Building Conceptual Understanding and Algebraic Reasoning in Algebra

by

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1. Introduction

With the recent "algebra for all" standards sweeping the United States, many mathematics researchers and educators have begun to take a closer look at the complex processes involved in the learning and teaching of algebra. In light of recent international studies, national publications, and federal and state legislations, algebraic concepts and relationships are creeping their way into primary mathematics curriculums as well as expanding their breadth and depth in secondary curriculums. New standards and expectations in mathematics education have motivated researchers to reexamine how students learn algebra.

Success in algebra is widely regarded as the "gatekeeper" to success in high school and upper level mathematics. In our increasingly technological society, skills such as abstract thinking and problem solving are necessary to the 21st century workforce. Algebra is a medium through which students can build these skills. This paper will examine algebra from a number of perspectives: cognitive science, historical development, types of knowledge, teacher beliefs, and the impact of instructional materials. Each perspective will build off each other to form a comprehensive view of learning and teaching algebra.

The New York State Mathematics Core Curriculum is broken down into three main components: conceptual understanding, procedural fluency, and problem solving. These key components help to guide the aim of this paper, by providing an overarching themes for the various perspectives addressed. Conceptual understanding is referred to as "...the understanding of mathematical ideas and procedures and includes the knowledge of basic arithmetic facts ... [the ability to] identify and apply principles, know and apply facts and definitions, and compare and contrast related concepts." Procedural fluency is defined as "...the skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" and problem solving is outlined as "...the ability to formulate, represent, and solve mathematical problems" ("Mathematics", 2005). Conceptual understanding, and problem solving will be addressed by examining the development of abstract thinking. The relationship between and development of conceptual understanding and procedural fluency will be examined through a number of different studies.

With a solid cognitive foundation, this analytic review will examine how students learn to think abstractly, by examining theories of abstraction. These theories attempt to shed light on how students develop conceptual understanding, to enable students to make the transition from arithmetic to algebra. This transition will also be examined from a historical perspective, to gain insight into student misconceptions.

Following the theoretical basis for the study will be a number of studies which investigate the relationship and nature of procedural and conceptual knowledge in algebra. This knowledge will be examined from a learner's perspective, and from a teacher's perspective to examine the interfaces of each. Lastly, this analytic review will examine the influence that teachers' beliefs and text organization have over student performance in algebra.

perspectives that one may gain a deeper understanding of how to foster conceptual understanding, procedural fluency, and problem solving skills.

2. Cognitive Framework: Schema Theory

Debra Johanning and Diana Steele conducted a study in 2004 to explore how 7th grade students developed problem-solving schemas in a pre-algebra math class. The theoretical basis for this study provides a sound foundation upon which to examine algebraic thinking and abstract thought.

Algebraic thinking is defined by Driscoll as the "capacity to represent quantitative situations so that relations among variables become apparent" (as cited in Johanning & Steele, 2004, p. 66). An alternative definition cited by Johanning and Steele, was defined by Keiran as "the use of any of a variety of representations that handle quantitative situations in a relational way" (2004, p. 65). Essentially, algebraic thinking involves one's ability to represent quantitative relationships. In order to identify and represent quantitative relationships, one must be able to think abstractly about the "quantitative situations" before them.

Johanning and Steele explored how abstract thinking can be developed within the cognitive framework of schema theory. Schema is defined as "a mechanism in human memory that allows for the storage, synthesis, generalization, and retrieval of similar experiences" (Johanning & Steele, 2004, p. 66). If the human mind was thought of as a file cabinet, the files would be the schemas that store and organize our experiences and conceptions of the world around us. These files, or schemas, can be altered and reorganized as new information and experiences are added to them. This “reorganization” of schemas is referred to as assimilation and accommodation. Assimilation is the process of adding new information to existing schema, and accommodation is the alteration of an existing schema to fit new information or experiences. If new information or experiences cannot assimilate, or be altered to accommodate into existing schema, a new schema must be developed (Johanning & Steele, 2004).

New schema is not often developed after an isolated experience, but rather with repetitive exposure to similarly organized events or information. This is not to be confused with memorization. There are important differences between “meaningful schema development” and memorization (Johanning & Steele, 2004, p. 67). Memorization can occur as rote, piecemeal process, with little or no understanding of the information being memorized. Processes can be memorized; true understanding of mathematical concepts, however, requires meaningful schema development. To develop schema, an individual must have a tangible understanding of the concepts involved in order to identify and classify patterns, connect to prior knowledge, and assimilate new information (Johanning & Steele, 2004). Schemas are constructed vertically, by “adding new layers that form a deeper and narrowly connected hierarchy of knowledge” (Johanning & Steele, 2004, p. 67). This vertical “hierarchy of knowledge” is analogous to the structure and organization of mathematics, not in the sense that one area of mathematics is more important than another, but rather in the cumulative and interconnected nature of its existence.

Johanning and Steele connect schema theory with the development of algebraic thinking through the process of generalization. Mason contends that "[g]eneralization is the heartbeat of mathematics" and draws parallels between assimilation and accommodation in schema development, and generalization in mathematics (as cited in Johanning & Steele, 2004, p. 67). When students make generalizations about a concept or idea in mathematics, they are either expanding or restructuring their existing schema and subsequent understanding of that concept. Dubinsky purported that, "when an individual learns to apply an existing schema to a wider range of experiences, then the schema has been generalized" (Johanning & Steele, 2004, p. 67). Consider the following tasks from a study conducted by Breiteig and Grevholm:

(a) Eva is thinking of two numbers. The sum of them is 19. The difference between them is 5. Find the numbers.
(b) Why is it always possible to find the two numbers when we know their sum and difference? (2006, p. 2 - 227)

Task (b) is a generalization of task (a). As students make the algebraic "leap" from (a) to (b) they are expanding their existing schema and deepening their mathematical comprehension to a more abstract level. The next section will examine how such a "leap", or transition is made.

3. Abstraction

Analogous to the idea of generalization in mathematics is abstraction. Hershkowitz, Schwarz, and Dreyfus define abstraction as "an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure" (as cited in Hazzan & Zazkis, 2005, p. 103). This "vertical reorganization" becomes extremely important as students transition from arithmetic to algebra. How students make this transition and deal with abstraction has received a great deal of attention in recent years. As students move from arithmetic to algebra they are entering a new world of abstract thought, unknown quantities, and latent relationships. This new world is encrypted in a syntax of unfamiliar symbols and notation. Terms and operations that were once familiar and tangible, take on a deeper, more complex meaning. Anne Sfard and Liora Linchevski (1994) provide an example of the complexity of an algebraic expression, 3(x + 5) + 1:

> In certain situations you will probably say this is a concise description of a *computational process*. 3(x + 5) + 1 will be seen as a sequence of instructions: Add 5 to the number at hand, multiply the result by three and add 1. In another setting you may feel differently: 3(x + 5) + 1 represents a certain *number*. It is the product of a computation rather than a computation itself...If the context changes, 3(x + 5) + 1 may become yet another thing: a *function* – a mapping which translates every number x into another. This time, the formula does not represent any fixed (even if unknown) value. Rather, it reflects a change. (p. 191)

The multiple perspectives of 3(x + 5) + 1 offered in this vignette exemplifies a cross section of the depth of knowledge secondary students are expected to master in their mathematics classes. It is also an example of what mathematics education researchers refer to as process/object duality. Process/object duality is a student's ability to view a mathematical concept as both a process and an object (Goodsonresearchers refer to as process/object duality. Process/object duality is a student's ability to view a mathematical concept as both a process and an object (Goodson-Espy, 1998). For example, when 3(x + 5) + 1 is viewed as a sequence of instructions, "add 5 to the number at hand, multiply the result by three and add 1", it is seen as a process, as a string of commands. However, when 3(x + 5) + 1 is seen as a number or function, then it is a singular object, rather than a set of commands.

