Advancing Student Learning in Trigonometry:

Purposeful Questioning and Investigative Problem Solving in an Inquiry-Based Classroom

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Abstract

This curriculum project was created by a teacher, for teachers, focusing on purposeful questioning for a trigonometric circular motion unit in an inquiry based Algebra II classroom. The curriculum project includes investigations that delve into the patterns of change of a coordinate point, and rotation of the central angle in circular motion. A hands on activity connects the unit circle to the sine and cosine curve. Purposeful questioning pushes students’ problem solving skills, and deep understanding of the material to further student learning. The curriculum is aligned to the Common Core State Standards for trigonometric functions.
Introduction

Purposeful questioning is critical to meaningful learning in the mathematics classroom. Teachers must prompt students with purposeful questions that further propel student learning and understanding. When extending thinking, students create meaning of the mathematical concepts and increase their base knowledge of material. With deep understanding comes the ability to apply knowledge to other situations. Cooperative group work keeps students accountable for their learning. Collaboration also strengthens student explanations, and discovery through multiple strategies (Barron & Darling-Hammond, 2008, p.11). Through questioning, explaining and collaboration, twenty-first century skills are strengthened, increasing ability to work as a team, to solve real-world, authentic problems, and to express one’s thinking.

This curriculum project is built to produce students that are successful problem solvers. It provides teacher questions to push student understanding in the topic of circular motion in trigonometry. Students will work through investigations collaboratively, working with small groups, and summarizing with the large group. The teacher will provide purposeful questions throughout. Students will study the pattern of motion a coordinate point takes while moving around a circle. The students will focus on the measure of the central angle. A culminating project is provided to push students’ understanding of how the sine and cosine curve are derived from the motion around the unit circle. They will connect specific points, and patterns of change back to the motion of the coordinate point around the unit circle as the central angle of rotation increases. Students will create and use a hands on manipulative to study these concepts.

The author is a teacher in a school where the mathematics curriculum is primarily inquiry based. Through experience in inquiry based teaching, the author has created a curriculum to push the mathematics to motivate student learning. The ideas behind this curriculum project is that if the students understand the importance of the material that they must learn, they will be more motivated in learning it. The following curriculum is provided to support instruction and to make the learning of math more
meaningful to students. This curriculum project focuses on utilizing a collaborative, inquiry based approach where students participate in mathematical discourse through teachers providing purposeful questioning. The goal is to get the students to understand the meaning behind the mathematics and seek generalized understandings of what the answers mean.

**Literature Review**

**Trigonometry Standards**

The Common Core State Standards (CCSS) for Trigonometric Functions (F-TF.1 and F-TF.2) state students must extend the domain of trigonometric functions using the unit circle. This includes understanding radian measures, and explaining how the unit circle can be used to extend all trigonometric functions to real numbers. The Next Generation Math Standards for New York State extend Trigonometry concepts from Geometry G-SRT.6 and G-SRT.7 and apply the same concepts stated above to Algebra II.

**A Background on Inquiry Based Learning**

Boaler (2002) examined how different teaching approaches effect knowledge, practice, and student understanding. Two schools with similar student populations and resources but with differing methods in which they teach mathematics were compared. The teachers at the first school, Amber Hill, taught mathematics with a traditional approach; demonstrating practice problems, followed by students practice the same problems. At the second school, Phoenix Park, teachers use multiple week long, open-ended projects where students work with small groups to teach mathematical concepts.

Boaler determined that students at Amber Hill did well when given textbook exercises. However, they struggled to use mathematics in “open, applied or discussion based situations” (Boaler, 2002, p.3). In comparison, the students from Phoenix Park were able to apply their learning to different types of situations, including conceptual questions as well as real world assessments. They outperformed the Amber Hill students on the National Exam (Boaler, 2002, p.3). Boaler also concluded that not only is a knowledge base formed in a situation like this, but also a set of practices that are used to apply knowledge. For the students from Phoenix Park, they were capable of applying mathematics in a variety
of situations. This is largely due to the fact that in the classroom, students engaged in a set of practices that were found in real life. In the end, they gained a deeper understanding of the material (Boaler, 2002).

Boaler (2002) promotes the idea that students should be working collaboratively to share thoughts, questions, and processes to solve problems. When students learn using an inquiry based approach, they were more willing to grapple with problem solving and in the end outperformed the student who learned the material in a traditional setting. This is due to the fact that the student will “try different methods, garner helpful resources and make use of the knowledge and practices she has learned” (Boaler, 2002, p.5).

Students must develop higher-order thinking skills in preparation for success in the chosen career path or post-secondary education. Such preparation should include engaging topics in real-world, authentic situations. Student collaboration should be a focus that leads to appropriate communication regarding topics in the curriculum. This communication should lead to students constructing and organizing their thinking. Additionally, students should be able to critically analyze their work, and the work of others in search of alternative approaches. These skills can be developed and honed through inquiry-based learning; specifically, project-based learning, and small group work (Barron & Darling-Hammond, 2008).

