# Gifted Students in Mathematics: A Problem Solving Approach to Enrichment 

## by

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This thesis is dedicated to my mom for her countless hours of proofreading, my dad for truly teaching me what it means to be a hard worker and my husband for being so supportive during my entire grad school experience.


#### Abstract

In the United States mathematically gifted students frequently go unnoticed and most often receive the same education as their at-level peers (Ysseldyke, Tardrew, Betts, Thill, Hannigan, 2004). There is very limited funding available for gifted students and the identification and classification varies by state, often being decided by school district (National Association, 2014). Gifted American students are exposed to less challenging problems than those in other countries and, as a result, are falling behind in academic performance (Ross, 1993). This curriculum is designed to supplement the existing Geometry curriculum by offering eight unique, challenging problems that can be used for gifted students in either heterogeneous or homogenous groups. The purpose of this curriculum project is to provide problems to support teachers in their instruction of gifted students so that these students are challenged by curriculum in a rigorous unit of mathematical material.


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## Chapter 1: Introduction

The United States (US) Department of Education defined gifted students to be "children and youth with outstanding talent who perform or show the potential for performing at remarkably high levels of accomplishment when compared with others of their age, experience, or environment" (Ross, 1993). In the United States, the classification and treatment of gifted students is at the discretion of each state, with no federal mandates or funding (National Association, 2014). In the 2013-2014 school year fourteen states did not provide any funding for gifted students and twenty-two states left all decisions about gifted students to the discretion of the school district (National Association, 2014). New York State defines gifted students as:
those who show evidence of high performances capability and exceptional potential in area such as general intellectual ability, special academic aptitude and outstanding ability in visual and performing arts. Such definition shall include those pupils who require educational programs or services beyond those normally provided by the regular school program in order to realize their full potential (New, 2009).

New York State does provide funding for gifted education and requires screening in school, including testing to determine whether a student fits under the definition of gifted, however the "the definition permits each district to determine the kinds of data to be used and procedures to be followed in identifying gifted students" (New, 2009).

Because of the broad definition and procedure for determining giftedness in the United States, gifted students are most often taught in the general education classroom and the majority of the time, the education they receive is not any different than their ator below- level peers. Because these students have no problem succeeding in the general
education classes, and often exceed expectations, the pitfalls of this method of educating them often go unnoticed. Research shows, however, that gifted students should receive an education that caters to their learning styles and learning needs (Ysseldyke, Tardrew, Betts, Thill, Hannigan, 2004). Gifted students often receive the exact same assignments as their peers, not being challenged in the way that they should and instead, "gifted students should be provided advanced learning activities. This is especially important in the domain of mathematics" (Ysseldyke, Tardrew, Betts, Thill, Hannigan, 2004, p. 295). Ibata-Arens (2012, p. 3) compared the education of gifted students in various countries in Asia to the gifted education in the United States. She claims that "the needs of U.S. gifted and talented students (are) not being met with the current practice."

The curriculum created in this study is designed for high school sophomores and aims to provide varied curriculum support for teachers of gifted students in heterogeneous classrooms. Wai, Lubinski, Benbow, \& Steiger (2010) discuss two main ways to challenge gifted students: exposure to many different topics and an in-depth look at curriculum. Although acceleration is one way to ensure that students are exposed to a large variety of mathematical content, enrichment is just as, if not more, important. "Speeding up learning and not going deeper or making it more complex would seem empty" (Wai, Lubinski, Benbow, \& Steiger, 2010, p. 860). The curriculum is designed to supplement existing Geometry curriculum, allowing students to take a more in-depth look at the content, by designing challenging problems that require higher-level thinking. Although it is designed to be implemented in a separate lab class for gifted students, it can easily be used as a modification for a heterogeneous class of general education students that contains gifted students.

The purpose of this curriculum project is to provide problems to support teachers in their instruction of gifted students so that these students are challenged by curriculum in a rigorous unit of mathematical material, where students work to solve complex mathematical problems in which the method of solving the problems is not obvious, all of the necessary information is not just given to the students and where there is more than one correct answer and correct method of solving the problem.

## Chapter 2: Survey of Literature

## Introduction

Much research has been done in the area of gifted students in mathematics (IbataArens, 2012, Winner, 1996, Ross, 1993). Researchers have analyzed how the United States compares to other counties in gifted education students', teachers' and parents' perceptions on how to best help gifted students succeed in school, and the success of gifted students beyond high school (Ibata-Arens, 2012). With the ever growing need for highly qualified workers in science, technology, engineering, and mathematics (acronym: STEM) fields, much research also focuses on the lack of students, particularly those gifted in mathematics, pursuing degrees in STEM fields and how educators can combat that issue (Wai, Lubinski, Benbow, \& Steiger, 2010).

## Comparison of the United States and Other Nations

Ibata-Arens (2012) found the education of gifted students in the United States to be lacking, particularly in the area of mathematics. Ellen Winner (1996) found that "only 2 cents of every $\$ 100$ spent on precollegiate education in 1990 went to gifted programs" (Winner, 1996). Unfortunately, with No Child Left Behind (NCLB) and the implementation of the Common Core State Standards (CCSS) there have been no significant changes in gifted education since the 1990s. Ibata-Arens referred to a report published in 1993 called "National Excellence: a Case for Developing America's Talent," which compared the education of gifted students in the United States to those in China, Taiwan and Japan, and states that nearly two decades later, nothing has changed (IbataArens, 2012). O’Connell Ross et al., (1993) claims that although the insufficiency of
gifted education in the United States is a known problem, not much has been done to solve it. Most gifted students are educated in general education classrooms with no particular attention paid to their needs. In comparison to other countries, the authors claim that
"Most top students in the United States are offered a less rigorous curriculum, read fewer demanding books, complete less homework, and enter the work force or postsecondary education less well prepared than top students in many other industrialized countries. These deficiencies are particularly apparent in the areas of mathematics and science" (Ross, 1993)

Numerous studies have shown that these problems in American education lead to low test results compared to other countries. In 2011 the Trends in Mathematics and Science Study (acronym: TIMSS) ranked the United States as eleventh out of seventy in fourth grade mathematics and ninth out of seventy in eighth grade mathematics (Mullis, Martin, Foy, \& Arora, 2012). Arguments can be made that this study is not a fair comparison since the United States often has a higher percentage of their students taking higher levels of mathematics, but even the top American students fall short of the top students in other countries. Data from Advanced Placement tests, which have remained relatively consistent over time in terms of difficulty, have shown the shortcomings of the highest achieving American students. When comparing the top one percent of American students to the top one percent of students from thirteen other nations, American students ranked last in algebra and twelfth in geometry and calculus (Ross, 1993). As a more recent study, the National Math and Science Initiative (2014) also points to disturbing data. In a ranking constructed by the Organization for Economic Cooperation and

Development, the United States ranked $27^{\text {th }}$ out of $34^{\text {th }}$ in mathematics, while the World Economic Forum ranked the United States as $48^{\text {th }}$ in the world in quality math instruction (National, 2014).

It was found that American students do not spend as much time in school, do not get as much homework and are not exposed to challenging problems as often as students in other countries (Ross, 1993). Although teachers do not have control over the length of the school day, they do have control over the other two discrepancies. Teachers can change the amount of homework that they assign, but there is conflicting research about the true benefits of homework (Marzano \& Pickering, 2007). Teachers should instead focus on exposing their students to challenging problems. Shoenfeld (1992) argues that often times teachers misuse the term problem solving, claiming that routine practice questions are problem solving. He instead claims that true problem solving involves questions that are difficult and perplexing. Sharma (2013) adds to Shoenfeld's notion that problem solving should be non-routine, arguing that effective problems have more than one method of solving them and often more than one solution. In order for students to truly learn from problem solving, they should not be given all relevant information and should, instead, learn to ask the right questions to get all the information that they need (Sharma, 2013). Every student should be given a chance to work on problems that require higher-level thinking, but this is especially important for gifted students.

## The Lack of American Students Choosing STEM Fields

While the disparities between students in the United States and other countries are disturbing, an even bigger problem is the lack of American students pursuing STEM
careers. While students in countries throughout the world continue to increase their focus on STEM fields, the United States is falling behind. With the aforementioned troubles in precollegiate education of gifted students in the United States, particularly in science and mathematics, it's no wonder that American students do not choose STEM majors in postsecondary school (National 2014). In other countries, mathematically gifted students are making breakthroughs in STEM fields, but in the United States, even when students do major in a math field, it is often business related (Nokelainen, Tirri, Campbell, 2004). Because of the poor quality of mathematics education and the lack of students pursuing STEM degrees, the National Math and Science Initiative claims the United States could be short as many as three million highly qualified workers by 2018 (National, 2014).

Jonathan Wai, David Lubinski, Camilla Benbow and James Steiger (2010) studied the precollegiate education and experiences of individuals who became successful in STEM fields. They found a direct correlation between students' exposure to STEM fields in primary and secondary school and their success in STEM career fields. "The number of precollegiate STEM educational opportunities beyond the norm that mathematically talented adolescents experience is related to subsequent STEM accomplishments achieved over 20 years later" (Wai, Lubinski, Benbow, \& Steiger, 2010, p. 865). The authors referred to these educational opportunities as "educational dose," and found that a higher mathematical educational dose in secondary schools led to postcollegiate success in STEM fields.

The lack of students pursuing degrees and careers in STEM fields is problematic. American STEM companies are forced to hire individuals from other countries, not
because they are necessarily more talented, but because they are better educated and the U.S. has a shortage of qualified individuals. Clearly, something needs to be done to combat this dilemma, but with much educational focus on the Common Core Curriculum, gifted students will most likely continue to see a lack of funding and a lack of government mandates. It is the job of teachers to ensure that all students are challenged. The curriculum in this paper is designed to offer free unique learning opportunities for gifted students in mathematics, exposing them to rich and new material, and showing them the usefulness of mathematics.