Traditionally, students view a mathematical concept as a process, before understanding it as an object (Hazzan & Zazkis, 2005). For example, when $y = 3x + 1$ is first introduced, it may be viewed as a computational process; substituting values for x to compute values of y. After repeated practice and exposure however, students may recognize $y = 3x + 1$ as an object: a linear function, or a line on a graph, rather than a set of commands. This recognition represents a deeper, more abstract understanding of the concept. The important juncture, however, is that students must be able to alternate between the operational (process) and structural (object) notions of algebraic concepts. Anne Sfard argues that "one needs to be able to think both operationally and structurally in order for one to develop meaning for higher level mathematical concepts" (as cited in Goodson-Espy, 1998, p. 223). Process/object duality in mathematics is a gradual evolution of abstract thought and algebraic reasoning. There are a number of theories that delineate how students develop this abstract reasoning within a mathematical framework. The following section will outline two major theories of abstraction, and their implications for mathematics education.

3.1 Theories of Abstraction

APOS theory and the theory of reification are two leading theories of abstraction within mathematics education. Both theories are closely related, and for the purposes of this paper will differ solely in terminology. They are both presented in this paper to validate the legitimacy of their interrelated infrastructure. These theories provide the framework upon which much other research concerning the development of conceptual understanding and abstract thought in algebra is built.

While initially used to examine how students learn abstract algebra in post-secondary courses, APOS theory and the theory of reification are also used to analyze how students transition from arithmetic to algebra. As defined earlier, abstraction deals with how one develops various levels of abstract thought to comprehend and internalize mathematical concepts. The realm of abstract thinking, however, extends far beyond the scope of mathematics, and is present in much of our daily lives. Ed Dubinsky (2000) parallels abstract thinking in society and abstract thinking in math and defines abstraction as, "any thinking that tries to 'deal with' phenomena to which we do not have access solely through our five senses but rather exist only in our minds and/or our interactions with others. By 'deal with,' I mean things like making sense out of, constructing (mentally), manipulating" (p. 3). APOS theory and the theory of reification shed light on the how one transitions from arithmetic to algebra, and how one develops abstract thought.

Much research shows that most students' difficulties with mathematics are rooted in their inability to work at the levels of abstraction necessary to understand certain algebraic concepts. APOS theory (actions, processes, objects, schemas) is a constructivist theory that illustrates how students may develop conceptual understanding in math.

The first stage in APOS theory is actions. To understand a math concept one must begin by performing actions, or operations on it. For example, when learning about negative numbers, one must practice adding, subtracting, multiplying and dividing negative and positive numbers to understand how they all interact, as well as the "rules" associated with each. Or, if given the equation $y = x^2$, a student may first compute values of x and y to gain an understanding of the relationship between to the two variables (Dubinsky, 2000).

After performing actions on a concept to familiarize oneself with it, a student may develop a higher level of understanding by viewing the concept as a process. In a process conception, students no longer have to perform explicit calculations, but can interiorize the actions, and imagine or run through them mentally. With the previous examples, a student at the process level of abstraction would be able to compute arithmetic with negative numbers mentally, or recognize the shape of a function by its equation (Dubinsky, 2000).

Once a process perspective of a concept has been developed, one may be able to delve deeper still into its algebraic structure. In order to do so, one must encapsulate the process to become an object, which can then be manipulated and altered by applying more actions and processes to it. For example, in order to invert or compose a function, one must first recognize the function as an entity onto itself, upon which actions may be performed. One may recognize $y = x^2$ as a parabola (object) that can be inverted or composed (processes) with other functions. At the object stage of conceptual development, one must be able to oscillate between process and object perspectives; hence process/object duality. In order to compose $y = x^2$ with another function, one must de-encapsulate it back into a process, perform the composition, and then encapsulate the end product as an object: a new function. Freudenthal speaks of this duality by characterizing mathematics as a "hierarchy of alternating perspectives". He states, "My analysis of mathematical learning process has unveiled levels in the learning process where mathematics acted out on one level becomes mathematics observed in the next" (as cited in Linchevski & Sfard, 1994, p. 194). Students at the object stage of abstraction are comfortable with various representations of a single concept, and know how to alternate between each (Dubinsky, 2000).

The final stage of abstraction is schema. As the name suggests, this is when a student develops a meaningful, complete schema of the math concept. Of course a schema may never be fully complete, but the math concept has been far more developed from when it was initially introduced. The schema is a collection of the various interpretations of the concept: its actions, processes, and objects, as well as its properties and relationships. All of this information has been stored, synthesized, generalized, and organized in such a way that is meaningful and relevant to the student. Thus, as students progress through the stages of abstraction, they are gradually building schema as they build their understanding of the concept. The final product is a holistic understanding, and organized schema of the concept at hand. With the case of $y = x^2$, once a student has reached schema level, they will have an overall understanding of parabolic functions, their properties and characteristics, the properties that distinguish parabolic functions from other functions, and many other nuances and relationships relevant to parabolic functions. APOS theory offers a framework for how one progresses through the various stages of abstraction, and strengthens their conceptual understanding in mathematics.

The theory of reification provides further support for the trajectory students follow in developing abstract thought in mathematics. Though the terminology is different, the progression is similar. The theory of reification posits the existence of three stages of concept formation in mathematics: interiorization, condensation, and reification. Interiorization is the initial stage in which students internalize a concept by performing operations on it until those actions become automatic. This is comparable to when a student's understanding of a concept progresses from actions to processes in APOS theory. The next stage is condensation. Condensation occurs when a "complicated process is condensed into a form that becomes easier to use and think about" (Goodson-Espy, 1998, p. 223). In this stage, the learner is able to generalize, make comparisons, and alternate between various representations of the concept. This is similar to the object stage in APOS theory. The final stage in the theory of reification is reification itself. Reification is achieved when "the solver can conceive of the mathematical concept as a complete object with characteristics of its own" (Goodson-Espy, 1998, p. 223). Similar to schema level in APOS theory, Linchevski & Sfard (1994) compares reification to when, "a person who is carrying many different objects loose in her hands decides to put all the load in a bag" (p. 198).

The theory of reification and APOS theory provide researchers and teachers with a better idea of how students understand mathematical concepts, and learn to think abstractly about the material. This information is particularly useful in examining how students transition from arithmetic to algebra. Learning to negotiate the many symbols and notation involved with algebra can be a large stumbling block for many students. These theories may help researchers and teachers understand how students can overcome these difficulties, and develop a deep understanding of algebra.

4. Transition from Arithmetic to Algebra

Trygve Breiteig and Barbro Grevholm (2006) conducted a study to examine if secondary students successfully transitioned from arithmetic to algebra. They analyzed students' answers to a specific question within the context of APOS theory. The following problem which was referenced earlier was given to 209 Norwegian 11th graders in 2005 (Breiteig & Grevholm, 2006, p. 2-227):

> Eva is thinking of two numbers. The sum of them is 19. The difference between them is 5.
> (a) Find the numbers.
> (b) How can you find the numbers?
> (c) Why is it always possible to find the two numbers when we know their sum and difference?