Inquiry Based Learning is an instructional practice where students explore the material being taught by investigation. The classroom is student centered for the students to pose and answer questions in a collaborative setting to construct their own meaning from the material being presented to them. The teacher’s role is not to demonstrate and lead instruction, but instead to guide and promote learning through student probing. The teacher should prepare questions that make the students reflect on their thought process. The teacher should also provide immediate feedback to students throughout the lessons (Caswell & LaBrie, 2017, p.4). One way of implementing an Inquiry Based environment is through Problem-Based Learning. Here, students are given problems, commonly without one specific answer, or that can be solved with multiple approaches. Students should evaluate what must be done in order to
accurately solve the problem, create strategies to do so, evaluate and then make appropriate changes to these strategies (Barron & Darling-Hammond, 2008, p.3).

Another aspect of Inquiry Based Learning is small group work. This can be defined as “students working together in a group small enough that everyone can participate on a collective task that has been clearly assigned” (Barron & Darling-Hammond, 2008, p.5). There are many benefits to working together in social and behavioral areas as well, including social interaction, student’s perception on self-worth, and positive feelings towards peers. Academically, the benefits also abound. Group work allows students to explain their own reasoning and strategies, as well as listen to other approaches to solving a problem. The students then can resolve differences in their problem solving through productive argument, as well as critiquing other’s ideas backed with reason (Barron & Darling-Hammond, 2008, p.10). Ultimately, group work allows students practice in working together appropriately to problem solve, which provide skills they will need throughout their lives. Additionally, it allows the students practice with explaining their mathematical reasoning. Effective group work can be difficult to implement and requires teachers’ knowledge and skill to manage well. Teachers must create classroom norms and accountability structure where it is not enough to just tell the students to work together, but the students must have a reason to take each other’s achievement seriously (Barron & Darling-Hammond, 2008, p.11).

The Importance of Purposeful Questioning

Mathematical discourse is important in any mathematics classroom. Classroom discourse is often thought of as the communication that occurs there. This can be extended to include representing ideas through multiple depictions, interpreting answers and methods, expressing thoughts, reflecting on work and strategies, and debating and explaining approaches. Not only does this discourse help student thinking, but also it helps to deepen it, as the students are required to go back and provide evidence to their thought process. Discourse with other students and teachers strengthen the interactions revolving around mathematics (Seely, 2017, p.2). However, with the stress of reflection, curiosity, and thought promotion of an Inquiry Based Classroom, purposeful questioning becomes valued. Questions are highly
important in determining what a student understands about the material being taught to them. Specifically, “purposeful questions should reveal students’ current understandings; encourage students to explain, elaborate, or clarify their thinking; and make the mathematics more visible and accessible for student examination and discussion” (Schwan, Raith, & Steele, 2017, p.80).

To begin, it is thought that most teacher questions should be open ended. This allows for students to elaborate, or explain their response, as well as shed light on their mathematical reasoning. There are four major categories of questioning to take place in a mathematics classroom. The first, gathering information questions, have students recall facts, definitions, or procedures that they may have previously learning. These are not open-ended questions. The second, probing thinking questions, which have students describe, add on, and clarify their thinking. Next, making the mathematics visible questions encourage students to talk about the mathematical structures shown in problems and make connections. Lastly, encouraging reflection and justification questions allow students to show understanding of their rational and to argue the validity of their work (Schwan, Raith, & Steele, 2017, p.81).

Further, there are two patterns of asking questions. A teacher can use the funnelling technique, which is when you use a series of questions to lead students to a desired answer. Answers that do not lead to the anticipated answer are not given much attention so that the conversation, and thus conclusion, is controlled. Unfortunately, this type of questioning does not allow for much time for students to build their own, individual meaning-making of the material. The other technique is focusing. Here, the goal should not be directed to one specific idea, but instead allow an open space for conversation and explanation of many problem solving methods. The teacher should ask many open ended questions in the hopes that students communicate and reflect on their own thinking, as well as the thinking of others in the classroom (Schwan, Raith, & Steele, 2017, p.82).

There is an additional break down of types of questions. First, assessing questions show a student’s knowledge by exploring the student’s thinking or gathering information. This allows the teacher to determine what the student knows about important mathematical ideas. Advancing questions can be
used to build upon current understanding of material. These questions help the students move beyond their present knowledge, and by supporting reflection and justification, take them to the end goal or solution (Schwan, Raith, & Steele, 2017, p.85).

Clearly, purposeful questioning is complex and takes a lot of preparation. However, the impact of good mathematical discourse and questioning is extremely beneficial to a student’s learning experience.

**Appropriate Assessment in the Inquiry Environment**

An important aspect of the Inquiry Based learning environment is the types of assessment given. Following the concepts of a student-centered classroom, a very important aspect of formative assessment is feedback, including both peer assessment, and self-assessment. Feedback has numerous positive attributes that lead directly to forming strong habits of mind. For example, feedback helps students to realize their mistakes, and motivate them to improve. Peer assessment benefits students as well by encouraging them to contemplate other suggestions, as well as make decisions, receive and give critique appropriately, and to engage in productive discussion with others. Ultimately, being able to work with others is critical to success in today’s world. Finally, self-assessment allows students to make connections between prior knowledge and new contexts. (Gloria, Sudarmin, Wiyanto, & Indriyanti, 2018. p.3). It is imperative that the teacher provides feedback to students throughout the unit of instruction. Observing student interactions, explanations, productive arguments, and questioning are all excellent means of providing feedback to students in an environment that makes students confident to share out their thinking.