## Women Are Underrepresented in STEM Fields

Among those that do choose STEM fields, women are underrepresented. Margaret O'Shea, Nancy Heilbronner, and Sally Reis (2010), studied this phenomenon and found that in 2005, women received over half of all bachelor's degrees (58\%) and master's degrees (59\%), but they only received $46 \%$ of undergraduate degrees in mathematics and even less in computer science, physics and engineering ( $O^{\prime}$ 'Shea, Heilbronner, \&Reis, 2010). Additionally, the National Math and Science Initiative recognizes that women hold $48 \%$ of all jobs, but only $24 \%$ of STEM jobs (National, 2014).

## Characteristics of Highly Effective Teachers of Gifted Students

In order to offer a rich, deep and unique curriculum to gifted students in mathematics, teachers need to be well educated (Sigle, DeVia Rubenstein, \& Mitchell, 2013). Not only should they be knowledgeable about the content that they teach, they
also need to ensure that they understand the pedagogy behind successfully teaching and challenging gifted students.

Research states that there are three main characteristics of teachers of gifted students (Sigle, DeVia Rubenstein, \& Mitchell, 2013). First and foremost, teachers must have an in-depth understanding of the content that they teach. Although we know that content knowledge is important for teachers of all students, it is especially important for teachers of the gifted. Teachers need to be able to answer student questions and challenge learners with examples beyond the level of the textbook. Del Sigle, Lisa DeVia Rubenstein, and Melissa S. Mitchell (2013) interviewed gifted students enrolled in college honors programs and found that students desired teachers who had a superior knowledge of the content, were able to apply it to genuine real-world situations, were "lifelong learners" and had enthusiasm for the subject matter that they taught (Siegle, DeVia Rubenstein, \& Mitchell, 2013, p. 37).

The second characteristic that gifted students valued in their teachers was a caring attitude (Sigle, DeVia Rubenstein, \& Mitchell, 2013). Students appreciated teachers who took an interest in them and cared about their being successful. Genuine care and concern for the students is often more important than content knowledge or pedagogy itself (Siegle, DeVia Rubenstein, \& Mitchell, 2013).

Finally, students cared about having teachers who understood pedagogy. They found that it did not matter which teaching style their teacher used, as long as he or she was good at that style of teaching. They also preferred that their teachers had high expectations for them. If expectations were low, students would do "enough" to get by,
but if expectations were high, students would rise to the challenge and would, in the end, learn more (Siegle, DeVia Rubenstein, \& Mitchell, 2013).

## Using Problem Solving to Challenge Gifted Students

Research on gifted education in mathematics demonstrates the need for students to be exposed to a variety of problems, particularly challenging problems that go beyond the normal curriculum (Ysseldyke, Tardrew, Betts, Thill, \& Hannigan, 2004, p. 293). John Threlfall and Melanie Hargreaves (2008) discuss the differences between gifted mathematicians and their average peers. They found that gifted students "have a broader and more inter-connected knowledge base, are quicker at solving problems, while spending more time planning...are more flexible in their use of strategies, [and] prefer complex, challenging problems" (Threlfall \& Hargreaves, 2008, p. 84). We would be doing a disservice to our students if we ignored these skills and desires.

Yogesh Sharma (2013) studied how gifted students show creativity in mathematics. He found that they are able to make up their own problems, come up with more than one solution to a problem, can see beyond previous methods to find new ways of solving a problem, and can identify missing information in a problem and ask the right questions to get the missing information (Sharma, 2013, p. 16). Sharma (2013) encourages teachers of gifted students to find ways to cater to these strengths. Providing challenging mathematical problems with potential for more than one correct answer with solutions or procedures that are not obvious, will bring out the best in mathematically gifted students.

## Group work in Problem Solving

Group work has recently become an integral part of mathematical pedagogy and may be particularly useful when working with gifted students. Students working with their peers tend to critique the reasoning of others, argue the validity of their answers and defend their own work. Working in groups is particularly useful in avoiding fixation on one method of solving a problem. As students discuss their approaches, they gain insight into new ways to approach a problem. When students work together toward a collective goal, their feelings about mathematics tend to become more positive. Edna Leticia Hernandez Gárduñzo (2001) argues that in group work, encouragement, discussion and support from peers fosters a more positive attitude about mathematics, which in turn, creates students, particularly females, more interested in pursuing mathematics degrees and careers (Hernandez Garduno, 2001). Since the goal of this curriculum is to not only challenge gifted learners, but also ensure that these students enjoy mathematics, many of the lessons will include group work (Higgens, 1994).

In order to ensure that group work is effective in our classrooms, we must look at a number of factors. In particular, we need to look at the effectiveness of the communication of the group. Much research claims that the effectiveness of group talk does not come automatically, but instead needs to be taught to students (Higgens, 1994).

Joanna Higgens (1994) agrees that group work is essential to mathematical success in the classroom and quotes the New Zealand curriculum, which states that children should be working in groups and discussing mathematics. She believes, however, that we need to look closely at how these groups work. Higgens realizes that simply placing students in groups and having them work on mathematics is not enough to
truly be effective. She argues that the teacher plays a vital role in small group work and needs to be aware of how the groups are working and be ready to intervene when necessary. If a particular group is not thinking mathematically as much as the teacher would like them to, he or she should ask them questions or ask them to explain each other's thinking (Higgens, 1994). Additionally, the teacher serves as a role model for how the students should be interacting with each other. "The way that the teacher models instructions of small-group activities is the key to the occurrence of benefits such as peer assistance...The children learn what to say and do in the questioner role from the teacher model" (Higgens, 1994, p. 341). Finally, Higgens notes that the nature of the problem plays a role in the effectiveness of group work, claiming that tasks need to be well connected to the lessons and sequential. Schoenfeld (1992) also claims that students learn a lot about mathematics from their teachers. Schoenfeld argues that the typical way that students see "problem-solving" in the classroom is that they are given a problem that has one solution and the way to get the solution is the method students learned in class. He claims that by the way mathematics is typically taught, "students learn that answers and methods to problems will be provided to them; the students are not expected to figure out the methods by themselves. Over time most students come to accept their passive role, and to think of mathematics as "handed down" by experts for them to memorize" (Shoenfeld, 1992, p. 27).

## Conclusion

The current method of educating gifted students in the United States is problematic in many ways. If the United States wants to keep up with other high-
achieving countries, American teachers need to ensure that our gifted students are being adequately challenged in mathematics. Although funding for gifted education is lacking, there is still a lot that teachers can do. Through problem solving, group work and a more in-depth curriculum, teachers can help their gifted students reach their true potential, enjoy mathematics and potentially continue in a field of mathematics beyond high school. This curriculum is designed to help teachers be successful in challenging their gifted students in either heterogeneous or homogenous classes. Using unique, challenging problems that go beyond the normal curriculum and allowing students to work in groups to discover and learn will bring out the best in gifted students and encourage them to love mathematics.

## Chapter 3: Body

This curriculum is comprised of several mini-lessons designed to challenge students who are gifted in the area of mathematics. Although it is a curriculum that could be used to supplement existing curriculum in a general education geometry classroom, it will be implemented in a separate "lab" class designed for students who are gifted in the area of mathematics. Much of the work will be done with partners or in small groups, but students will have the opportunity to work alone if they choose. The literature states that group work is beneficial for many gifted students, but some are more successful if they work alone (Hernandez Gardunio, 2001). These sophomore students are capable of deciding which method works best for them.

The lessons in this curriculum will be mostly rooted in geometry and will be based, in part, on the geometry common core standards. They are, however, nonsequential and therefore can be done in any order. Although much of the content of the curriculum is in the common core learning standards for Geometry, the more significant concepts that students should learn while completing these problems are what common core designates the "mathematical practices." Most importantly, students will "make sense of problems and persevere in solving them" (National, 2010, p. 5). The curriculum is designed so that the solutions are not obvious. Students may try several different methods for solving the problem before they find one that will actually work. The other important mathematical practice that will be covered in this curriculum is that students will learn to model with mathematics. The common core modeling standards for high school mathematics breaks modeling down into six steps.
"The basic modeling cycle...involves (1) identifying variables in the situation and
selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them (National, 2010, p. 61).

As students complete these problems, they will work with a variety of different models. Student may choose to represent the situations in two-dimensional pictures or three-dimensional models. Regardless of how they choose to represent the situations, students will need to use extreme care and will need to spend time analyzing whether their representations match the problem, whether or not they have all the information they need in their model, how they can use their model to better understand the problem, that their models are done to scale and represent the situation accurately and whether or not their final solution makes sense based on the model. Once they arrive at an answer, they will need to check their answer against the model to ensure that the answer makes sense.

Students will complete many of the lessons in groups of two or three. As mentioned above, group work has many benefits for students' mathematical learning. When students spend time discussing and arguing about the potential methods for solving a problem, the problems with their approach, the additional information needed to solve the problem and the validity of their final answers, they will greatly gain insight into the best way to approach the problem and why their solutions work. Groups will present
their solutions to their classmates, offering them a chance to compare their answers, seeing that there is often more than one correct way to solve a problem.

## Lesson One: Soccer Goalie Reaction Time

The first lesson of this curriculum is titled Soccer Goalie Reaction Time and asks students to find the amount of time an average height soccer goalie has to get to the ball if the ball is kicked to the upper corner of the net. This problem is one that can be done with students who are just beginning their geometry course. The lesson requires that students use prior knowledge to solve for the missing sides of a right triangle using the Pythagorean Theorem and how to convert between units. In eighth grade, the students learn how to use and apply the Pythagorean Theorem. Common Core learning standard 8.G.B. 7 claims that students should be able to "apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real world and mathematical problems in two and three dimensions" (National, 2010, p. 48). Much of the foundation of this lesson will be based on this standard. Additionally, students are required to use the Pythagorean Theorem in the Geometry curriculum. Standard G-SRT.C. 8 requires that students "use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems" (National, 2010, p. 65). The other task that students need to be proficient in is converting units. Students worked heavily with ratios and rates in seventh grade and should be well aware of the steps needed to convert between units. Standard 7.RP. 3 requires students to "use proportional relationships to solve multi-step ratio and percent problems" (National, 2010, p. 41).