The problem represents a range of competencies and abstract thinking the researchers wished to measure amongst the students. Task (a) can be solved using a variety of numeric and algebraic approaches. Task (b) is designed to shed insight on students' awareness of the algebraic structure of the problem, and their ability to accurately articulate how they solved the problem. Task (c) offers the students an opportunity to generalize the problem, and deal with the concept of parameter.

The answers for each part were analyzed and categorized as correct, fail, and no answer; the researchers then further examined both the correct and incorrect answers to gain further insight into the students' thought processes. 73 percent of students answered part (a) correctly, 28 percent of students answered part (b) correctly, and 4 percent of students answered part (c) correctly, showing a steep decline in the number of students able to articulate and generalize their methods (Breiteig & Grevholm, 2006).

For those students who answered part (a) correctly, their explanations in part (b) fell into three different categories: explains the solution mainly verbally (21%), shows the calculation by numbers (6%), and shows the calculation by using symbols (1%). This demonstrates the students' strong preference for verbal, or rhetorical explanations, as opposed to numeric or symbolic. Some of the numeric explanations were a mixture of numbers and words, which is referred to as syncopated algebra; the least frequent explanation utilized symbols only, indicative of symbolic algebra and a deep understanding of the algebraic structure of the problem. Detailed examples of explanations offered for parts (b) and (c) are listed in the table below, as well as the APOS level the researchers assessed the students' comprehension to be at (Breiteig & Grevholm, 2006, p. 2 - 230).

Type of explanation or justification used by students, who gave a correct answer in (a)	*2005*	APOS level
1. Guess and check	72	Action
2. Table search: two conditions	13	
3. Calculate (19 - 5)/2, add 5 for the 2nd number	10	Process
4. Halving: 19/2 + 5/2 and 19/2 - 5/2	17	
5. One equation $x + (x + 5) = 19$ and solved	7	
6. Two equations $x + y = 19$; $x - y = 5$ and solved	11	
7. General solution $x = \frac{s+d}{2}$ $y = \frac{s-d}{2}$	9	Object
Others, no explanation	16	

Table 1: Ways of explaining in part (b) or (c) (when (a) is correct), number of students.

Guess and check, and using a table, were the two most popular strategies for solving the problem. This indicates an action level of abstraction, as the students performed operations on the numbers to determine the answer, as opposed to developing an algorithm. Guess and check and using a table are problem solving strategies that can be applied to a variety of problems; it does not signify an understanding of the underlying algebraic concepts Strategies 3 through 6 represent process level as students utilized a systematic mathematical process to solve the problem. The general solution shown in method 7 indicates the students' understanding of the algebraic structure of the problem. In this case, the students have generalized the problem for any sum and difference, and expressed it in terms of these parameters. This signifies an object level understanding of the problem (Breiteig & Grevholm, 2006).

The students who participated in this study had exposure to solving systems of linear equations prior to the experiment, though only a small minority utilized this method for solving the problem. The heavy reliance on guess and check and table methods indicates that most students had not successfully made the transition from arithmetic to algebra, as they were only at the action level of abstraction. In addition, less than 30 percent of students were able to explain or justify their answers in part (b). The researchers concluded that this sort of meta-cognitive activity—reflecting on and justifying one's answers—may be lacking in the classroom, and may contribute to the low number of students who fully transitioned from arithmetic to algebra. As is consistent with other literature in this domain, the researchers suggest that more practice with reflecting on, discussing, and writing how and why one solved the problem the way they did, would help students progress from arithmetic to algebra, and further develop their algebraic thinking.

Though these students had been exposed to algebraic methods for solving this problem, the results indicate that most of the students were at a low level of abstraction, relative to their instruction. This exemplifies the difficulty students have in developing a deep conceptual understanding of algebra, and making the transition from arithmetic to algebra. There is also an interesting parallel between the reliance on rhetorical explanations, and the historical development of algebra. The next section will examine the development of algebra from a historical perspective.

4.1 Historical Development of Algebra

"...the development of the long sequence of possible approaches to algebra and to its symbolic constructs took thousands of years. Today, to solve one little problem from a standard textbook, the learner must often resort to all the different perspectives together" (Linchevski & Sfard, 1994, p. 202). This point illustrates the divergence between the historical development of algebra and the logical development of algebra. As one writer put it, "it is but a myth that '[t]he [logical] structure of mathematics accurately reflects its history'" (as cited in Linchevski & Sfard, 1994, p. 195). Many authors caution that the history of algebra is not the history of symbols. As Linchevski and Sfard (1994) state, "...in history and in the process of learning mathematics, algebraic thinking appears long before any special notation is introduced" (p. 196). It is upon this perspective that researchers have compared student learning with the historical development of algebra.

As noted earlier there are many factors that contribute to students' difficulties with algebra, one of which being the symbolic nature of algebra. This point is illustrated in the previous study, where the majority of students provided rhetorical (verbal), as opposed to symbolic, explanations of their answers. This does not mean however, that the students' methods were void of algebraic thinking. Rhetoric algebra is the root of modern algebra, and had been practiced for centuries before symbolic algebra was developed (Linchevski & Sfard, 1994). Consider the following Babylonian problem and solution (Linchevski & Sfard, 1994, p. 196):

Babylonia, second(?) millennium B.C.

The problem:	Find the side of the square if the area less the side is 14,30 (the numbers are presented on basis 60).
Solution:	Take half of one, which is 0;30, and multiply 0;30 by 0;30 which is 0;15; add this to 14,30 to get 14,30;15. This is the square of 29;30. Now add 0;30 to 29;30, and the result is 30, the side of the square.

The solution is presented in rhetoric form, and the calculations are purely numeric instead of symbolic. Despite the absence of symbolic notation, there are still algebraic concepts involved in its solution. Today this problem could be solved using a quadratic equation; however manipulation of a quadratic equation to solve this type of problem took centuries to develop- this speaks to the complex and often mystifying nature of algebraic notation.

Algebra existed far before formal algebraic notation existed. As evidenced by the above problem, complex algebraic problems were solved verbally, or rhetorically, without the use of formal notation. Formal algebraic notation was developed out of necessity, as problems became too complex to address with traditional rhetoric algebra. The introduction of formal algebraic notation transformed the face of algebra from primarily operational to allow for a number of different perspectives of algebra, including: "(1) algebra as a generalized arithmetic, (2) algebra as a problem solving tool, (3) algebra as the study of relationships, (4) and algebra as the study of structures" (Van Amerom, 2003, p.64). This multiplicity of perspectives is the source of much of algebra's strength and flexibility, as well as the source of much confusion for students.

Algebra is often introduced as a discipline of formal symbolic notation. Researchers believe that this notation often runs counter to students' intuitive problem solving skills, which can stunt their conceptual understanding of algebra. For centuries rhetoric algebra was used to solve complex algebraic problems, which highlights the intuitive nature of such approaches (Linchevski and Sfard,1994). Barbara Van Ameron (2003) recommends utilizing students intuitive algebraic notions, such as rhetoric explanations of solutions, to build a deeper understanding of algebraic symbols and concepts. She contends that algebraic reasoning and algebraic symbolizing do not necessarily develop concurrently, as illustrated by the historical development of algebra. Linchevski and Sfard (1994) echo these sentiments, stating that, "The curriculum literally reverses the order in which algebraic notions seem to be related to each other, [and] the order in which they developed through the ages" (p. 224). Both authors contend that the historical development of algebra can shed light on some of the difficulties students may encounter when learning algebra. It is a valuable lens through which to examine student misconceptions, and perhaps use as a guide to build intuitive algebraic notions into deep mathematical understanding.