Boaler and Anderson (2017) advances our thinking on feedback. It is important that mistakes are an extremely important part of the learning process, and that the brain has larger growth when mistakes are made due to the pattern that synapses take. Thus, “there is greater brain activity and growth when people have a growth mindset than when people have a fixed mindset.”(Boaler, & Anderson, 2017, p.2)
ADVANCING STUDENT LEARNING IN TRIGONOMETRY

This all leads to the idea that many teachers “use summative assessment formatively—that is, they give student’s scores or grades, which summarize their learning, when students are in the middle of the learning processes” (Boaler, & Anderson, 2017, p.3) This is a problem because if assessment is used to benefit learning, students need to be aware of where they are now, where they need to reach, and what needs to be done to close the gap between the two places. By grading students, it doesn’t give them information on what they do or do not know and it doesn’t help them with what they still must learn. Instead, it tells them how they are doing in relation to other students. (Boaler & Anderson, 2017, p.4)

Alternatively, teachers should provide feedback that “highlights ways students can build on their current understandings” (Boaler & Anderson, 2017, p.4). This feedback can be specific to one area of the assignment, or broad to cover general conceptual errors, but it needs to allow for student’s to process what they did incorrectly, and give them time to fix their mistakes (Boaler & Anderson, 2017, p.5).

Curriculum Project

The following curriculum project was designed to support critical questions and the understandings and uses of the unit circle, radians, and circular motion curriculum in Algebra 2. Within this curriculum project, purposeful questions have been created to push student thinking and motivate learning within inquiry-based lessons and activities.

The curriculum project was inspired by the Core-Plus Mathematics Project (CPMP). Some of the materials are from the Core-Plus Course 3 Textbook. Much of the work has been created by the author. The Core-Plus Mathematics Project is an inquiry based, problem focused curricula that is aligned with the Common Core State Standards (CCSS) for mathematics. The investigations place standards in real world contexts, and the problems were designed to engage students to productively struggle and reason through mathematical concepts. The Core-Plus textbooks contain investigations that are meant to be completed by groups where students work collaboratively to reason through the problems, and with it, concepts. Additionally, the following curriculum has a discovery piece that applies the Navigator system on
students’ TI-Nspires. This use of technology allows students to rapidly manipulate visual models to determine patterns. There are also additional discovery-based questions that fit well in the problem-centered classroom and supports student to student, and student to teacher communication. All interactions, both student to student, and teacher to student, are based upon reasoning with problems, making and refining conjectures, questioning one another, and reflecting on work (Core-Plus Mathematics Project, 2019). Students are encouraged to use multiple representations to work through the mathematics until they find methods that enable them to make progress in their work.

Teacher use of purposeful questioning can support students’ reasoning with mathematical concepts and ideas presented in the Core-Plus’ text book questions, and that is one purpose of this project. The other is to examine how the connections between the unit circle and the sine curve can push student learning of foundational trigonometric concepts. The hands-on activity is designed to solidify students’ understanding of degrees and radians regarding placement along the unit circle. This activity will relate to the graphs of the sine and cosine curves.

The curriculum project below is created to take 11 class periods that are 45 minutes long. The work compiles two investigations and a culminating project that is nestled into the large Circular Motion unit. It comes after a short investigation on angular velocity and linear velocity, and before an investigation on graphing and transforming trigonometric functions. All purposeful questions are bulleted. Typical student answers are italicized. Notes to teachers are italicized.
### Curriculum

#### Modeling Circular Motion

The Ferris wheel was invented in 1893 as an attraction at the World Columbian Exposition in Chicago, and it remains a popular ride at carnivals and amusement parks around the world. The wheels provide a great context for study of circular motion.

Ferris wheels are circular and rotate about the center. The spokes of the wheel are radii, and the axis are like points on the circle. The wheel has horizontal and vertical lines of symmetry through the center of rotation. This is a natural coordinate system for describing circle motion. As you work on the problems in this investigation, look for answers to the following questions:

1. What are the coordinates of a point on a rotating circle object?
2. How are they determined from the radius and angle of rotation?

**Figure 1**: Hirsch et. al. (2015) p.425

#### Coordinates of Points on a Rotating Wheel

To aid your thinking about positions on a rotating circle, it might be helpful to make a Ferris wheel model that uses a disk to represent the wheel with $x$- and $y$-coordinate axes on a fixed background.

Connect the disk to the coordinate axes background with a fastener that allows the disk to turn freely while the horizontal and vertical axes remain fixed in place.

![Diagram of Ferris wheel](image)

Imagine that your model represents a small Ferris wheel that has radius 1 decimeter (about 3 feet) and that your seat is at point A when the wheel begins to turn counterclockwise about its center at point C.

- a. How does the $x$-coordinate of your seat change as the wheel turns?
- b. How does the $y$-coordinate of your seat change as the wheel turns?

**Figure 1**: Hirsch et. al. (2015) p.425

### Purposeful Questions

- In this unit, we will be thinking about the seat on a Ferris wheel going around and around. We will devise a way that will allow us to find the position of the seat at any time during the ride.
- Has anyone been on a Ferris wheel before?
- Where have you gone on the Ferris wheels?
- Do you think that all of those Ferris wheels were exactly the same?
- How did they differ?
  - **Height, speed (angular velocity)**
  - Does the Ferris wheel rotate clockwise or counterclockwise?