The lesson is based on a problem asked by a student to the Khan Academy. The

Khan Academy has a video solution to the problem and the additional information that students will need available at https://www.khanacademy.org/math/basic-geo/basic-geo-pythagorean-topic/basic-geo-pythagorean-theorem/v/soccer-thiago.

## "Soccer Goalie Reaction Time" Lesson Plan

I. Objective

Students will persevere in solving a problem, asking for relevant information, analyzing the situation and arguing the validity of their final answer.
Students will apply the Pythagorean Theorem multiple times to answer a real-life situation question.
II. Preparation

Standards
Addressed

Mathematical Students will make sense of problems and persevere in Practices

G-SRT.C.8: "Students use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems" solving them

Students will learn to model with mathematics

Materials Needed Videos, pencils, paper, calculators, modeling materials

Source

Khan Academy

Khan, S. (Producer). (2013, December 9). Khan Academy. Thiago ask: How much time does a goal keeper have to save a penalty kick. Video retrieved from https://www.khanacademy.org/math/basic-geo/basic-geo-pythagorean-topic/basic-geo-pythagorean-theorem/v/soccer-thiago
III. Procedure
A. Anticipatory Set
a. Students will watch a short tutorial video on how to save a penalty kick
https://www.youtube.com/watch?v=V47SmRfB5t0
b. Students will then watch a video of soccer goalie penalty kick save highlights.
https://www.youtube.com/watch?v=hRaO5A5dGnc
(1:45, 2:15, 2:50, 4:00, 5:20)
B. Body
a. Students will be given the question: "How much time does an average
height goalie have to get to the ball if a penalty kick is kicked to the upper corner of the net?"
b. Students will work with a partner or a group of three
c. They will first need to figure out the relevant information that they need to know in order to solve the problem. Unlike typical word problems, students will need to ask to get the information they need to know to solve the problem
i. The following pieces of information will be useful for students when solving this problem. If possible, students should not be given the information, they should instead ask questions in order to obtain it:

1. The ball is 36 ft from the center of the goal
2. The goal is 24 ft wide
3. The goal is 8 ft tall
4. An average goalie stands 7.5 ft tall with his reach*
5. A penalty kick travels $60 \mathrm{mph}^{*}$
6. An average person can jump at a speed of $15 \mathrm{mph}^{*}$ *Note: these numbers are based on averages or estimates. The process of solving the problem is much more important than a perfect numerical answer.
d. The students will spend a lot of time working on a plan to solve the problem, trying various methods and asking for more relevant information if necessary
e. When all groups have a solution, they will present their solutions to the class.
f. After hearing all potential solutions and explanations, students will get a chance to revise their solutions.
C. Closing
a. There will be a class lead discussion about problems students had while solving the problems.
b. Additionally, students will discuss what went well while they were working with their groups.
D. Follow-up
a. For homework, students will write clear and concise steps to solve the problem at hand.

Name: $\qquad$ Date: $\qquad$

## Soccer Goalie Reaction Time Classwork

How much time does an average height goalie have to get to the ball if a penalty kick is kicked to the upper corner of the net?


Khan, S. (Producer). (2013, December 9). Khan Academy. Thiago ask: How much time does a goal keeper have to save a penalty kick. Video retrieved from
https://www.khanacademy.org/math/basic-geo/basic-geo-pythagorean-topic/basic-geo-pythagorean-theorem/v/soccer-thiago

## Soccer Goalie Reaction Time Homework

Write clear and concise steps to answer the following question. Include all relevant information a problem solver needs to know in order to answer the question. How much time does an average height goalie have to get to the ball if a penalty kick is kicked to the upper corner of the net?


## Lesson Two: Square It Up

The second lesson of the curriculum requires some prior knowledge of geometry assumptions and facts. Students must know the definition of midpoint, how to find the area of various figures without given measurements and how to compare sizes of figures. This lesson starts out with a basic question, one that students should be able to answer with relative ease and then modifies the question to make it more complex. Students work in a group to solve this problem and justify their answers mathematically.

Each solution method may require different prior knowledge and understanding. In the solution posed in this curriculum, students need to use several ideas from the Geometry curriculum. Students must understand how to prove that two triangles are similar using the angle-angle similarity postulate. In Geometry, standard G.SRT.A. 3 requires students to "use the properties of similarity transformations to establish the AA criterion for two triangles to be similar" (National, 2010, p. 65). Additionally, G.SRT.B.5, requests that students apply the AA similarity postulate to solve problems. It claims that students should, "use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures" (National, 2010, p. 65). Finally, once problem solvers have established that two of the triangles in the square are similar, they need to know the relationship between their areas.

The lesson is based on a lesson found on Dan Meyer's blog. The questions can be found at http://blog.mrmeyer.com/wp-content/uploads/squareitup.pdf.

## "Square It Up" Lesson Plan

I. Objective

Students will persevere in solving a problem, asking for relevant information, analyzing the situation and arguing the validity of their final answer.
Students will use the information they know about area, and relationships between different figures to determine the fraction that each piece represents.
II. Preparation

Standards
Addressed

Mathematical Students will make sense of problems and persevere in Practices
G.SRT.B.5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures solving them

Materials Needed Problem worksheet, pencils, paper, dynamic geometry software

Source
Dan Meyer's Blog

Meyer, D. (Producer). (2011, August 18). Dy/Dan. Some
Really Obscure Geometry Problem. Retrieved from http://blog.mrmeyer.com/2011/3acts-some-really-obscure-geometry-problem/
III. Procedure
A. Anticipatory Set
a. Students will be shown a picture of a square with the diagonals drawn on the smart board
b. They will be asked what fraction each part of the box separated by the diagonals represents
B. Body
a. After students give their answers, they will be given the actual question:
i. What fraction of the square would each portion represent if the endpoint of one of the diagonals was moved from the vertex to
the midpoint of the side?
b. Students will work as a group on this question. They will discuss their answers and come up with a solution.
c. Groups will present their solution(s)
d. Groups will reconvene to revise their solutions
e. After groups have their revised solutions, they will be shown an example on geometer's sketchpad. They will watch how the areas of the portions change as the size of the square changes.
f. They will once again be given a chance to revise their solution
C. Closing
a. We will discuss what students learned as they worked on solving the problem.
D. Follow-up
a. Students will be given a follow-up question for homework. What if the endpoint of the diagonal was moved so that it cut the side in a ratio of two to one?

Name: $\qquad$ Date: $\qquad$

## Square It Up - Classwork

. In square $A B C D$ to the right, point $M$ is the midpoint of side $C D$. Find the ratios of the areas of the four regions ( $\triangle M P C, \triangle B P C$, $\triangle A P B$, and quadrilateral $A P M D$ ) that are formed. Justify your result.


Meyer, D. (Producer). (2011, August 18). Dy/Dan. Some Really Obscure Geometry Problem. Retrieved from http://blog.mrmeyer.com/2011/3acts-some-really-obscure-geometry-problem/

Name: $\qquad$

## Square It Up-Homework

Instead of $M$ being the midpoint of side $C D$, suppose $M$ cuts side $C D$ so that $M D=2 \cdot C M$, as seen to the right. What are the ratios of the areas of the four regions?


Meyer, D. (Producer). (2011, August 18). Dy/Dan. Some Really Obscure Geometry Problem. Retrieved from http://blog.mrmeyer.com/2011/3acts-some-really-obscure-geometry-problem/

## Lesson 3: Seven Squares

The third lesson in this curriculum is also rooted in the Pythagorean Theorem, but requires knowledge of operations with radicals, and therefore should be done after students learn about how to simplify and combine radicals in their geometry course. According to Common Core Learning Standard G-SRT.C.8, students need to use "the Pythagorean Theorem to solve right triangles in applied problems" (National, 2010, p. 65). Although simplifying radicals was formerly an Algebra topic, the common core learning standards do not begin to cover operations with radicals until Geometry. According to Module 2 of the Common Core Curriculum, simplifying radicals is part of standard N.RN.A.2, where students "rewrite expressions involving radicals and rational exponents using the properties of exponents" (National, 2010, p. 51). This standard will be mastered in full in an Algebra 2 course, but the basics of operations with radicals begins in Geometry. Additionally, for success with this lesson, students will need to know how to find the perimeter of a square, the definition of midpoint, and some algebra including combining like terms.

In this problem, students will be given a perimeter for a square and will be told that another square will be drawn inside the first such that the vertices of the second square are the midpoints of the sides of the first square. Then a third square will be drawn in the same fashion, such that the vertices of the third square are the midpoints of the sides of the second square. This pattern will continue until there are seven squares in total and students will be asked to find the sum of the perimeters of the squares. Students will solve the same problem twice. First, they will solve the problem given a number for
the perimeter of the first square. Then, once they arrive at the correct answer, they will face the same problem with a variable for the perimeter of the first square.

The lesson is based on a problem from MATH 605 taught by professor Gabriel Prajitura of SUNY Brockport.