A similar disconnect between the development of rhetorical (verbal) algebra and symbolic algebra will later be presented in two studies by Kenneth Koedinger and Mitchell Nathan (2000). These studies take a slightly different perspective on the matter, but their findings are consistent with the idea that the historical development of algebra is an important resource for analyzing and improving algebra instruction to foster a deep understanding of mathematics.

5. Procedural and Conceptual Knowledge

Mathematics education must address not only conceptual understanding of the material, but also procedural fluency. To be successful in algebra one must know both why something works the way it does, and how. Two dominant types of knowledge in mathematics are procedural knowledge and conceptual knowledge. Procedural knowledge is defined as the "ability to execute action sequences to solve problems" (Alibali, Rittle-Johnson, & Siegler, 2001, p. 346). Conceptual knowledge refers to the comprehension of ideas or generalizations that govern a particular domain, and "connect mathematical constructs" (Capraro & Joffrion, 2006, p. 149). This paper has already addressed a number of ways to build conceptual understanding in math, but success in mathematics hinges on an individual's ability to connect concepts and procedures. Competence in one domain over another, results in significant gaps in understanding, and an unstable foundation upon which to build deeper mathematical knowledge.

In a study published in 2001, Martha Alibali, Bethany Rittle-Johnson, and Robert Sieger examined the development of conceptual and procedural knowledge, and the relationship between the two types of knowledge. Much previous research on this topic focused on which type of knowledge developed first: conceptual or procedural. Alibali et al., however, proposed that conceptual and procedural knowledge develop in an iterative fashion, with the development of one type of knowledge influencing the development of the other (Alibali, et al., 2001, p. 346).

They purported a cyclic development of conceptual and procedural knowledge, rather than unilateral.

With the iterative model, either conceptual or procedural knowledge may develop first. The initial development relies on the context of instruction, whether a conceptual or procedural approach is implemented. The iterative model suggests then, that as one type of knowledge develops the other does as well (Alibali, et al., 2001).

To examine the implications of the iterative model, Alibali et al. designed an experiment to measure the changes in conceptual and procedural knowledge of students after an instructional intervention. They worked with 74 5th grade students from two rural public elementary schools, to assess their conceptual and procedural understandings of decimals. The students took two pretests to measure their conceptual and procedural understanding of decimals, and then participated in a brief lesson on decimals with the researchers. After the lesson the students participated in game to reinforce the lesson, and then the students were presented with a conceptual and procedural posttest (Alibali, et al., 2001). The results were coded and analyzed to determine patterns in conceptual and procedural knowledge.

The results of the experiment were consistent with the iterative model. The analysis showed that students with strong conceptual knowledge at the pretest, showed a significant increase in procedural knowledge on the posttest. Alternatively, scores on the procedural pretest accurately predicted gains in conceptual posttest results. Therefore, students' initial conceptual understanding influenced learning of procedures, and students' initial procedural understanding influenced improvements in conceptual understanding (Alibali, et al., 2001).

Alibali, Rittle-Johnson, and Siegler urge that conceptual and procedural knowledge are not two separate entities, but instead should be incorporated into instruction in such a way that utilizes their bidirectional development. Many current reforms in mathematics emphasize conceptual development over procedural. Alibali et al. recommends giving students opportunities to reflect upon and communicate the math processes they employed in solving a problem to build both types of knowledge.

The findings from Alibali, Rittle-Johnson, and Sieger's study show that well-rounded mathematics instruction must build upon both conceptual and procedural knowledge. In a study of middle school pre-algebra students, Mary Capraro and Heather Joffrion examine how conceptual and procedural approaches influence students' understanding of translating verbal statements into mathematical expressions.

Capraro and Joffrion administered a 15 question assessment to over 650 middle school students dispersed between 25 different schools, to measure their facility at translating verbal statements into mathematics expressions (2006). The assessment was administered on a pre/posttest basis, with 3 of 15 questions on the posttest specifically analyzed for data (questions included in appendix). In addition, an error analysis was conducted on 60 of the incorrect responses, to gain further insight into the students' misconceptions. The researchers also interviewed five students to explain their solution strategies, which were then classified as either conceptual or procedural (2006).

Of the three test items, only 9% of the students answered all three correctly. The incorrect responses to the questions revealed a variety of conceptual and procedural errors. In examining student errors, 22% of incorrect answers showed evidence of a strong conceptual understanding, which may indicate an error in procedure. Alternatively, other students' errors revealed a reliance on procedure, without an understanding of the underlying concepts (Capraro & Joffrion, 2001).

These results further support the interconnected nature of conceptual and procedural knowledge. Capraro and Joffrion noted that a reliance on procedural knowledge of translating verbal statements prevents students from generalizing the process used to new situations, stating that "students may lack the cognitive structures necessary for the level of abstraction required by this skill" (Capraro & Joffrion, 2001, p. 162), these cognitive structures being schema. To further develop the schema, Capraro and Joffrion recommended gradual increased exposure to translation problems, as well as teaching students to check their equations to make sure they "make sense". Reflection on processes used and answers given is consistent with the suggestions that Alibali, Rittle-Johnson, and Siegler gave to increase conceptual and procedural knowledge.

6. Content Knowledge of Teachers

The focus on knowledge now shifts to the conceptual and procedural knowledge of mathematics teachers. Motivated largely by the Third International

Mathematics and Science Study (TIMSS), much research has focused on the relationship between teacher competence and student achievement. The TIMSS revealed significant discrepancies in mathematics performance between American students and students from Shanghai, Hong Kong, and Korea. In 2002, Frederick Leung and Kyungmee Park conducted a study measuring the competencies of elementary math teachers from Hong Kong and Korea, stating that "it is reasonable to expect that teachers' competence in mathematics and pedagogy should be a major factor in influencing student achievement" (Leung & Park, 2002). They modeled their study after large-scale study that Liping Ma conducted for her book, *Knowing and Teaching Elementary Mathematics.* Leung and Park investigated a small sample of East Asian teachers' knowledge of mathematics and pedagogy using the Teacher Education and Learning to Teach Study (TELT), to determine whether or not the teachers has a profound understanding of mathematics (Leung & Park, 2002).

After analyzing the data, Leung and Park concluded that most Hong Kong and Korean teachers "good grasp" of underlying elementary mathematics concepts, but did not have a profound understanding of fundamental mathematics (Leung & Park, 2002, p. 125). They also concluded that most teachers taught in a very procedural directed manner. Leung and Park attributed this procedural instruction to two main factors: lack of time and lack of desire. Under pressure to cover all the content, many teachers did not feel they had the time to teach conceptually. Additionally, many teachers expressed that a conceptual understanding of mathematics was not necessary for elementary students (Leung & Park, 2002).

Despite the procedural nature of instruction, East Asian students still greatly outperform their U.S. counterparts in mathematics. This may be due to the beliefs and philosophies their teachers hold about learning. Leung contends, "understanding is not a yes or no matter, but a continuous process or continuum" and that "procedural teaching does not necessarily imply rote learning or learning without understanding...the assumption that one must first understand before one can have meaningful practice may not be valid" (p. 127). Leung also contends that these practices must be based on a solid conceptual framework, within a well-designed curriculum. Leung and Park contribute Hong Kong and Korean students' success in math to procedural instruction embedded within a solid conceptual foundation, along with their teachers' "good grasp" of fundamental mathematics concepts.