Purposeful Questions to introduce #1 and give clear instructions for your expectations on the problem:

- So we have studied angular and linear velocity as a circular object rotates. Thus, we have looked at how fast a point moving around a circle has travelled and how far it has travelled. We will now focus on the position of the point and the pattern that makes as it moves around the circle.
- In Algebra 1, when describing patterns of change, what did we include in our descriptions?
  - Variables in context
  - Whether the dependent variable was increasing or decreasing
  - At what rates the variables were increasing or decreasing
- Can I write one description for the entire circle?
  - **No**
- Why not?
  - **The point doesn’t follow the same pattern of change for the motion around the entire circle.**
- Great, so I would split it up?
  - **I think that the four quadrants would be a good place to start.**
- I agree! So you have a pretty clear outline of what we need to include in our descriptions of the patterns of change here. Take some time and work with your groups on #1 a and b. When you finish you can continue on to #2. We will pull back together to share out in about twenty minutes.

*Students should be given one copy of the master for part a and one copy for part b (master found in appendix, under Rotating Wheel). It is helpful for a large stack of these masters for students to grab more to use throughout this investigation.*

Purposeful Questions for #1 a.:

- What are we looking at here in quadrant one?
  - **The motion of the $x$ coordinate**
- How can we show a visual representation of this horizontal distance, or the value of the $x$-coordinate?
Students come up with a variety of ways to depict this change. Two examples are given below:

- Great! So, what is happening to this horizontal distance as the seat moves around the Ferris wheel?
- The distance by which your $x$-coordinate changes is getting bigger and bigger.
The amount you change horizontally is increasing by more and more each time.

- So now for our first quadrant, we have “the x-coordinate decreases at an increasing rate.” Does this involve both an independent and dependent variable?
  - No
- You’re right, so which variable are we focusing on here?
  - Many students will say: independent, because it is the x-coordinate.
- Well what do we know about the definition of an independent variable?
  - It changes without being impacted by another variable
    - It changes consistently
    - It is on the x-axis.
- I agree! So if you jump on a Ferris wheel, does the x-coordinate change consistently? Is it impacted by any other variable?
  - Oh, no!
- Right! So what is happening independently of anything else on the Ferris wheel?
• Think about hopping on the ride and this variable is changing consistently.
• What does your position on the ride depend on?
  o How far you’ve rotated?
• And how is this measured?
  o The angle of rotation!
• Yes! Exactly! So if the angle measure is the independent variable, what depends on that?
  o The \( x \)-coordinate! So the \( x \)-coordinate is the dependent variable in this context.
• Perfect! So can you write your final description for the first quadrant here?
  o As the angle increases by a constant rate of 1, the \( x \)-coordinate decreases at an increasing rate.

**Measuring the \( y \) coordinate is just finding the vertical distance**

Conversation for after #1:

• Let’s pull back together and share out a descriptive sentence for the first quadrant in part a.
• Can somebody walk us through the way in which they found this pattern?
• I saw other groups depict this horizontal distance, or the value of the \( x \)-coordinate differently. Would someone else like to share out their thinking?
• Would anyone like to comment on these visuals? Would anyone like to add anything else to the descriptions that we have heard?

**Students will struggle to determine the independent variable. This makes sense, because normally, the \( x \)-coordinate is the independent variable and the \( y \)-coordinate is the dependent variable. If necessary, discuss this more in depth when pulling together to share out patterns.**

<table>
<thead>
<tr>
<th>2 Find angles of rotation between 0° and 360° that will take the seat from point ( A ) to the following special points.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Maximum and minimum distance from the horizontal axis</td>
</tr>
<tr>
<td>b. Maximum and minimum distance from the vertical axis</td>
</tr>
<tr>
<td>c. Points with equal ( x )- and ( y )-coordinates</td>
</tr>
<tr>
<td>d. Points with opposite ( x )- and ( y )-coordinates</td>
</tr>
</tbody>
</table>

Hirsch et. al. (2015) p.425

<table>
<thead>
<tr>
<th>#2a.</th>
</tr>
</thead>
<tbody>
<tr>
<td>What does it mean to be furthest from the horizontal axis?</td>
</tr>
<tr>
<td>What do we know about the degree measure of 0 degrees and 360 degrees?</td>
</tr>
</tbody>
</table>
  o They both bring us to the same position on the wheel. |

**Trigonometry Review:**

Students often need a review on the work that they did with trigonometric ratios in Geometry. A good way of presenting the
1. The image below depicts an angle in standard position.

![Diagram of angle in standard position](image)

Figure 2: Hirsch et. al. (2015) p.425

a. Given the following list of terminology, label the image: initial side, terminal side, central angle, vertex.
b. State a definition of Standard Position.

2. Explain why $r$ is equivalent to $\sqrt{x^2 + y^2}$.

3. Define the Trigonometric functions in terms of $x$, $y$ and $r$.
   a. $\sin \theta$
   b. $\cos \theta$
   c. $\tan \theta$

4. For the point (9,12)
   a. Provide a labeled sketch of an angle in standard position, where the terminal side extending through the given point.
   b. Find each of the trigonometric ratios in exact form.
   c. Find the measure of $\theta$.

