 4$$

b) $f(x) = 1 - 3x^2$ at $(0, 1)$

$$m(0) = \lim_{\Delta x \rightarrow 0} \frac{1 - 3(0 + \Delta x)^2 - (1 - 3(0)^2)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{1 - 3(\Delta x^2) - 1}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-3\Delta x^2}{\Delta x}$$

$$\lim_{\Delta x \rightarrow 0} -3\Delta x = -3(0) = 0$$

$$\therefore m = 0$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 0(x - 0)$$

$$y = 1$$

c) $f(x) = \frac{1}{2x}$, at $x = 1$

$$m(1) = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{2(1+\Delta x)} - \frac{1}{2(1)}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{2} \left(\frac{1}{1+\Delta x} - 1 \right)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2 - (2 + 2\Delta x)}{2(2 + 2\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2 - 2 - 2\Delta x}{4 + 4\Delta x} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-2}{4 + 4\Delta x} = \frac{-2}{4 + 4(0)} = \frac{-2}{4} = \frac{-1}{2}$$

$$f(1) = \frac{1}{2(1)} = \frac{1}{2}$$

$$y - \frac{1}{2} = -\frac{1}{2}(x - 1)$$

$$y - \frac{1}{2} = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + 1$$

d) $f(x) = x + \sqrt{x}$, at $x=1$

$$m \text{ at } x=1 = \lim_{\Delta x \rightarrow 0} \frac{(1+\Delta x) + \sqrt{1+\Delta x} - (1 + \sqrt{1})}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(1+\Delta x) + \sqrt{1+\Delta x} - (1 + \sqrt{1})}{\Delta x} \cdot \frac{(1+\Delta x) - \sqrt{1+\Delta x}}{(1+\Delta x) - \sqrt{1+\Delta x}}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(1+\Delta x)(1+\Delta x) - (1+\Delta x)}{\Delta x (-1+\Delta x - \sqrt{1+\Delta x})} = \lim_{\Delta x \rightarrow 0} \frac{1 - 2\Delta x + \Delta x^2 - 1 - \Delta x}{\Delta x (-1+\Delta x - \sqrt{1+\Delta x})}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-3\Delta x + \Delta x^2}{\Delta x (-1+\Delta x - \sqrt{1+\Delta x})} = \lim_{\Delta x \rightarrow 0} \frac{-3 + \Delta x}{-1+\Delta x - \sqrt{1+\Delta x}} = \frac{-3}{-1-\sqrt{1}} = \frac{-3}{-2}$$

$m = 3/2$

$$f(1) = 1 + \sqrt{1} = 1 + 1 = 2$$

$$y - 2 = 3/2 (x - 1)$$

$$y - 2 = 3/2 x - 3/2$$

$$\begin{array}{r} y - 2 \\ + 2 \\ \hline y = 3/2 x + 1/2 \end{array}$$

Lesson 8: Quiz

Quiz:

Directions: For the following functions, find the derivative of the function, and create the equation of its tangent line at the given point.

1. $f(x) = 5x + 3$, $x = 4$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{5(x+\Delta x) + 3 - (5x+3)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{5x + 5\Delta x + 3 - 5x - 3}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{5\Delta x}{\Delta x} = 5$$

$f'(4) = 5$

$f(4) = 5(4) + 3 = 23$

EQ = $y - 23 = 5(x - 4)$

$$\begin{array}{r} y - 23 = 5x - 20 \\ + 23 \quad + 23 \\ \hline \end{array}$$

$y = 5x + 3$

2. $f(x) = x^2 - 3x + 5, x = 1$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^2 - 3(x+\Delta x) + 5 - (x^2 - 3x + 5)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 3x - 3\Delta x + 5 - x^2 + 3x - 5}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + \Delta x^2 - 3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} 2x + \Delta x - 3 \quad f'(x) = 2x - 3$$

$f(1) = 1^2 - 3(1) + 5 = 3$

$f'(1) = 2(1) - 3 = -1$

EQ: $y - 3 = -1(x - 1)$

$$\begin{array}{r} y - 3 = -x + 1 \\ +3 \quad +3 \\ \hline y = -x + 4 \end{array}$$

3. $f(x) = \sqrt{x-5}, x = 9$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x+\Delta x-5} - \sqrt{x-5}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x-5} + \sqrt{x-5}}{\sqrt{x+\Delta x-5} + \sqrt{x-5}}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x-5) - (x-5)}{\Delta x(\sqrt{x+\Delta x-5} + \sqrt{x-5})}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{x+\Delta x-5 - x+5}{\Delta x(\sqrt{x+\Delta x-5} + \sqrt{x-5})}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x-5} + \sqrt{x-5})}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x+\Delta x-5} + \sqrt{x-5}}$$

$f'(x) = \frac{1}{2\sqrt{x-5}}$

$f(9) = \sqrt{9-5} = \sqrt{4} = 2$

$f'(9) = \frac{1}{2\sqrt{9-5}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

EQ: $y - 2 = \frac{1}{4}(x - 9)$

$$\begin{array}{r} y - 2 = \frac{1}{4}x - \frac{9}{4} \\ +2 \quad +2 \\ \hline y = \frac{1}{4}x - \frac{1}{4} \end{array}$$

Side bar

$$-\frac{9}{4} + \frac{2 \cdot 4}{1 \cdot 4}$$

$$-\frac{9}{4} + \frac{8}{4} = -\frac{1}{4}$$