7. Teacher Beliefs

Much like teachers' knowledge of a subject area, their beliefs about students' ability to achieve in that subject area also has a great influence on student performance. Kenneth Koedinger and Mitchell Nathan studied the relationship between teachers' beliefs about the development of algebraic reasoning, and students' performances. Koedinger and Nathan wished to enlighten current beliefs about the development of algebraic reasoning held by teachers and researchers.

Koedinger and Nathan had a group of teachers and researchers rank six different types of problems on their perceived difficulty level for students. The problems were classified by two different types of criteria: (a) the position of the unknown quantity in the problem, and (b) the linguistic representation of the problem

(Koedinger & Nathan, 2000, p. 169). For the first criterion, a problem could either be classified as a result-unknown problem, or start-unknown problem depending on the placement of the variable or unknown problem. Result-unknown problems were considered arithmetic in nature, while start-unknown problems were considered algebraic. Almost all teachers and researchers surveyed ranked the start-unknown problems as more difficult than the result unknown problems (Koedinger & Nathan, 2000).

For the second criterion, there were three different types of linguistic representation: story problem, word equation, or symbolic equation. Story problems were presented in a verbal format with relevant contextual information that could be used in solving the problem. Word equations verbally described the relationship between numeric quantities with no contextual information, and symbolic equations were strictly numeric equations. With both types of criteria, there were six different types of problems for teachers and researchers to rank (Koedinger & Nathan, 2000).

Most teachers and researchers ranked the start-unknown (algebraic) word and story problems as the most difficult, and result unknown (arithmetic) symbolic problems as the easiest. The arithmetic word and story problems were ranked as easy as well, with algebraic symbolic and algebraic story problems falling in the middle. Most teachers and researchers considered algebraic story problems more difficult than algebraic symbolic problems (Koedinger & Nathan, 2000).

The same problems were given to a group of students who had completed Algebra I. Their performance however, was not consistent with the teachers and researchers predictions. As determined by the percentage of students who answered the question correctly, the algebra symbolic problem proved to be the most difficult problem, with only 29% of students answering it correctly. The easiest problems were the arithmetic story and word problems with 80% and 74% of students respectively, answering the problems correctly. The symbolic arithmetic problem, which most teachers and researchers ranked as easiest, students found to be of medium difficulty level, as well as the algebraic story and word problems (Koedinger & Nathan, 2000).

The largest discrepancy between teacher and researcher predictions, and student performance, was the difficulty of symbolic algebraic problems over story and word problems. This discrepancy calls into question the traditional sequencing patterns in instruction, and actual algebraic development. From these results, the authors hypothesized two competing models in the development of algebraic reasoning; Symbol Precedence Model (SPV) and Verbal Precedence Model (VPM) The SVM suggest that students first develop the algebraic reasoning skills through symbolic arithmetic, and then extend those skills to algebraic symbolic problems. Development of verbal arithmetic and algebraic skills follow symbolic manipulation. The VPM outlines the reverse trajectory of development, suggesting that verbal competence in arithmetic and algebraic problems precedes symbolic competence. Koedinger and Nathan examined student solutions against the two models of development, and found that 91% of students fit the Verbal Precedence Model, 62%

of students fit the Symbolic Precedence Model, and 55% of students shared aspects of both (2000).

Overall, the study exposed a number of widely held teacher and researcher misconceptions about the algebraic reasoning development of students. The precedence of verbal algebraic development over symbolic algebraic development leaves some interesting questions about the sequencing of algebra instruction. Nathan further explores this area in a study on textbook organization.

8. Text Organization

Following his study on teachers and researchers beliefs about algebraic reasoning, Mitchell Nathan along with Martha Alibali and Scott Long examined the sequence of problem-solving activities in ten widely used pre-algebra and algebra textbooks. The authors investigated whether or not the text exhibited a symbolic precedence view of algebraic development (Alibali, Long & Nathan, 2002).

The authors chose ten textbooks that were used by the teachers who participated in the algebra ranking study. The textbooks included a pre-algebra and algebra text from each of the following publishers; Harcourt Bruce Jovanovich, Houghton/Mifflin, UCSMP, McDougal, Littell, and Glencoe. They examined the contents of the books, to determine the sequencing of the material. After analysis, the authors found that the texts all had a strong preference for the Symbolic Precedence Model of algebraic development; algebra topics were first introduced in a symbolic format, and then verbal. The authors also found that algebra texts were more inclined to strict symbolic preference than pre-algebra texts. This was consisted with the previous study, in which middle school teachers were more likely than high school teachers to predict symbol problems to be more difficult than symbolic. According to a belief survey, middle school teachers also held their students' problem solving intuitions at a higher regard than high school teachers, believing that they could invent effective non-symbolic problem solving methods. The authors attribute this discrepancy in beliefs to the fact that middle school teachers have more opportunities than high school teachers do to observe their students utilizing these intuitions as they transition from arithmetic to algebra (Alibali et al., 2002).

The symbolic precedence organization of the textbooks runs counter to Nathan's previous study that suggested that students' verbal reasoning skills develop before their symbolic. This discrepancy has important implications for instruction. The correlation between textbook structure and teacher beliefs indicates the influence that textbooks can hold over sequencing and instruction. Alibali et al. hold that "understanding the beliefs held by educators is central to the improvement of instruction" (2002, p. 16).

9. Conclusion

These studies and theoretical background tie in a number of relevant perspectives on the development of conceptual understanding and the development of abstract thought. They show important relationships between schema theory and abstraction, the historical development of algebra, conceptual and procedural knowledge, as well as teacher beliefs and student performance. These studies provide a well-rounded view of the various factors and influence how students learn algebra.

To begin, teachers must be aware of how students learn; through the vertical development of meaningful schema, after multiple exposures to similarly organized events and information. Building schema is far more complex than memorizing facts. Building schema requires an understanding, and internalization of the information, on behalf of the student. Teachers must shape instruction to foster such a development.

In shaping instruction, it is important to have an understanding of how students build their abstract thinking skills. The multiplicity of perspectives in algebra can be a stumbling block for many students, but utilizing students' intuitive problem solving strategies may help to overcome such difficulties. Teachers may look to the historical development of algebra for such guidance.

Teachers must also strive to have a balance of both a procedural and conceptual instruction. These two types of knowledge build together, not separately. Teachers and students must both be aware of how concepts and procedures relate. A reliance on one type of knowledge over another will result in significant gaps in student understanding and achievement.

As we strive for higher achievement and a more stable education system, we must also be aware of how our beliefs as teachers influence student performance. A willingness to reexamine these beliefs can lead to improved instruction. Collectively, these studies provide a comprehensive view of the ways in which educators can build conceptual understanding and abstract reasoning into their instruction.

10. Supportive Lessons

The following three lessons offer instructional support for the research outlined in this paper. The three lessons cover important topics in algebra: Lesson 1 investigates linear models, Lesson 2 explores the coordinate plane, and Lesson 3 takes a deeper look at the Pythagorean Theorem. The lessons also reflect a variety of pedagogical strategies to motivate and engage students. These lessons are intended to offer some ideas for how to build abstract reasoning, and conceptual and procedural fluency into instruction. They require the students to actively problem solve, investigate, and form conceptions and schemas along the way.