   When a circle like that modeling the Ferris wheel is placed on a rectangular coordinate grid with center at the origin $(0,0)$, you can use what you know about geometry and trigonometry to find the $x$- and $y$-coordinates of any point on the circle.

3. Find coordinates of points that tell the location of the Ferris wheel seat that begins at point $A(1, 0)$ when the wheel undergoes the following rotations. Record the results on a sketch that shows a circle and the points with their coordinate labels.
   a. $\theta = 30^\circ$
   b. $\theta = 70^\circ$
   c. $\theta = 90^\circ$
   d. $\theta = 120^\circ$
   e. $\theta = 140^\circ$
   f. $\theta = 180^\circ$
   g. $\theta = 220^\circ$
   h. $\theta = 270^\circ$
   i. $\theta = 310^\circ$

Hirsch et. al. (2015) p.426

5. For the following points,
   a. Provide a labeled sketch of an angle in standard position, where the terminal side extending through the given point.

   Purposeful questions when walking around to groups:
   - If I want to find the coordinate on a right triangle, what do I do?
   - What are we given in this problem?
     o An angle
   - What else are we given?
     o A side length
   - What have we used in math in the past that uses angle and side length?
     o Trigonometric functions

Here, a continuation on the Nearpod presentation could be used, or handouts can be passed out. Either way, students should be given time to work through the questions and examples with their groups before coming together for a full class conversation.
b. Find each of the trigonometric ratios in exact form.
c. Find the measure of $\theta$.
   i. (-9,12)
   ii. (-9,-12)
   iii. (9,-12)

6. You began on a ferris wheel ride at the point (1,0) on a coordinate plane. You have rotated 40 degrees and are now at the point (.77,.64). Explain how the symmetry of the circle allows you to find…

   a. your angle measurement when your seat continues to rotate on the Ferris wheel.
   b. Your seat location as your ride continues

#5:

- How does the triangle in part i relate to the triangle that you drew in #4?
  o They both are right triangles
  o They both are drawn down to the x-axis.
- What angle are you finding when using your trigonometric ratios?
  o The angle created by the terminal side and the horizontal axis in the right triangle.
- How does the angle in #4 relate to the angle in i?
  o They both have an angle of 53 degrees
- So what do we know about these triangles?
  o They are congruent
- What about the other triangles?
  o They are also congruent
- How can you use this angle to help you find your central angle in parts i, ii, and iii?
  o Depending on the location of the point, you can add, or subtract the angle to 180 degrees or 360 degrees.
- We call this a reference angle, which is a part of this reference triangle.
- Why is it important that they are drawn to the horizontal axis and not back to the initial side of the angle?
  o They must be right triangles in order to use the trigonometric ratios.
- Try using all three ratios to calculate the angle. What do you notice?
  o They don’t all give the same value.
- Why do you think these angles are not all the same?
  o Most likely, the students will be confused here.
- Think about the position of the point, and what this position does to the x and y value.
  o The x and y values can be made negative or positive.
- And how would this impact the trigonometric ratios?
  o They would become negative or positive!
- So why should you always use the trigonometric ratio that has the positive value?
- Where would the triangle have all positive values?
  o In the first quadrant.
- And what is the measure of the central angle in the first quadrant?
  o The reference angle!
- And then once you have found the value of the reference angle, what must you do to find the central angle?
  o You just add or subtract the value from 180 or 360 depending on where you are on the circle!
Suppose that $P(x, y)$ is a point on the Ferris wheel model with $m \angle C A = \theta$ in degrees.

a. What are the coordinates $x$ and $y$?

b. How will the coordinate values be different if the radius of the circle is $r$ decimeters?

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#5b:

- What was the radius in #3?
- How did we calculate the coordinates in #3?
  - Using $\cos(\theta) = \frac{x}{r}$ or $\sin(\theta) = \frac{y}{r}$
- So what if the radius wasn’t 1? How would we undo the division by the radius?
  - Multiply both sides by that value.

Investigation 2 Summary Conversation:

1. How do I find the coordinates when the angle of rotation is 50 degrees?
2. What about the coordinates for 130 degrees? 230 degrees? 310 degrees? 410 degrees?
3. What is special about 45 degrees?
4. What is special about 30 degrees and 60 degrees?
5. Where is the value of cosine positive? Negative? What about the value of sine? Where do you see this on a Ferris wheel?

Students begin by answering these posted questions with their group members. Then, the class will come together to discuss their answers. Students should not be required to take notes at this time. Instead, the teacher should ask the questions to the left. If students want to add notes to their work, they can. However, this discussion is more based upon conversation amongst students. The teacher merely poses the questions, and the students should agree, disagree, add on, and provide other explanations themselves. Students should be encouraged to draw visuals on whiteboards for the class to see, or come up to the board and write down their work.

This section is important. Throughout the investigation, the teacher should direct students back to this paragraph when they are confused.

As students collect data, have them write it in a place that is accessible to all students for viewing.