## "Seven Squares" Lesson Plan

I. Objective

Students will apply the Pythagorean Theorem and their knowledge of simplifying radicals to solve a problem. Students will persevere in solving a problem, asking for relevant information, analyzing the situation and arguing the validity of their final answer.
II. Preparation

| Standards |  |
| :--- | :--- |
| Addressed | G-SRT.C.8: "Students use trigonometric ratios and the <br> Pythagorean Theorem to solve right triangles in applied <br> problems" |
| N.RN.A.2: "Students rewrite expressions involving radicals <br> and rational exponents using the properties of exponents" |  |
| Mathematical <br> Practices | Students will make sense of problems and persevere in <br> solving them |
|  | Students will learn to model with mathematics |

Materials Needed Problem worksheet, pencils, paper

Source
Gabriel Prajitura's MATH 605 Geometry Problem Set 6

Prajitura, G. (May 2013). Geometry Problem Set 6.
Retrieved from angel.brockport.edu, MTH 605.
III. Procedure
A. Anticipatory Set
a. Students will be given the perimeter of a square and will be asked to find the length of the diagonal of the square. They will be asked to express their answer in simplest radical form.
b. We will discuss their approaches to solving the problem, focusing on their use of the Pythagorean Theorem and their knowledge of the sides of a square being equal in length
B. Body
a. Students will be given the following question:
i. A square, MATH, has a perimeter of 16. Another square has its vertices at the midpoints of the sides of the first. A third
square has its vertices at the midpoints of the sides of the second. Continuing in the same way, there are 7 squares in total. Find the sum of the perimeters of each square.
ii. Students will have the option of working in groups or working alone to solve the problem
iii. When they have a solution, it will be checked. If their solution is correct, they will move on to the next problem.
b. The second problem will be given to students/groups who have a correct answer to the first problem and requires the use of algebra:
i. A square, GEOM, has a perimeter of $a$. Another square has its vertices at the midpoints of the sides of the first. A third square has its vertices at the midpoints of the sides of the second. Continuing in the same way, there are 7 squares in total. Find the sum of the perimeters of each square.
C. Closing
a. Students will discuss the patterns that they found while working on the two problems.
D. Follow-up
a. Students will be given a follow-up question for homework. How will the answers to the questions change if the perimeter of the original square was doubled?

Name: $\qquad$ Date: $\qquad$

## Seven Squares - Classwork

Warm-up: A square has a perimeter of 32 . What is the length of a diagonal of the square? Express your answer in simplest radical form.

## Problem 1

A square, MATH, has a perimeter of 16. Another square has its vertices at the midpoints of the sides of the first. A third square has its vertices at the midpoints of the sides of the second. Continuing in the same way, there are 7 squares in total. Find the sum of the perimeters of each square.


## Problem 2

A square, GEOM, has a perimeter of $a$. Another square has its vertices at the midpoints of the sides of the first. A third square has its vertices at the midpoints of the sides of the second. Continuing in the same way, there are 7 squares in total. Find the side length and perimeter of each square.

| Square Number | Side Length | Perimeter |
| :--- | :--- | :--- |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |
| 5 |  |  |
| 6 |  |  |
| 7 |  |  |

Prajitura, G. (May 2013). Geometry Problem Set 6. Retrieved from angel.brockport.edu, MTH 605.

Name: $\qquad$ Date: $\qquad$

## Seven Squares Homework

In class today you looked at finding the sum of the perimeters of seven squares whose vertices are midpoints of the other squares. How would the lengths of the sides change if the perimeter of the original square doubled?

Hint: Try doubling the perimeter of MATH or GEOM and compare the lengths of the sides to the original answers.

## Lesson 4: Once Upon a Time

The fourth lesson in this curriculum is built around the central angles in a circle. Students may be able to complete this lesson early on in their Geometry course because it simply requires the knowledge that the sum of the central angles in a circle is three hundred sixty degrees. Many students may come into the course with this prior knowledge. In the Geometry Common Core curriculum, standard G-C. 2 requires that students "identify and describe relationships among inside angles" in a circle.

This lesson has students analyzing the angles formed by the hands of a clock. Students will start out by finding a time of day when the hands of a clock form a specific (forty-seven degree) angle. They will then be asked to find other times in the day when the hands of a clock form the same angle. Students will ultimately need to find the number of degrees in the angle formed by one minute and how many degrees the hour hand moves with each passing minute. Using this knowledge, students may be able to come up with an equation or simply use trial-and-error.

The lesson is based on the "Once Upon a Time" problem of the month created by insidemathematics.org. The basis for the problem can be found at http://www.insidemathematics.org/assets/problems-of-the month/.

## "Once Upon a Time" Lesson Plan

I. Objective

Students will apply their knowledge of the central angles formed inside a circle.

Students will persevere in solving a problem, asking for relevant information, analyzing the situation and arguing the validity of their final answer.
II. Preparation

Standards
Addressed

Mathematical
Practices

G-C.2: "Students identify and describe relationships among inside angles radii and chords"

Students will make sense of problems and persevere in solving them

Students will learn to model with mathematics

Materials Needed Problem worksheet, pencils, paper, compass

Source

Inside Mathematics - Problem of the Month

Noyce Foundation (2013). Problem of the month: Once upon a time. Retrieved from http://www.insidemathematics.org/assets/problems-of-the-month/once\ upon\ a\ time.pdf
III. Procedure
A. Anticipatory Set
a. Students will be shown an analog clock.
b. They will be asked at what time of the day the hands form a $90^{\circ}$ angle. Students will write their answers down on their own. We will discuss the answers as a class and see how many possible times we can get.
B. Body
a. Students will be given the following questions:
i. The minute hand and the hour hand on a clock form a $47^{\circ}$ angle. What time is it?
ii. At what other times during the day do the hands on the clock
form a $47^{\circ}$ angle?
iii. How many times in a day ( 24 hour period) do the hands form a $47^{\circ}$ angle? Explain your reasoning.
iv. The students will have the option of working on the problems as a group or individually. If students choose to work individually, they will have to discuss their work/answers occasionally with another person working alone or with a group of students.
b. We will come back as a whole group and discuss the results. We will create a class list of the times groups discovered with an angle of $47^{\circ}$
C. Closing
a. As a class, we will discuss the patterns students saw when looking at the angles represented by different times.
D. Follow-up
a. Students will be given a follow-up question for homework. Find the angle formed by the hands of the clock for the ten-minute intervals between 3:00, 4:00 and 5:00. They will write about any patterns they see between the three hours.

Name: $\qquad$ Date: $\qquad$

## Once Upon a Time

Warm-up:
On an analog clock (see right), what is the measure of the angle formed by the hands at the following times?
a) $2: 00$
b) $4: 30$
c) $5: 45$
d) $7: 10$
e) $8: 32$

f) $9: 09$

## Problems

1. The minute hand and hour hand on a clock form a $110^{\circ}$ angle. What time could it be?

2. At what other times during the day do the hands on the clock form a $110^{\circ}$ angle? (Write your answers to the nearest second)

Noyce Foundation (2013). Problem of the month: Once upon a time. Retrieved from http://www.insidemathematics.org/assets/problems-of-themonth/once\ upon\ a\ time.pdf

Name:
Date: $\qquad$
Once Upon a Time Homework
Fill in the following table for the angles formed at each time.

| Time | Angle <br> formed by <br> hands | Time | Angle <br> formed by <br> hands | Time | Angle <br> formed by <br> hands |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $3: 00$ |  | $4: 00$ |  | $5: 00$ |  |
| $3: 10$ | $4: 10$ |  | $5: 10$ |  |  |
| $3: 20$ |  | $4: 20$ |  | $5: 20$ |  |
| $3: 30$ |  | $4: 30$ |  | $5: 30$ |  |
| $3: 40$ |  | $4: 40$ |  | $5: 40$ |  |
| $3: 50$ |  | $4: 50$ |  | $5: 50$ |  |

What patterns do you notice?

## Lesson 5: Toothpick Math

The fifth lesson in the curriculum is one that is not based on any specific part of the Geometry curriculum, but instead, is one that is meant to get students to think logically. Students will work on the mathematical practices, focusing on modeling with mathematics. Students will need to use their knowledge of various shapes and their spatial reasoning abilities.

The lesson involves ten toothpick puzzles. The puzzles start with a picture formed by toothpicks and ask the problem solver to move or remove a certain number of toothpicks to change the picture in a specific way. While working on spatial reasoning and problem solving skills, the puzzles are also a great way to help overcome fixation in mathematical problem solving. Yogesh Sharma (2013) discussed the challenges students face with fixation in mathematical problem solving. When students are faced with a new problem and cannot see past a method that worked for solving a previous problem, even after realizing it will not work, they struggle to become effective problem solvers. Sharma recommends that teachers "try to provide certain exercises to test fixation in mathematics (Sharma, 2013, p. 18). This lesson is designed to do just that.

The lesson is based on education.com's Toothpick math which can be found at http://www.education.com/activity/article/Toothpick_Math/.

## "Toothpick Math" Lesson Plan

I. Objective

Students will apply their spatial reasoning and critical thinking skills to overcome fixedness in solving problems. Students will persevere in solving a problem, asking for relevant information, analyzing the situation and arguing the validity of their final answer.
II. Preparation

Mathematical
Students will make sense of problems and persevere in
Practices solving them

Students will learn to model with mathematics

Materials Needed Toothpicks, problem worksheet, pencils, paper

Source Education.com - Toothpick Math

Donaldson, C (2014). Toothpick math. Retrieved from http://www.education.com/activity/article/Toothpic k_Math/
III. Procedure
A. Anticipatory Set
a. Students will be given 24 toothpicks. They will be given two minutes to create the most creative shape they can with the toothpicks.
B. Body
a. Students will keep their 24 toothpicks to be used for the toothpick problems.
b. Students will complete the toothpick puzzles on their own.
c. They will try to solve as many as they can, using their toothpicks and the worksheet.
d. On the worksheet they will mark which toothpicks they moved or removed.
e. With ten minutes left, we will go over the toothpick questions. Students will be able to present their solutions to the class.
C. Closing
a. We will discuss fixation and how the problems helped students overcome fixation
D. Follow-up
a. Students will be asked to create their own toothpick puzzle for homework

Name: $\qquad$ Date: $\qquad$

## Toothpick Math Classwork

1) Remove one toothpick to leave three squares.