Lesson 1: Testing Paper Bridges – Investigating Linear Models

Rational:

As an introductory lesson to linear models and relationships, this group project is intended to give students the opportunity to build their own understanding of linear relationships through this experiment. The lesson begins by connecting students' previous understanding of what a "model" is, to an extended mathematical understanding of the term.

The students are working in groups to foster a strong learning community, and utilize each others strengths and intuitions in the project. Each student is designated a specific role, so that they can all play an active part in building their knowledge.

To familiarize themselves with models, students will actually perform "actions" on a mathematical model of a bridge that they have built themselves. The construction of the bridge gives student more ownership of their work. Following the collection of data, the students are prompted with a number of questions to reflect upon their work. Students are expected to reflect, search for patterns, and make predictions about the data. This probes the students to think deeper about the data, and build conceptual understanding.

After conducting the experiment and graphing their data, the students will share their findings with the entire class. This repetitive exposure to various linear models should help students begin to construct their schema of what a linear relationship is.

As an introductory lesson, the investigation focuses on building students conceptual knowledge of linear relationships, but supports with practice graphing (procedural). This will help students understand the various representations of linear relationships.

I. Anticipatory Set

Generate a discussion with the students about models and relationships in mathematics.

"What do you think of when you hear the word model? What different types of models are there? How could we use a model in math? How can we have a relationship in math? What could relationships in math be used for?"

II. Objectives

In this lesson, the students will construct a mathematical model of a bridge and analyze the relationship that exists between the thickness of the bridge and its strength. The students will work in cooperative groups to set up the mathematical model (the bridge) and collect, tabulate, and graph their data. This is an introductory lesson, intended to familiarize students with mathematical models, and linear relationships. By the end of the lesson, students will:

- Have a basic understanding of a linear relationship (Knowledge/Comprehension)

- Determine the independent and dependent variable in a given situation (Analysis)
- Collect and organize data into a table and graph (Application)
- Make conjectures based on data (Synthesis/ Evaluation)
- Work cooperatively in groups to complete the task

This lesson addresses the following **New York State Standards**:
8.PS.4 Observe patterns and formulate generalizations
8.PS.5 Make conjectures from generalizations
8.CM.4 Share organized mathematical ideas through the manipulation of objects, numerical tables, drawings, pictures, charts, graphs, tables, diagrams, models and symbols in written and verbal form
8.CN.4 Model situations mathematically, using representations to draw conclusions and formulate new situations
8.R.8 Use representation as a tool for exploring and understanding mathematical ideas

III. Input
Each student will get a packet which explains their task. Together the class will review the definition of linear relationship, independent variable and dependent variable. The class will go over the directions for the experiment. Before forming groups the class will discuss the different roles each group member will have in the group, as defined in their packets. The students will then count off to form groups of four, and determine within their groups what their roles will be.

IV. Model
The teacher will set up an example bridge in the front of the class. The teacher will model for the students step by step how to set up the bridge, both the right way and the wrong way. Before conducting the experiment, the teacher will ask the students to predict how many pennies the bridge will hold. The teacher will call a student volunteer to help with the experiment and run through the experiment twice, with a different number of layers each time. Possible discussion questions include:
"How should we add the pennies?"

"Should we reuse our paper bridges once they have failed?"

"Does it matter where we place our cup? Where we place our bridge? How will this affect our data?"

V. Guided Practice/ Check for Understanding
Once each group has collected their materials, they will begin their experiments. Each student will record their data in their packet. When the students have completed the experiment they will graph their data and answer the accompanying questions.

The teacher will circulate the room, monitoring student progress, assisting with questions. Possible guiding questions include:
"What patterns do you see in your data?"

"If you dropped the pennies from a different height how would that affect your data?"

"What types of material do you think would make a stronger bridge?"

"Does this pattern appear to be linear? Why or why not?"

VI. Evaluate/ Closure
The groups will share their data, and discuss what patterns they found in their data. The class will discuss whether or not they believe the relationship between the thickness of the bridge and its strength is linear, and defend their arguments. The teacher will review the characteristics of a linear relationship, and how linear relationships can be represented.

VII. Independent Practice
The teacher will review independent and dependent variable, and the components of a complete graph. For homework, the students will graph and label the data they collected and tabulated.

Name______________________________ Date___________

Group Project

Testing Paper Bridges:
Investigating Linear Models

In this project, each group will construct and analyze a **mathematical model** to determine the relationship that exists between the thickness and strength of a bridge. A **mathematical model** is a mathematical representation, such as a table, graph, or equation, of the relationship in a set of data. This project will investigate whether or not a **linear relationship** exists between the thickness and strength of a bridge.

Before you begin, let's review some important terminology:

- A **linear relationship** is a relationship where there is a ___________________________ between two variables. A linear relationship can be represented by a ______________, ______________, or an ______________________.
- An **independent variable** _____________________ rely on another variable; it belongs to the ___- axis.
- A **dependent variable** _____________ rely on another variable, it *depends* on another variable, and belongs to the ____-axis.

The Task:

Your group is going to conduct an experiment by building a paper bridge, and testing the strength of the bridge depending on its thickness. You will create a table and a graph of your data, and analyze your findings. In your analysis, you will identify patterns in your data, and use these patterns to make predictions about your bridge.

Materials:

- 16 paper strips for bridges
- about 50 pennies
- 1 small cup
- 2 sets of notebooks

The Group: Write each group member's name on the lines below.

______________________________ ______________________________

______________________________ ______________________________

Group Name:______________________________ (Come up with a name that is creative and catchy that your group can identify with.

Group Roles:

Each person in the group will assume one role. On the line below each role, write the name of the group member who will fulfill that role.

- **Go-Getter:** The Go-Getter will be responsible for getting all the materials for the group. This person will also be responsible for making sure that ALL materials are returned to their proper location at the end of the investigation.

  ____

- **Penny Dropper:** The Penny Dropper will be responsible for dropping the pennies into the cup at a constant rate in a consistent manner each time. They will also be responsible for keeping track of the number of pennies used in each trial.

  ____

- **Checker:** The Checker will recount the number of pennies it took to break the bridge (to double check the Penny Dropper). The Checker will also be responsible for setting up the bridge correctly for each new trial.

  ____

- **Recorder:** The Recorder will be responsible for collecting and recording the data from the experiment. The recorder should have neat, legible handwriting and will make sure that all other members of the group copy the data onto their worksheets.

  ____

Directions:

Step 1: The Go-Getter should pick up the paper strips for their group, and distribute them to his or her group members. Following the instructions on the sheet, each group member should fold the papers until all bridges have been made. Wait for further instructions.

Step 2: Once the bridges are made, the Go-Getter can pick up the rest of the groups' materials. The Checker should suspend one bridge between the two sets of notebooks. The bridge should overlap each set of notebooks by about 1 inch. Place the cup in the center of the bridge (as indicated on the bridge).

Step 3: The Penny Dropper should put pennies into the cup, one at a time, until the bridge crumples. The Recorder should record the number of pennies that was added to the cup in the table provided. This number is the **breaking weight** of the bridge.

Step 4: The Checker should now put two strips together to make a bridge of double thickness. Find the breaking weight for this bridge the same way. Repeat this experiment to find the breaking weight for bridges made from three, four, and five strips of paper. Record your findings.