Purposeful Questions for after the experiment has been completed:

- What do you notice about the numbers that we have on the board?
  - It should be all the same angle, or very similar.
- Why do you think that they should all be the same? I see different sized circles!
  - All circles are similar, angles are the same in similar figures.
• So why are the angles not all the same?
  o Human error
• If the angles should all be the same, then what should the angle measure be?
  o Students tend to say 60 degrees.
• If it was 60, what would that mean about the circle in terms of what we did with the pipe cleaner?
  o It should go around 6 times, 6 radii fit around the circle perfectly.
  o If students do not have an answer here, have students go back through the introductory paragraph and read.
• Okay, try it. Do 6 radii go perfectly around the entire circle?
  o No, that is just short.
• So what does that say about our prediction of 60 degrees for our angle?
  o It would be less than 60 degrees because we need more than 6 radius lengths around the circle.
• What were we finding as we went around the circle with the radii?
  o Circumference
• Quick review- what is the circumference equal to?
  o $2\pi(r)$
• What does r represent?
  o Radius
• What is the numerical value of $2\pi$?
  o 6.28
• So what does $2\pi(r)$ represent here?
  o There are 6.28 radii that fit around the circle.
• So what are we originally trying to find?
  o The angle that forms the opening of the length of one radii
• And we have always measured angles in degrees, so how many degrees does it take to travel around the entire circle?
  o 360 degrees
• And to go all away around is 6.28 radii, so how do we determine what the measure is of the angle that is created when the arc length is the length of just one radii?
  o Give them think time within their groups to determine this calculation.
  o $\frac{360}{6.28} \approx 57.3$
• Now this actual degree measurement isn’t what is terribly important. What you need to understand from this conversation is that the angle is not 60 degrees and why the angle is not 60 degrees. You now have a picture in your head of what one radian measure would look like. Finally, we know that in every single circle, the measure of one radian is approximately 57.3 degrees no matter the size of the circle.
Can somebody rephrase what a radian is in their own words?

Students should state and teacher should reinforce, that the radian measures an angle that intercepts an arc equal in length to the radius.

Figure 4: Hirsch et. al. (2015) p.428

The online manipulative should be used to answer the following questions:

**Angles Measured in Degrees and Radians**

<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Adjust the manipulation so that the arc length and the radius are equal. What do you notice about the radian measure of the central angle?</td>
</tr>
<tr>
<td>2. What do you notice about the central angle when the length of the arc is twice that of the radius?</td>
</tr>
<tr>
<td>3. Predict what the length of the arc would be if the angle measure is 3 radians. Explain your thinking.</td>
</tr>
<tr>
<td>4. What relationship do you see between arc length, central angle, and radius?</td>
</tr>
<tr>
<td>5. A circle with center O has radius ( r ) units. Central angle (&lt; AOB) has positive radian measure ( \theta ) and intercepts ( \overline{AB} ) of length ( s ) units.</td>
</tr>
<tr>
<td>a. If ( r = 7 ) and ( s = 21 ), determine the measure ( \theta ) of (&lt; AOB).</td>
</tr>
<tr>
<td>b. If ( r = 6 ) and ( \theta = 2.5 ), determine ( s ).</td>
</tr>
<tr>
<td>c. Write ( s ) in terms of ( r ) and ( \theta ). If ( r = 1 ), how are ( s ) and ( \theta ) related?</td>
</tr>
</tbody>
</table>

Students should be sent the link to the left to work through on their phones or one to one devices. Questions can be posed on the board, or printed out for a worksheet. Students will work through questions with their groups on paper or a large whiteboard.

As students are working, teacher should walk around, encouraging use of the manipulation and checking in on the relationship that students found in part 4.
Radian Worksheet:
1. Every central angle in a circle intercepts an arc on the circle. The length of the arc is some fraction of the circumference of the circle. A radian is the measure of any central angle in a circle that intercepts an arc equal in length to the radius of the circle. Sketch and label a diagram that illustrates this definition.

2. Describe in general how the radian measure of an angle can be found if the length of the radius and the arc length are known.

3. In circle O, imagine that ray OB makes a complete revolution from the starting point coinciding with OA.
   a. Sketch a picture and label the initial and terminal side.
   b. What would be the length of the intercepted arc AB?
   c. What would be the radian measure of central angle AOB?

4. In circle O, imagine that ray OB makes \( \frac{1}{2} \) revolution from the starting point coinciding with OA.
   a. Sketch a picture and label the initial and terminal side.
   b. What would be the length of the intercepted arc AB?
   c. What would be the radian measure of central angle AOB?

#3b and c:
- What relationship do you see between the arc length and the circumference in this example?
- The arc length is equal to the circumference.
- So how many radii span the entire arc length?
- \( 2\pi \) radii
- What is the measure of the central angle if it is intercepted by an arc length equal to one radius? How do you know?
  - 1 radian, because the arc length is equal to the radius and therefore the central angle is 1 radian.
- With this in mind, how many degrees is the angle that encompasses the arc length in your sketch?
  - 360 degrees
- And therefore, how many radians?
  - \( 2\pi \) radians
- So if there are \( 2\pi \) radii in the circumference, how many radians are there in 360 degrees?
  - \( 2\pi \) radii are created by an angle of 360 degrees.

A similar conversation can take place for #4 if necessary.

Benchmark Angles:
- a.)