2) Move two toothpicks to make the pig go the opposite way.

3) Remove six toothpicks to leave four triangles.

4) Take away two toothpicks and leave two squares.

5) Make the fish swim the opposite way by moving three toothpicks and the coin.

6) Remove six toothpicks and leave two squares.

7) Move four toothpicks and leave three equilateral triangles.


Donaldson, C (2014). Toothpick math. Retrieved from
http://www.education.com/activity/article/Toothpick_Math/

Name: $\qquad$ Date: $\qquad$

Create your own toothpick problem.
The problem must fit the following criteria:

- The greatest number of toothpicks that can be used is 24
- You may use up to two coins in addition to the toothpicks
- The problem must have just one way of solving it
- The solution must be obtained from either moving or removing a given number of toothpicks
- You must draw the problem and the solution


## Lesson 6: Angles in Triangles

The sixth lesson focuses on finding missing angles in triangles. In order to be successful with these problems, students must know a number of theorems about the angles in triangles and angles formed by parallel lines. They may be required to use theorems such as the triangle angle sum theorem, the isosceles triangle theorem, the exterior angle theorem, the quadrilateral angle sum theorem, linear pairs sum to onehundred eighty degrees and the parallel postulate. The Geometry Common Core curriculum covers each of these theorems in standard G.CO-9 and G.CO-10. According to standard G-CO.9, "students will prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints." Additionally, according to G.CO-10, students should "prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to $180^{\circ}$, [and] base angles of isosceles triangles are congruent" (National, 2010, p. 65).

In this lesson, students will work through two triangle angle problems. In addition to being able to use the aforementioned angle theorems, students will also need to work to overcome fixation in mathematical problem solving. Although these problems will be centered on finding the angles in triangles, students will need to see past the previous problems in order to be successful with each subsequent problem. Students will first find the missing angle measure in a triangle, knowing which sides are congruent, but knowing none of the angle measures. They must be competent in their ability to use algebra in order to solve this problem. The next question requires that students draw
several auxiliary lines. As students work through the problem, several hints will be given to ensure that students make it to the final answer.

The first question in this lesson came from the University of Washington's Math 444 course and can be found at http://www.math.washington.edu/~lee/Courses/4442010/challenge.pdf. The second question in the lesson was named "The World's Hardest Easy Geometry Problem" and can be found at http://thinkzone.wlonk.com/MathFun/ Triangle.htm. The solution was provided by Matthew Daly, a master's thesis student at the State University of New York College at Brockport.

## "Angles in Triangles" Lesson Plan

I. Objective

Students will apply their knowledge of angles formed inside triangles.
Students will persevere in solving a problem, asking for relevant information, analyzing the situation and arguing the validity of their final answer.
II. Preparation

Standards
Addressed

Mathematical Students will make sense of problems and persevere in Practices solving them

Students will learn to model with mathematics
Materials Needed Problem worksheet, pencils, paper

Source University of Washington - Math 444 - Geometry for Teachers
University of Washington Math 444 (2010). Geometry for teachers: Some challenge problems. Retrieved from http://www.math.washington.edu/~lee/Courses/4442010/challenge.pdf.

Keith's Think Zone - World's Hardest Easy Geometry Problem

Keith Enevoldsen. World's Hardest Easy Geometry Problem. Retrieved from http://thinkzone. wlonk.com/MathFun/Triangle.htm

## III. Procedure

A. Anticipatory Set
a. Students will be given some basic triangle missing angle questions.
b. We will discuss the answers
B. Body
a. Students will be given a worksheet with two missing angle questions
i. They will work with a partner or as a group to determine the measure of the missing angles for the first problem
ii. When groups have a solution, they will get their solution checked.
iii. If their solution is approved, they will go on to the next question. They will be given about ten minutes to work on the question on their own.
iv. After ten minutes, a hint will be given to students as to which auxiliary line should be drawn. Then they will be given more time to work on their own. When they are at a standstill more hints will be given.
v. Students will work through the question, with various hints given throughout the process. After each hint, students will be given time to think about the hint and discover more missing information in the problem

1. Possible hints: Draw auxiliary line $\overline{D F} \| \overline{A B}$. Draw in AF and let G be the intersection of $\overline{A F}$ and $\overline{D B}$. Draw in $\overline{C G}$. Look for known angles due to theorems such as the angles from when parallel lines are cut by a transversal, the isosceles triangle theorem, congruent triangle theorems, and corresponding parts of congruent triangles are congruent
b. We will come back as a whole group and discuss the results. We will discuss the solution and how to obtain the solution
C. Closing
a. We will discuss problem-solving strategies that groups used to obtain solutions to the final problem.
D. Follow-up
a. Students will fill in missing information in the proof of the problem.

Name: $\qquad$ Date: $\qquad$

## Angles in Triangles Classwork

Warm-up: Find the measure of the angle marked $\theta$ in each triangle below.


Problem Set
In the following diagram, $A B=B C=C D$ and $A D=B D$. Find the measure of angle $D$.


University of Washington Math 444 (2010). Geometry for teachers: Some challenge problems. Retrieved from http://www.math.washington.edu/~lee/Courses/4442010/challenge.pdf.

Determine the measure of angle $x$.


Keith Enevoldsen. World's Hardest Easy Geometry Problem. Retrieved from http://thinkzone. wlonk.com/MathFun/Triangle.htm


Keith Enevoldsen. World's Hardest Easy Geometry Problem. Retrieved from http://thinkzone. wlonk.com/MathFun/Triangle.htm

## Lessons 7 and 8: Sectors in Circles and Cutting the Cake

The final two problems in this curriculum are centered on sectors in circles. According to Common Core Geometry standard G-C.5, students should be able to "find arc lengths and area sectors of circles" (National, 2010, p. 66). However, the Geometry Curriculum overview claims that only a small portion (2-8\%) of the examination in geometry will be on circles. Teachers may see this as a category to skim or skip, but there are a lot of ways to use this topic to challenge gifted students. The questions shown in this curriculum can be done with an understanding of circle sectors, the Pythagorean Theorem, percentages and algebra.

The first sector lesson consists of two different applications of sectors. The first two problems ask students to compare the areas of sectors and circles. Students will need to use the Pythagorean Theorem and algebra to complete the problems. The final problem has two quarter circles overlapping in a rectangle. Students are to use the areas of these sectors to find the missing side of a rectangle. These problems require students to have a thorough understanding of algebra including combining like terms and multiplying binomials.

The second sector lesson is based on a real life problem involving a circular cake. Students are told that a circular cake with a ten inch diameter is divided into twelve equal pieces. Students must first find the area of each slice of cake. Then they learn that the already sliced cake must be cut again so that it can serve twenty-four people. Students must determine where the cut should be made so that each person gets an equal amount of cake and how much cake each person will get. Students will need to apply their knowledge of sectors, finding areas of circles and operations with radicals.

The first lesson is a combination of questions found from two sources. The first is from the NRICH Project from the University of Cambridge. The first two questions from their problem set "Round and Round" were used in the lesson and can be obtained at http://nrich.maths.org/634/. The latter part of the lesson was based on a problem posed by the blog Five Triangles. It can be obtained at http://fivetriangles.blogspot.com/ 2014/09/184-overlapping-sectors.html. The second lesson on sectors was based on a worksheet from the Virginia Department of Education and can be obtained at http://www.doe.virginia.gov/testing/solsearch/sol/mat/G/m_ess_g-11bc.pdf.

## "Sectors of Circles" Lesson Plan

I. Objective

Students will apply their knowledge of finding the area of circles and sectors to compare the size of sectors. Students will persevere in solving a problem, asking for relevant information, analyzing the situation and arguing the validity of their final answer.

## II. Preparation

Standards G-C.5: "Students will find the arc lengths and areas of

Addressed

Mathematical
Practices sectors"

Students will make sense of problems and persevere in solving them

Students will learn to model with mathematics

Materials Needed Problem worksheet, pencils, paper

Source
NRICH - University of Cambridge

NRICH (2014). Round and round. University of Cambridge. Retrieved from
http://nrich.maths.org/634.

Five Triangles (2014). Overlapping Sectors. Retrieved from http://fivetriangles.blogspot.com/ 2014/09/184-overlapping-sectors.html
III. Procedure
A. Anticipatory Set
a. Students will find the area of various sectors of circles in anticipation of applying the concept to more complex problems
B. Body
a. Students will be given the worksheet with all three sector questions.
b. They will be given the option to work alone or with a partner.
c. As they complete each question, they will get their solution approved before moving on to the next question.
C. Closing
a. We will discuss the solutions to the problems as a group

Name: $\qquad$ Date: $\qquad$

## Sectors of Circles Classwork

Warm-up: Circle A is shown to the right. The length of diameter $\overline{A B}$ is 8 . Find the area of each sector.


Problem Set:
Prove that the shaded area of the semicircle is equal to the area of the inner circle.


What percentage of the sector OAB is taken up with the inner circle?


NRICH (2014). Round and round. University of Cambridge. Retrieved from http://nrich.maths.org/634.

The diagram below shows rectangle ABCD with height 10 cm . An arc with center at point B is drawn from point A to side $\overline{B C}$. An arc with center at point C is drawn from point D to side $\overline{B C}$. Given that the shaded regions a and b have equal area, determine the length of $\overline{B C}$.