Step 5: Once you have recorded all your findings, the Go-Getter should return all materials as they found them.

Data:

Thickness (layers)	Breaking Weight (pennies)

Questions:

Use your group's data above to answer the following questions.

1. Which column of data in the table (Thickness or Breaking Weight) represents the independent variable? Using complete sentences, explain how you know which is the independent variable.

____
____
____
____

2. Which column of data in the table (Thickness or Breaking Weight) represents the dependent variable? Using complete sentences, explain how you know which the dependent variable is.

____
____
____
____

3. Do you see any patterns in the data? Describe what patterns you see.

____
____
____

4. Do you believe that a linear relationship exists between the thickness of the bridge and its breaking weight? Explain in complete sentences why or why not. Is the relationship close to a linear relationship?

____

____

____

____

5. Based on your data, predict the breaking weights for bridges with 6 and 7 layers. Explain how you arrived at your answer.

____

____

____

____

Graph:

Using the attached graph paper, graph your group's data from the table.

Lesson 2: Review of the Coordinate Plane

Rationale:

This lesson is intended as a review of important concepts and terminology involved with the coordinate plane. It is essential that students are comfortable with the coordinate plane in order to understand higher-level mathematical concepts, such as functions. This lesson is also incorporating technology, the TI Navigator System and Smartboard to entice and engage the students.

One of the benefits of the technology is that students are able to explore the coordinate plane in an innovative way, alongside their peers. Each student will have an individualized cursor, which is displayed on the screen along with each of their classmates cursors. The teacher will give prompts in such a way to scaffold the students understanding of the coordinate plane, and examine it from a number of perspectives. The prompts represent a variety of levels from Bloom's Taxonomy.

Following the prompts the students will be asked to answer a number of questions on Quick Poll. The Quick Poll is a great assessment tool because it allows immediate feedback on student answers. After each person has answered the class will examine the results of the Quick Poll (which will be anonymous) and analyze misconceptions. This is a powerful tool for reflecting on one's own work.

For practice graphing, the students will choose which picture they would like to graph for homework. The pictures will be categorized by difficulty level, and students can pick accordingly. The homework is meant to reinforce the procedures from the day, and will be displayed on the walls in the classroom when finished.

Anticipatory Set:

"Take a moment to play with your calculators and practice moving about the graph. As you explore, I want you to think about what the different parts of a graph are; and some of the words we use in graphing. In a minute, I am going to ask you what you remember about graphing and we are going to see what you remember about the coordinate plane."

The students will use their graphing calculators to explore the coordinate plane. Their collective work will be displayed on the screen in the front of the room, where they will view their own movement, and the movement of their peers. The students will then follow a series of prompts pertaining to the coordinate plane.

Objectives:

- Given a series of prompts, the students will correctly identify the components of a coordinate plane: x-axis, y-axis, origin, quadrants I, II, III, and IV. ***(Knowledge)***
- Students will review the term ordered pair, and accurately plot four points in their notes following guided instruction from the teacher ***(Knowledge, Comprehension, Application)***

State Standards:

- A.CM.3 Present organized mathematical ideas with the use of appropriate standard notations, including the use of symbols and other representations when sharing an idea in verbal and written form
- A.CM.12 Understand and use appropriate language, representations, and terminology when describing objects, relationships, mathematical solutions, and rationale

Materials:

- TI-83 or 84 Plus graphing calculator
- TI-Navigator System
- SmartBoard
- Overhead and transparencies with questions
- Coordinate Plane Notes
- Graphing Worksheet

Purpose:

"Today we are going to review the basics of graphing as we begin our new unit. Before we get into the fun of graphing, we need to make sure that everyone is an expert on the different parts of a graph; so today we will get to practice labeling a graph and plotting points."

This is an introductory lesson to our unit on graphing lines. The material covered in this lesson should be review for the students, and is intended to overview important terminology, and let the students practice labeling a graph and plotting points.

Input:

- Students will log into their calculators and practice moving about the graph *(1 minute)*
- Students will find their individualized cursors on the activity center screen *(1 minute)*
- Students will follow a series of prompts and move their cursors to specific location on the coordinate plane, the prompts will guide a discussion of the components of the coordinate plane *(10 minutes)*
- Students will complete guided notes which summarize their discussion of the coordinate plane and plotting points *(10 minutes)*
- Students will converse with their group members and ensure that they are all "experts" on the coordinate plane *(2 minutes)*
- Students will set their notes aside and answer a number of "Quick Poll's" on their calculators *(5 minutes)*
- The class will analyze the results of the Quick Poll to assess their collective knowledge *(5 minutes)*

- Teacher will review the criteria for the homework, and students will chose which graphing activity they wish to complete for homework (easy, medium, or hard) *(2 minutes)*
- Students will begin their assignment *(remaining class time)*

Model:
The teacher will demonstrate how to label a coordinate plane, and offer a strategy to remember the quadrants. The teacher will also demonstrate how to plot an ordered pair.

Check for Understanding:
Quick Poll will be used to assess the students knowledge and comprehension of the coordinate plane and plotting points.

Guided Practice:
The class will complete guided notes with the teacher that summarizes their findings from exploring the coordinate plane.

Closure:
The students will assess and discuss the (anonymous) results of the quick poll. Students will analyze why certain misconceptions may have occurred.

Independent Practice:
Students will choose the difficulty level of their graphing homework (easy, medium, or hard) and begin it in class.

Assessment:
The Quick Poll will assess the students' knowledge and comprehension of the coordinate plane, and the graphing homework will assess the students' ability to accurately plot points and follow directions.

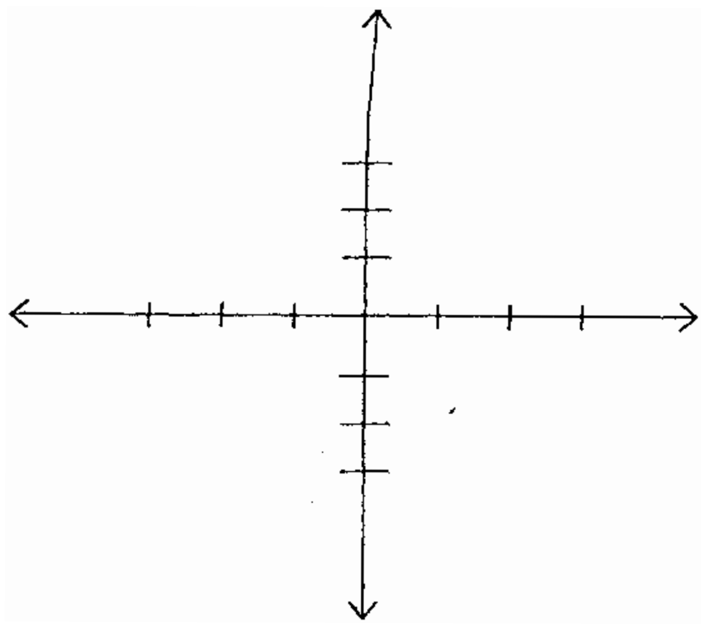
Questions:
- What would we label the origin, and why? ***(Knowledge, Evaluation)***
- What is special about the value of x and y in the first quadrant? ***(Comprehension)***
- In which region is the first quadrant? ***(Knowledge)***
- (2, -3) is in which quadrant? ***(Application, Analysis)***
- (-6, 7) is in which quadrant? ***(Application, Analysis)***
- (0, 4) lies on what line? ***(Application, Analysis)***
- In which quadrant are both the x and y values negative? ***(Knowledge, Comprehension)***