Using the idea of benchmark angles, and without the use of your calculator, convert the following angle measures from degrees to radians. Use of a circle as a visual is encouraged to solve these problems.

Benchmark Angles:
- What are the ways of measuring angles?
- So let’s use what we just learned in order to help us convert angles measured in degrees to angles measured in radians without the use of your calculator.
- So if there are \( 2\pi \) radii spanning the arc of the circle, how many degrees and radians does the central angle measure?
  - 360 degrees, \( 2\pi \) radians
- What about if we only go around half way?
  - 180 degrees, \( \pi \) radians.
- How did you determine this?
  - I divided both sides by 2.
1. 135 degrees
2. 120 degrees
3. 210 degrees
4. 300 degrees

b.)

Using the idea of benchmark angles, and without the use of your calculator, convert the following angle measures from radians to degrees. Use of a circle as a visual is encouraged to solve these problems.

5. $\frac{2\pi}{3}$
6. $\frac{7\pi}{4}$

- These are called benchmark angles. They are angles that the circle can be split into nicely. There are other benchmark angles. Can anyone think of another?
  - $90$ degrees $= \frac{1}{2} \pi$ radians
- Great! How did you know this?
  - I divided both sides by 2.
- A mathematical norm is that we would write this as $\pi/2$.

a.)

- Now I want you to take 2 minutes and look at these two examples that I have on the board. When you finish your individual think time here, you will share out your ideas with your group. Each group will show their work on a large whiteboard. At the end of this, we will share each strategy out to the class. Get started!

- Students should be given a chance to look through problems on their own. Enough time should be given so that everyone has a chance to process at least the first example. Then, students should be given time to discuss their strategies with their group members. Finally, the strategies should be modeled on a large whiteboard. Full class discussion should follow.

- Look around the room at other group’s strategies to solve the first problem. What do you see? What do you think of these strategies? What wonderings do you have?
- What similarities do you see between the way your group solved the problems and other group’s methods?

Some methods that should be shared out include:

  - Splitting the circle into 90 degrees and then half of 90 degrees, which is 45 degrees, and half of $\pi/2$ is $\pi/4$. Then adding these two measures together.
  - Splitting the circle into 45 degree segments. Each 45 degree segment is $\pi/4$ radians. To get 135 degrees, you need 3 of the $\pi/4$ radian measures; $3\pi/4$ radians!

Be sure to provide examples where the angle is larger than 180 degrees or $\pi$ radians. Students shouldn’t just be doing fractions of $\pi$.

b.)

- Now, I want you to take these strategies and we are going to try to use similar reasoning to convert radians to degrees!
Some methods shared here should include:

- Split $\pi$ into 3rds and shade in 2 of them.
- Split the entire circle, $2\pi$, into 3rds and shade in one of them.

Students should be reminded throughout this activity and conversation that they will be exposed to other strategies to convert, and that this is to create meaning of how they are going about these conversions.

Begin this problem by allowing students to read through both Kadijah and Jacy’s reasoning in converting degrees to radians. Then discuss as a class.

- What do you notice about Kadijah’s method?
- What does $1/3$ tell you in Kadijah’s Method?
  - You’ve gone on third the distance around the circle
- Remind me, how many radians are there in an entire circle?
  - $2\pi$ radians
- So how much of the $2\pi$ radians have you spanned along the arc with this angle measure?
  - $2\pi/3$ radians
- How did Jacy get that 1 degrees = $\pi/180$ radians?
  - Divide both $2\pi$ radians and 360 degrees by 360 degrees.
- How does she use this fact to help her to convert degrees to radians?
  - Now we know what 1 degree is equal to.
    - If they don’t explain why they want this, ask for someone to explain their thinking.
- Emphasize this concept by asking the following questions: If it costs $4 to buy 5 cupcakes, how much does it cost to buy 8 cupcakes? How does this connect to the idea of degrees to radians?

Allow students to work through both Kadijah and Jacy’s method.

Many strategies should come out of parts d and e. Some of the suggested solutions are below:

- $\pi/4$ is equivalent to $2\pi/8$ which can be read as $1/8$ of $2\pi$. Since 360 degrees is equivalent to 2$\pi$ radians, by substituting in 360 degrees for 2$\pi$ radians, you can then just multiply these two values together and get you angle measure of 45 degrees.
e.) When we were converting degrees to radians, we needed to know what one degree was equal to in order to make an equal conversion. Now we must determine what one radian is equal to in order to convert to degrees. It is equal to 180/π. Now multiply your radians by this value and you will be left with 45 degrees as an outcome.

We know π radians is equal to 180 degrees. Thus, substituting 180 where you see π can allow a conversion between degrees to radians.

Since there are so many methods to share out for converting radians to degrees, and so much meaningful conversation to occur, this is a great time to circle up all tables. The teacher sits in this “mega table” and allows the students to go to the board and share out their reasoning. Students stay engaged listening to, and commenting on one another’s strategies.