Five Triangles (2014). Overlapping Sectors. Retrieved from http://fivetriangles.blogspot.com/ 2014/09/184-

## "Cutting the Cake" Lesson Plan

I. Objective Students will apply their knowledge of finding the area of circles and sectors to compare the size of sectors. Students will persevere in solving a problem, asking for relevant information, analyzing the situation and arguing the validity of their final answer.
II. Preparation

| Standards | G-C.5: "Students will find the arc lengths and areas of |
| :--- | :--- |
| Addressed | sectors" |

Materials Needed Problem worksheet, pencils, paper

Source $\quad$ Virginia Department of Education - Activity Sheet 2: Cake Problem

Virginia Department of Education (2011). Activity Sheet 2: Cake Problem. Mathematics Enhanced Scope and Sequence. Retrieved from http://www.doe.virginia.gov/testing/solsearch/sol/m at/G/m_ess_g-11bc.pdf.
III. Procedure
A. Anticipatory Set
a. Students will be given a question about pizza. They will look at a pizza with a sixteen inch diameter. They will be asked the area of each slice of pizza if it's cut into 6 slices.
b. Students will work on the questions on their own and we will come back together as a group to discuss the answers.
B. Body
a. Students will be given the Cake problem worksheet. They will have the opportunity to work on it in small groups or on their own.
b. As students work, the teacher will walk around the room, monitoring student progress.
C. Closing
a. Students will exchange solutions with another group/person. They will then critique the solution
D. Follow-up
a. Students will be given a follow-up question: what if, instead of equal pieces, you want to split the cake into twenty-four pieces such that half of the pieces are twice as big as the other pieces.

Name: $\qquad$ Date: $\qquad$

Cutting the Cake - Classwork

Warm-up: You order a pizza for your friends. The pizza has a diameter of twenty-four inches. If you cut the pizza into six equal slices, what will the area of each piece be?


Problem Set
You are halfway through a party and twelve people are there. You decide to cut the circular cake with a diameter of 10 " into 12 equal pieces.

1. What is the area of the top of each piece of cake?

2. Before you get a chance to serve the cake, twelve more people arrive. You already cut the twelve pieces out of the cake, but decide you can still make equal slices by cutting out a concentric circle in the cake to make 24 pieces. How far from the center of the cake should the circle be made so that all 24 people get the same amount of cake?

3. What is the area of each segment of cake? How much cake will each person receive?

Virginia Department of Education (2011). Activity Sheet 2: Cake Problem. Mathematics Enhanced Scope and Sequence. Retrieved from http://www.doe.virginia.gov/testing/solsearch/sol/mat/G/m_ess_g-11bc.pdf.

Cutting the Cake - Homework
As you are just about to make your concentric circle cut, one lady speaks up. "Women usually eat less than men. Since there are 12 women and 12 men here, maybe you should make the pieces for the men twice as big as the pieces of the women." You decide this is a good idea. How far from the center of the cake should the circle cut be made so that the outside pieces are twice as big as the inside pieces?


## Chapter 4: Discussion, Summary and Reflection

The curriculum is designed to offer a resource for teachers to reach their gifted learners despite limited funding for gifted students in the United States. Although it is designed for students who are currently taking New York State Common Core Geometry, it is suitable for any gifted learner with a geometry background. Gifted students benefit greatly from solving a variety of problems, particularly those in which the answers or solution methods are not obvious, all the information is not simply given to the students, and problems that have more than one correct answer. By offering these types of problems this curriculum can truly challenge gifted students and help them reach their full potential.

Although the lessons in the curriculum focus on a variety of topics, the common theme is problem solving. The lessons focus on the Common Core Mathematical Practice, students will "make sense of problems and persevere in solving them" (National, 2010, p. 5). As Ross (1993) pointed out, gifted students in the United States are not given opportunities to solve challenging problems as often as those in other countries. As a result, gifted students in the United States are often well behind gifted students in other countries (National, 2014). Through the use of this curriculum, teachers can expose their gifted students to unique, thought-provoking problems.

Gifted students benefit greatly from being challenged and actually enjoy solving challenging problems (Threlfall \& Hargreaves, 2008). They are able to ask for relevant information, find more than one correct answer and more than one method of solving problems and make up their own problems (Yogesh Sharma, 2013). When teachers offer
these opportunities to their students, not only do the students excel, but they also tend to appreciate and enjoy mathematics more (Threlfall \& Hargreaves, 2008).

## Appendix

## Soccer Goalie Reaction Time Classwork

How much time does an average height goalie have to get to the ball if a penalty kick is kicked to the upper corner of the net?

Let $b=$ distance from the ball to the bottom corner of the net
$36^{2}+12^{2}=b^{2}, b=37.95 \mathrm{ft}$
Let $u=$ distance from the ball to the upper corner of the net
$37.95^{2}+8^{2}=u^{2}, u=38.78 \mathrm{ft}$
Let $r=$ distance from the middle of
 the goal to the upper corner of the net
$12^{2}+8^{2}=r^{2}, r=14.42 \mathrm{ft}$
Assume the height of the goalie from the ground to the tip of his fingers when he reaches is 7.5 ft . Then the goalie needs to travel 6.9 ft .

Assume the ball travels 60 mph and therefore it travels $88 \mathrm{ft} / \mathrm{sec}$. Assume the goalie travels 15 mph and therefore he/she travels 22 feet per second.

Since distance equals rate times time, the ball makes it to the upper corner of the goal in 0.44 seconds and the goalie makes it to the upper corner of the net in 0.31 seconds.

Therefore, the goalie has 0.13 seconds to react to the kick and begin his or her jump.

## Soccer Goalie Reaction Time Homework

Write clear and concise steps to answer the following question. Include all relevant information a problem solver needs to know in order to answer the question. How much time does an average height goalie have to get to the ball if a penalty kick is kicked to the upper corner of the net?


Answers will vary

## Square It Up - Classwork

. In square $A B C D$ to the right, point $M$ is the midpoint of side $C D$. Find the ratios of the areas of the four regions ( $\triangle M P C, \triangle B P C$, $\triangle A P B$, and quadrilateral $A P M D$ ) that are formed. Justify your result.

The ratio of areas of region ADMP: APB: CPB: CMP is 5:4:2:1. The area of ADMP is $5 / 12$ of the area of the entire square The area of APB is $1 / 3(4 / 12)$ of the area of the entire square The area of CPB is $1 / 6(2 / 12)$ of the area of the entire square
 The area of CPM is $1 / 12$ of the area of the entire square

Justification
Since ABCD is a square, we know that $A B=B C=C D=A D$ and that $A B \| D C$ and $A D \| B C$.

We can prove that $\triangle A P B \sim \triangle C P M$. We know that $\Varangle A B P \Varangle \cong C M P$ since when parallel lines are cut by a transversal, alternate interior angles are congruent. We also know that $\Varangle A P B \cong \Varangle C P M$ since vertical angles are congruent. Therefore, by the angle angle similarity postulate, $\triangle A P B \sim \triangle C P M$. Since M is the midpoint of $\overline{D C}$ and $D C=A B$, $M C=\frac{A B}{2}$. Therefore the ratio of the sides of $\triangle A P B: \triangle C P M=2: 1$, hence the ratio of their areas is $4: 1$.

Since $\overline{A C}$ is the diagonal in a square, we know that the sum of the areas of $\triangle A P B+$ $\triangle C P B$ is half the area of the entire square. We also know, that $\triangle M C B$ is one-quarter of the area of the entire square. We can let the area of $\triangle A P B=a$, the area of $\triangle B P C=b$ and the area of $\triangle M P C=c$. Then we know that $a+b=\frac{1}{2}(\operatorname{area} A B C D)$ and $c+b=$ $\frac{1}{4}$ (area $\left.A B C D\right)$. Therefore by substitution $4 c+b=\frac{1}{2}($ area $A B C D)$ and hence $c=$ $\frac{1}{12}$ (area of $A B C D$ ). Then we know that $b=\frac{1}{6}($ area $A B C D)$ and $a=\frac{1}{3}$ (area $\left.A B C D\right)$. Then by subtraction, the area of $A D M P=\frac{5}{12}$ (area of $A B C D$ ).

Name: $\qquad$

## Square It Up-Homework

## Instead of $M$ being the midpoint of side $C D$, suppose $M$ cuts

 side $C D$ so that $M D=2 \cdot C M$, as seen to the right. What are the ratios of the areas of the four regions?The ratio of areas of region ADMP: APB: CPB: CMP is 11:9:3:1. The area of ADMP is $11 / 24$ of the area of the entire square The area of APB is $3 / 8(9 / 24)$ of the area of the entire square The area of CPB is $1 / 8(3 / 24)$ of the area of the entire square The area of CPM is $1 / 24$ of the area of the entire square

Justification
Since ABCD is a square, we know that $A B=B C=C D=A D$ and that $A B \| D C$ and $A D \| B C$.

We can prove that $\triangle A P B \sim \triangle C P M$. We know that $\Varangle A B P \Varangle \cong C M P$ since when parallel lines are cut by a transversal, alternate interior angles are congruent. We also know that $\Varangle A P B \cong \Varangle C P M$ since vertical angles are congruent. Therefore, by the angle angle similarity postulate, $\triangle A P B \sim \triangle C P M$. Since M divides $\overline{D C}$ so that $\overline{D M}$ is twice the length of $\overline{M C}$ and $D C=A B, M C=\frac{A B}{3}$. Therefore the ratio of the sides of $\triangle A P B: \triangle C P M=3: 1$, hence the ratio of their areas is 9:1.

Since $\overline{A C}$ is the diagonal in a square, we know that the sum of the areas of $\triangle A P B+$ $\triangle C P B$ is half the area of the entire square. We also know, that $\triangle M C B$ is one-sixth of the area of the entire square. We can let the area of $\triangle A P B=a$, the area of $\triangle B P C=b$ and the area of $\triangle M P C=c$. Then we know that $a+b=\frac{1}{2}(\operatorname{area} A B C D)$ and $c+b=$ $\frac{1}{6}(\operatorname{area} A B C D)$. Therefore by substitution $9 c+b=\frac{1}{2}($ area $A B C D)$ and hence $c=$ $\frac{1}{24}$ (area of $A B C D$ ). Then we know that $b=\frac{1}{8}$ (area $A B C D$ ) and $a=\frac{3}{8}$ (area $A B C D$ ). Then by subtraction, the area of $A D M P=\frac{11}{24}$ (area of $A B C D$ ).