Nam Date_______

Unit 4: Graphing Lines

Day 1 - Coordinate Plane

COORDINATE PLANE:



Key Components:

- x-axis
- y-axis
- origin
- quadrants I,II, III, and IV

**The quadrants begin in the ______________________________ and continue

________________________. ((C-Rule))

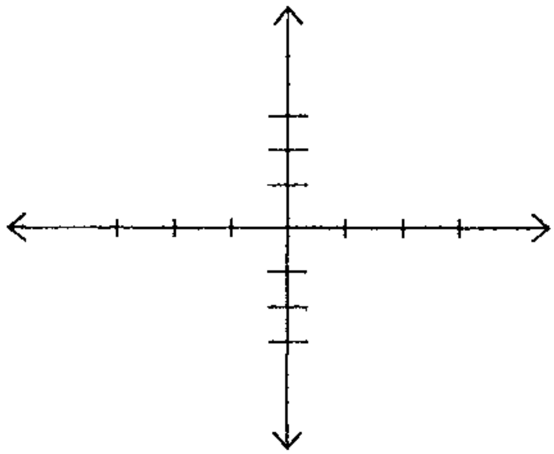
PLOTTING POINTS:

- Points on a graph are called ________________ ______________
- An ordered pair is made up of an ____________________ and

____

Ex:

Plot and label the following points on the graph.



A(2, 1)
B(-1, 2)
C(-2, -2)
D(1, -2)

Lesson 3: Exploring the Pythagorean Theorem

Rationale:

This lesson is intended to help students to understand the geometric and algebraic connections in the Pythagorean Theorem. By connecting these two areas of math students will have a deeper understanding of the theorem, and appreciation for its mathematic significance.

The students will use the internet to explore various applets of the Pythagorean Theorem. The lesson is guided by questions which probe the students to think deeper about the concepts involved. The students are required to make observations, search for patterns, and draw conclusions based on their explorations. They must reflect upon and articulate these findings on the packet provided. This lesson is also designed for students to move at their own pace through the material.

1. Anticipatory Set

Draw a right triangle on the board, ask the students to draw it in their notes and identify it.

Give the students one minute to write in their notebooks everything they know about right triangles.

Have students share what they recorded in their notebooks, and record it on the board. Make sure concepts such as hypotenuse, legs, and Pythagorean Theorem are included.

Lead students to discussion on Pythagorean Theorem. Explain that we are going to explore what this theorem really means.

II. Objectives

The objective of this lesson is to help student gain a deeper understanding of the Pythagorean Theorem. Students must be able to understand both how and why the Pythagorean Theorem works, and be able to communicate it in words, symbols, and pictures. Students will also explore their understanding of the theorem to incorporate other mathematical concepts such as transformations and fractals. The objectives of this lesson correlate with the following New York State Standards:

9.PS.3 Understand and demonstrate how written symbols represent mathematical ideas

9.CM.4 Share organized mathematical ideas through the manipulation of objects numerical tables, drawings, pictures, charts, graphs, tables, diagrams, models and

symbols in written and verbal form
9CM. 11 Draw conclusions about mathematical ideas through decoding, comprehension, and interpretation of mathematical visuals, symbols, and technical writing
G.G.43 Investigate, justify, and apply the Pythagorean theorem and its converse

III. **Purpose**
The purpose of this lesson is to help students understand the geometric representation of the Pythagorean Theorem. By understanding what this theorem means, students will gain a deeper appreciation for the geometric and algebraic connection. Students need to seek to understand important mathematical concepts, not simply memorize them.

IV. Input
Discuss with the students what the Pythagorean Theorem means to them. Ask them to explain, in their own words, what the Theorem means. Now, we will try to represent this geometrically. On the board, and in the students' notebooks, we will break down the Pythagorean Theorem piece by piece. Ask the students what each term means, and how they can represent that with a picture. May use guiding questions such as;

"What does the term a^2 mean? How can we represent this with a picture?"

Draw a diagram on the board corresponding to each term in the theorem. Once the diagram is completed, ask the students if the Pythagorean Theorem has taken on new meaning, and what it now means to them.

Using the Internet, we will now explore if this theorem holds true for all right triangles.

V. **Model**
Hand out Exploring the Pythagorean Theorem packet.

Have students draw the diagram from their notes on the front page of the packet.

Direct students to the website indicated, click on Pythagorean Theorem (1). Go through question 1 a with the students. Then, the students will complete the rest of question I individually.

Once students have completed question 1, we will go over their findings as a class. Have students share their answers allowed, and summarize the findings.

VI. **Guided Practice**

After discussing student findings, students will then explore the extensions following question 1. Allow students ample time to explore at their own pace. Circulate the room, answer students' questions, and provide guiding questions. The depth to which the students explore will vary depending on their background knowledge. Help students make connections between the Pythagorean Theorem and other math concepts.

VII. Closure

Once students have completed the rest of the packet, reconvene as a whole class and have students share their findings. Extend the discussion to other concepts such as transformations and fractals, ask students when and where they have seen these other concepts. Discuss how these concepts relate to the Pythagorean Theorem. Have students reflect in their notebooks what new information they have learned about the Pythagorean Theorem; what they found clear, and what they found confusing.

VIII. Independent Practice

Hand out application worksheet for homework. If there is time left in the class, students may begin working on it. The worksheet will be collected during the next class period.

*A mathematical formula is meaningless unless you can understand **how** and **why** it works.*

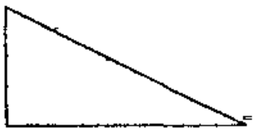
Name:___________________________ Date:_______

Exploring the Pythagorean Theorem

www.ies.co.jp/niath/jva/geo/pythigoras.html

Draw the diagram from your notes below

$a^2 + b^2 = c^2$



1. Pythagorean Theorem (1)

a. In the diagram shown, what is the hypotenuse labeled? How do you know this is the hypotenuse?

b. In the applet, click on the red dot and drag it around. What happens to the diagram? What changes, what stays the same?

c. Press the Define button and drag the pieces into the empty square. What happens?

What does this represent?

d. Press the Init button and drag the red dot to a different position. Follow the same procedure, and record your observations below. Repeat until you have found consistent results.

Based on your observations, what can you conclude about the Pythagorean Theorem? Answer in complete sentences.

____
____
____
____
____

Extensions

2. Pythagorean Theorem (3)

a. In Quadrilateral mode, follow the directions to the applet. Record your observations below.

What can you conclude?

b. Now try the applet **in Triangle mode. Record** your observations below.

What can you conclude?

c. What is the difference between Quadrilateral and Triangle mode?

3. **Pythagorean Theorem (4)**
   a. Observe (4). What is the difference between (3) and (4)?

4. **Pythagorean Theorem (5)**
   a. Try (5). What geometric concepts are examples (5), (4), and (3) demonstrating?

5. Explore some of the other applications on this website, in particular; Hyppocrates' Lunar, Pythagoras Tree, Minimum Distance and Shortest Distance.
   How do these applications relate to the Pythagorean Theorem? What other mathematical concepts are involved?

For some other interesting puzzles and demonstrations, check out
http://nlvm. usu.edu/en/nav/vlibrary.html
Click on the box in the grid for Geometry and 9—12
Scroll down and click on the link for Pythagorean Theorem
Enjoy!

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