Name:
Date: $\qquad$

## Seven Squares - Classwork

Warm-up: A square has a perimeter of 32 . What is the length of a diagonal of the square? Express your answer in simplest radical form.

Since the diagonal of the square forms a right triangle with the sides, we can use the Pythagorean Theorem to solve for the length of the diagonal, which we will call $d$. Therefore, we know $d^{2}=8^{2}+8^{2}$ and hence $d=8 \sqrt{2}$.

## Problem 1

A square, MATH, has a perimeter of 16. Another square has its vertices at the midpoints of the sides of the first. A third square has its vertices at the midpoints of the sides of the second. Continuing in the same way, there are 7 squares in total. Find the sum of the perimeters of each square.

When we connect the midpoints of a square to form another square, it forms right triangles. Therefore, can find the side of each subsequent square using the Pythagorean Theorem. Below is the length of each side with 1 representing the original square, counting up until the final square.

| Square Number | Equation to solve for side | Side Length | Perimeter |
| :---: | :---: | :---: | :---: |
| 1 | $32 / 4$ | 8 | 32 |
| 2 | $4^{2}+4^{2}=a^{2}$ | $4 \sqrt{2}$ | $16 \sqrt{2}$ |
| 3 | $(2 \sqrt{2})^{2}+(2 \sqrt{2})^{2}=b^{2}$ | 4 | 16 |
| 4 | $2^{2}+2^{2}=a^{2}$ | $a=2 \sqrt{2}$ | $8 \sqrt{2}$ |
| 5 | $(\sqrt{2})^{2}+(\sqrt{2})^{2}=b^{2}$ | $b=2$ | 8 |
| 6 | $1^{2}+1^{2}=c^{2}$ | $c=\sqrt{2}$ | $4 \sqrt{2}$ |
| 7 | $\left(\frac{\sqrt{2}}{2}\right)^{2}+\left(\frac{\sqrt{2}}{2}\right)^{2}=d^{2}$ | $d=1$ | 4 |

Sum of Perimeters:
$32+16 \sqrt{2}+16+8 \sqrt{2}+8+4 \sqrt{2}+4=28 \sqrt{2}+60$

## Problem 2

A square, GEOM, has a perimeter of $a$. Another square has its vertices at the midpoints of the sides of the first. A third square has its vertices at the midpoints of the sides of the second. Continuing in the same way, there are 7 squares in total. Find the side length and perimeter of each square

| Square Number | Equation to solve for side | Side Length | Perimeter |
| :--- | :---: | :---: | :--- |
| 1 | $\frac{a}{4}$ | $\frac{a}{4}$ | $a$ |
| 2 | $\left(\frac{a}{8}\right)^{2}+\left(\frac{a}{8}\right)^{2}=b^{2}$ | $\frac{a \sqrt{2}}{8}$ | $\frac{a \sqrt{2}}{2}$ |
| 3 | $\left(\frac{a \sqrt{2})}{16}\right)^{2}+\left(\frac{(a \sqrt{2})}{16}\right)^{2}=c^{2}$ | $\frac{a}{8}$ | $\frac{a}{2}$ |
| 4 | $\left(\frac{a}{16 \sqrt{2}}\right)^{2}+\left(\frac{a}{16 \sqrt{2}}\right)^{2}=e^{2}$ | $\left.\frac{a}{16}\right)^{2}=d^{2}$ | $\frac{a}{8 \sqrt{2}}=\frac{a \sqrt{2}}{16}$ |
| 5 | $\left(\frac{a}{32}\right)^{2}+\left(\frac{a}{32}\right)^{2}=f^{2}$ | $\frac{a}{2 \sqrt{2}}=\frac{a \sqrt{2}}{4}$ |  |
| 6 | $\left(\frac{a}{32 \sqrt{2}}\right)^{2}+\left(\frac{a}{32 \sqrt{2}}\right)^{2}=g^{2}$ | $\frac{a \sqrt{2}}{32}$ | $\frac{a}{32}$ |
| 7 |  | $\frac{a \sqrt{2}}{8}$ |  |
| 7 |  | $\frac{a \sqrt{2}}{8}$ |  |
|  |  |  |  |

Name:
Date: $\qquad$

## Seven Squares Homework

In class today you looked at finding the sum of the perimeters of seven squares whose vertices are midpoints of the other squares. How would the lengths of the sides change if the perimeter of the original square doubled?

Hint: Try doubling the perimeter of MATH or GEOM and compare the lengths of the sides to the original answers.

Answers will vary.

| Square Number | Side Length | Perimeter |
| :---: | :---: | :---: |
| 1 | 16 | 64 |
| 2 | $8 \sqrt{2}$ | $32 \sqrt{2}$ |
| 3 | 8 | 32 |
| 4 | $4 \sqrt{2}$ | $16 \sqrt{2}$ |
| 5 | 4 | 16 |
| 6 | $a=2 \sqrt{2}$ | $8 \sqrt{2}$ |
| 7 | $b=2$ | 8 |


| Square Number | Side Length | Perimeter |
| :--- | :---: | :---: |
| 1 | $\frac{a}{2}$ | $2 a$ |
| 2 | $\frac{a \sqrt{2}}{4}$ | $a \sqrt{2}$ |
| 3 | $\frac{a}{4}$ | $a$ |
| 4 | $\frac{a \sqrt{2}}{8}$ | $\frac{a \sqrt{2}}{2}$ |
| 5 | $\frac{a}{8 \sqrt{2}}=\frac{a \sqrt{2}}{16}$ | $\frac{a}{2 \sqrt{2}}=\frac{a \sqrt{2}}{4}$ |
| 6 | $\frac{a}{16}$ | $\frac{a}{4}$ |
| 7 |  |  |

Name: $\qquad$ Date: $\qquad$

## Once Upon a Time

Warm-up:
On an analog clock (see right), what is the measure of the angle formed by the hands at the following times?
a) $2: 0060^{\circ}$
b) $3: 3065^{\circ}$
c) $5: 4597.5^{\circ}$
d) $7: 10155^{\circ}$
e) $8: 3264^{\circ}$
f) $9: 09220.5^{\circ}$

**The minute hand moves $6^{\circ}$ every minute. The hour hand moves $0.5^{\circ}$ every minute.**

## Problems

1. The minute hand and hour hand on a clock form a $110^{\circ}$ angle. What time could it be? See list below
 hands on the clock form a $110^{\circ}$ angle?
(write your answers to the nearest second)
$12: 20,12: 45: 20,1: 25: 20,1: 50: 55,2: 30: 55,2: 56: 22,3: 36: 22,4: 01: 49,4: 41: 49$,
5:07:16, 5:47:16, 6:12:44, 6:52:44, 7:18:11, 7:58:11, 8:23:38, 9:03:38, 9:29:05, 10:09:05, 10:34:33, 11:14:32, 11:40

Name: $\qquad$ Date: $\qquad$

## Once Upon a Time Homework

Fill in the following table for the angles formed at each time.

| Time | Angle <br> formed by <br> hands | Time | Angle <br> formed by <br> hands | Time | Angle <br> formed by <br> hands |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $3: 00$ | $90^{\circ}$ | $4: 00$ | $120^{\circ}$ | $5: 00$ | $150^{\circ}$ |
| $3: 10$ | $35^{\circ}$ | $4: 10$ | $65^{\circ}$ | $5: 10$ | $95^{\circ}$ |
| $3: 20$ | $20^{\circ}$ | $4: 20$ | $10^{\circ}$ | $5: 20$ | $40^{\circ}$ |
| $3: 30$ | $75^{\circ}$ | $4: 30$ | $45^{\circ}$ | $5: 30$ | $15^{\circ}$ |
| $3: 40$ | $130^{\circ}$ | $4: 40$ | $100^{\circ}$ | $5: 40$ | $70^{\circ}$ |
| $3: 50$ | $185^{\circ}$ | $4: 50$ | $155^{\circ}$ | $5: 50$ | $125^{\circ}$ |

What patterns do you notice?
Answers will vary.

Name: $\qquad$ Date: $\qquad$

## Toothpick Math Classwork

1) Remove one toothpick to leave three squares.

2) Remove six toothpicks to leave four triangles.

3) Take away two toothpicks and leave two squares.

4) Make the fish swim the opposite way by moving three toothpicks and the coin.

5) Remove six toothpicks and leave two squares.


6) Move two toothpicks to make the pig go the opposite way.

7) Remove three toothpicks and leave three squares.

8) Move two toothpicks to get the ball out from between the posts.

$$
s t: d e \rightarrow \frac{Y_{0} 1}{T_{1}}
$$

9) Remove eight toothpicks and leave three squares.

10) Move four toothpicks and leave three equilateral triangles.


Donaldson, C (2014). Toothpick math. Retrieved from http://www.education.com/activity/article/Toothpick_Math/

Name: $\qquad$ Date: $\qquad$

Create your own toothpick problem.
The problem must fit the following criteria:

- The greatest number of toothpicks that can be used is 24
- You may use up to two coins in addition to the toothpicks
- The problem must have just one way of solving it
- The solution must be obtained from either moving or removing a given number of toothpicks
- You must draw the problem and the solution

Answers will vary

Name: $\qquad$ Date: $\qquad$

## Angles in Triangles Classwork

Warm-up: Find the measure of the angle marked $\theta$ in each triangle below.


Problem Set
In the following diagram, $A B=B C=C D$ and $A D=B D$. Find the measure of angle D .
$m \Varangle d=36^{\circ}$


Determine the measure of angle $x$.

$$
m \nleftarrow x=20^{\circ}
$$

See proof below


A This diagram is drawn to scale.

Prove $m \not \measuredangle x=20^{\circ}$.

Proof created based on a proof by Matthew Daly (2014)
Construct $\overline{D F} \| \overline{A B}$ and let the intersection of $\overline{B D}$ and $\overline{A F}$ be $G$.


| Statement | Reason |
| :---: | :---: |
| $\text { 1. } \begin{aligned} & \Varangle B A C=80^{\circ}, \\ & \Varangle A B C=80^{\circ}, \\ & \Varangle B A E=70^{\circ}, \\ & \Varangle D A E=10^{\circ}, \\ & \Varangle A B D=60^{\circ}, \\ & \Varangle D B E=20^{\circ}, \\ & \Varangle A C B=20^{\circ}, \end{aligned}$ | 1. Given |
| 2. $\overline{D F} \\| \overline{A B}$ | 2. Given |
| 3. $\Varangle A E B=30^{\circ}$ | 3. Angles in a triangle sum to $180^{\circ}$ (1) |
| 4. $\Varangle C D F=80^{\circ}$ | 4. When parallel lines are cut by a transversal, corresponding angles are equal in measure $(1,2)$ |
| 5. $\Varangle D F C=80^{\circ}$ | 5. Angles in a triangle sum to $180^{\circ}(1,4)$ |
| 6. $\Varangle B D F=60^{\circ}$ | 6. When parallel lines are cut by a transversal, alternate interior angles are equal in measure (1) |
| 7. $\triangle A C B$ is isosceles | 7. Triangles whose base angles have equal measures are isosceles (1) |
| 8. $\overline{A C} \cong \overline{B C}$ | 8. Isosceles triangles have congruent legs (7) |
| 9. $\triangle F D C$ is isosceles | 9. Triangles whose base angles have equal measures are isosceles $(5,6)$ |
| 10. $\overline{C D} \cong \overline{C F}$ | 10. Isosceles triangles have congruent legs (9) |
| 11. $\overline{A D} \cong \overline{B F}$ | 11. Partition postulate ( 8,10 ) |
| 12. $\overline{B A} \cong \overline{A B}$ | 12. Reflexive property |
| 13. $\triangle B A D \cong \triangle A B F$ | 13. SAS (1, 11, 12) |
| 14. $\Varangle B A F \cong \Varangle A B D$ | 14. СРСТС (13) |
| 15. $m \Varangle B A F=60^{\circ}$ | 15. Congruent angles have equal measures (14) |
| 16. $m \nrightarrow A F D=60^{\circ}$ | 16. When parallel lines are cut by a transversal, alternate interior angles are equal in measure $(2,15)$ |
| 17. $m \nrightarrow D G F=60^{\circ}$ | 17. Angles in a triangle sum to $180(6,16)$ |
| 18. $\triangle D F G$ is equilateral | 18. A triangle with all equal angles is equilateral ( 6,16 , 17) |
| 19. $\overline{D G} \cong \overline{F G} \cong \overline{D F}$ | 19. An equilateral triangle has all congruent sides (18) |
| 20. $\overline{C G} \cong \overline{C G}$ | 20. Reflexive Property |
| 21. $\triangle C D G \cong \triangle C F G$ | 21. SSS (10, 19, 20) |
| 22. $\Varangle D C G \cong \Varangle F C G$ | 22. СРСТС (21) |
| 23. $m \Varangle A C G=10^{\circ}$ | 23. If two congruent angles form an angle, each is half |


|  | the measure of the angle ( 1,22 ) |
| :---: | :---: |
| 24. $m \Varangle C A F=20^{\circ}$ | 24. Partition postulate ( 1,15 ) |
| $\text { 25. } \begin{aligned} \measuredangle C A F & \cong A B C \\ & \Varangle A C G \end{aligned} \begin{gathered} \\ \cong \end{gathered}$ | 25. Angles with the same measure are congruent (1,23, 24) |
| 26. $\overline{C A} \cong \overline{C A}$ | 26. Reflexive Property |
| 27. $\triangle A C G \cong \triangle C A E$ | 27. SAS $(25,26)$ |
| 28. $\overline{A G} \cong \overline{C E}$ | 28. СРСТС (27) |
| 29. $\triangle A C F$ is isosceles | 29. Triangles with base angles that are equal in measure are isosceles $(1,24)$ |
| 30. $\overline{A F} \cong \overline{C F}$ | 30. Isosceles triangles have congruent legs (29) |
| 31. $\overline{F G} \cong \overline{F D}$ | 31. Partition postulate ( 28,30 ) |
| 32. $\overline{F G} \cong \overline{F E}$ | 32. Transitive Property (19) |
| 33. $\triangle F E D$ is isosceles | 33. A triangle with two congruent sides is isosceles |
| 34. $\Varangle E D F \cong \Varangle D E F$ | 34. Isosceles triangles have congruent base angles |
| 35. $m \Varangle D E F=50^{\circ}$ | 35. Angles in a triangle sum to $180(5,34)$ |
| 36. $m \Varangle A E D=20^{\circ}$ | 36. Partition postulate ( 3,35 ) |

Name:
Date: $\qquad$

## Sectors in Circles Classwork

Warm-up: Circle A is shown to the right. The length of diameter $\overline{A B}$ is 8 . Find the area of each sector.
Area of circle $\mathrm{C}=16 \pi$
Sector CAE: $\frac{30}{360} \cdot 16 \pi=\frac{4}{3} \pi$
Sector CEF: $\frac{90}{360} \cdot 16 \pi=4 \pi$
Sector CFB: $\frac{60}{360} \cdot 16 \pi=\frac{8}{3} \pi$
Sector CBD: $\frac{60}{360} \cdot 16 \pi=\frac{8}{3} \pi$
Sector CAD: $\frac{120}{360} \cdot 16 \pi=\frac{16}{3} \pi$


## Problem Set:

Prove that the shaded area of the semicircle is equal to the area of the inner circle.


Let $r$ be the length of the radius of the semicircle. Then, $r$ is the diameter of the inner circle and $\frac{r}{2}$ is the radius of the inner circle. Therefore the area of the semicircle is $\frac{\pi r^{2}}{2}$ and the area of the inner circle is $\frac{\pi r^{2}}{4}$. The area of the shaded region is the difference of the area of the semicircle and the area of the inner circle and hence can be found as follows:

$$
\begin{gathered}
\frac{\pi r^{2}}{2}-\frac{\pi r^{2}}{4} \\
=\frac{\pi r^{2}}{4}
\end{gathered}
$$

Therefore, the area of the shaded area of the semicircle is equal to the area of the inner circle.

What percentage of the sector OAB is taken up with the inner circle?


Let $r$ be the radius of the circle and let Y be the center of the circle. Then OY can be found using the Pythagorean Theorem and therefore has length $r \sqrt{2}$. We then know that the radius of the quarter circle is $r \sqrt{2}+r=r(\sqrt{2}+1)$.

Then we know that area of the semi-circle is $\pi r^{2}$, while the area of the quarter circle is $\frac{\pi(r \sqrt{2}+r)^{2}}{4}$. Therefore, the area of the inner circle is $\frac{4}{3+2 \sqrt{2}}=$ 68.6\%.

The diagram below shows rectangle ABCD with height 10 cm . An arc with center at point B is drawn from point A to side $\overline{B C}$. An arc with center at point C is drawn from point D to side $\overline{B C}$. Given that the shaded regions a and $b$ have equal area, determine the length of $\overline{B C}$. Let the regions a and b have area $x$. Then we know that the area of both quarter circles is $\frac{100 \pi}{4}=25 \pi$ and hence the nonshaded region of each quarter circle is $25 \pi-x$. Therefore the area of the entire rectangle is $25 \pi-x+25 \pi-x+x+x=$ $50 \pi$. If the area of the rectangle is $50 \pi$ and the height is 10 , the length
 of BC must be $5 \pi$.

Name: $\qquad$ Date: $\qquad$

Cutting the Cake - Classwork

Warm-up: You order a pizza for your friends. The pizza has a diameter of twenty-four inches. If you cut the pizza into six equal slices, what will the area of each piece be?

Area of pizza $=144 \pi$
Each piece $=\frac{144 \pi}{6}=24 \pi$


## Problem Set

You are halfway through a party and twelve people are there. You decide to cut the circular cake with a diameter of 10 " into 12 equal pieces.

1. What is the area of the top of each piece of cake?

Area of cake $=25 \pi$
Area of each piece $=\frac{25 \pi}{12}$

2. Before you get a chance to serve the cake, twelve more people arrive. You already cut the twelve pieces out of the cake, but decide you can still make equal slices by cutting out a concentric circle in the cake to make 24 pieces. How far from the center of the cake should the circle be made so that all 24 people get the same amount of cake?

The inside must have half the area of the entire circle and therefore must have an area of $\frac{25 \pi}{2}$. We can solve for the radius of the inside portion of the cake as follows:

$$
\frac{25 \pi}{2}=\pi r^{2}
$$

Therefore the radius of the inside portion must be $\frac{5 \sqrt{2}}{2}$.

The cut should be made $\frac{5 \sqrt{2}}{2}$ inches from the center of the cake.

3. What is the area of each segment of cake? How much cake will each person receive?
Each person will receive $\frac{25 \pi}{24}$ square inches of cake.

## Cutting the Cake - Homework

As you are just about to make your concentric circle cut, one lady speaks up. "Women usually eat less than men. Since there are 12 women and 12 men here, maybe you should make the pieces for the men twice as big as the pieces of the women." You decide this is a good idea. How far from the center of the cake should the circle cut be made so that the outside pieces are twice as big as the inside pieces?


The inside must have one-third the area of the entire circle and therefore must have an area of $\frac{25 \pi}{3}$. We can solve for the radius of the inside portion of the cake as follows:

$$
\frac{25 \pi}{3}=\pi r^{2}
$$

Therefore the radius of the inside portion must be $\frac{5 \sqrt{3}}{3}$.

The cut should be made $\frac{5 \sqrt{3}}{3}$ inches from the center of the cake.

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