### #4c:
- What relationship do we see between the revolutions and the radians?
  - The radians have a denominator half that of the radians, and the numerator is multiplied by π.
- How do the revolutions and radian relationship relate to Kadijah’s strategy?
  - The revolutions are the portion of the circle travelled. The radians are the angle measure of this portion.
- How is this done mathematically?
  - Multiply the revolutions by 2π

### #9a:
- i: What is the numerical value of 2π?
  - 6.28
- So if 5 is less than this, what does this mean about the radian measure within the circle?
  - It will be close to a complete circle, but not quite, so quadrant 4.
- ii: What does a negative mean in the context of this problem?
  - Going backwards around the circle- clockwise!
    - If they don’t get this, prompt with what does a negative mean along a number line?
• iv: What are we doing here?
  o Going back to degrees!
• How can we use Kadijah’s method to help here?
  o Multiply by 1 radian!

b:
• Encourage students to draw this out!
Culminating Project

Students have discovered multiple ideas throughout the previous two investigations. First, they study the patterns of change of the x and y coordinate points and the motion of these points as the central angle changes. They then relate these patterns to the calculations that are necessary to be made to compute the values. Here, they are studying the right triangles that allow the students to make these calculations. Next, students learn about the radian; what it is and how they can reason between converting degrees to radians and vice versa.

However, the students have not yet been exposed to the connection between the patterns of change as the coordinate moves around a circular object and its relation to the graphs of trigonometric functions. That is where this culminating activity comes in to play. Here, students will create the sine and cosine graphs using the unit circle. They will see that as a point moves around a circle, the motion of the y coordinate will be depicted by the sin(x) graph, and the motion of the x coordinate can be seen in the cos(x) graph. Through this project, students will have more practice in converting radians to degrees.

Students will work in groups throughout this project. The teacher role is to use critical questions and allow for productive struggle within their groups. It is important to recognize the balance between productive struggle and student frustration and provide more support as needed by the students. As groups finalize their exploration, the teacher should transition back to whole class and facilitate whole class discussion. The key for the activity can be found in the appendix.
Teacher Guide to Culminating Activity:

The list below is key points to make sure to focus on while walking around the classroom and watching the students partake in this activity.

1. When creating the number line for the students, get 8 ½ by 14 inch paper. Put a dash at the beginning of this number line and place a point on the dash. This point will represent the point (0 degrees/radians, y=0).

2. Cut a piece of string equal in length to the circumference of the wheel. Make sure that when this string is unwrapped from the circle it fits on the piece of paper. The size of the current wheel in the appendix will fit.

3. In the appendix, the wheel from Investigation #2 has been provided to be consistent with the student's past work. However, if the teacher wanted, they could use a wheel that includes the 45 degree angle measure as well.

4. When walking around the room, be sure that the students are making the connection that the number line represents the central angle measure, and thus, how far the point has travelled around the circle, or how the position has changed based on the angle of rotation.

5. Around #8 you may have to stop the class and demonstrate what they should be measuring, if they are not understanding the directions at this point.

6. The key to this activity is that students will be measuring the horizontal and vertical distances on each fold, representing the x- and y-coordinate at that specific angle measure. When the student goes around the unit circle doing this at every 30 degree increment, they will create the sine and cosine curve.

7. #15 and #27- Students should see the patterns that they found in Investigation 2 #1 in these graphs! If they do not see this, be sure to discuss this point with each group. This should also be shared out in the summary at the end of this activity.
ADVANCING STUDENT LEARNING IN TRIGONOMETRY

8. #20- Students should recognize that in order to create a cosine curve, they are looking at the horizontal distances, or the distance to the y axis. This will provide the value of the x-coordinate. This question needs to be answered correctly in order to get the correct graph for cosine. Be aware of this while walking around the classroom.

9. #21- students may need to be brought together here for a demonstration of what they are measuring.

10. #23- Students should recognize that this pattern will continue on in a cyclic pattern. This is important for further investigations.

11. #25- If the students do not see that the cosine curve is just the sine function shifted horizontally to the right 90 degrees, this is fine. If a student does, this is a good point to bring up at the end to launch in to the next investigation on transforming trigonometric functions.

12. #35- This will give the students a chance to gather their thoughts before the full class conversation.

13. Pictures and answer key have been included in the appendix to see the outcome of this activity.
Circular Motion

Purpose:

This activity allows you to explore the relationship between the unit circle, angle measures, and trigonometric functions.

Materials:

- Two 8 ½ by 14 inch piece of paper with a horizontal line drawn through the middle
- Circle
- String, cut to the length of the circumference of the circle
- Marker

Activity:

1. **Consider** the pattern of change that you would expect to see in the cosine and sine functions. Get out the wheels from #1, Investigation 2 to access discoveries from past work.
2. Begin by **wrapping** the given unit circle with the string, **marking** off each segment from the circle, on to the string with the marker.
3. **Lay** this string flat on the straight line given and record on the line where you see marks on the string.
4. What are you creating as you unwind the string from the circle? What do the dashes represent in the context of circular motion?

5. What is the independent variable in the context of a point moving around a circle? So where could we see this number line in a graph?

6. In what units would it make sense to label this axis?

7. **Label** this number line in both degrees and radians. *This is a great chance for you to practice the strategy you liked best for converting radians and degrees!*  


ADVANCING STUDENT LEARNING IN TRIGONOMETRY

8. Once the number line has been created, take the paper with the number line on it, and at the first angle measure, fold it.

9. Line the angle measure on the number line up with the x axis and place a dot on the fold where the paper intersects the unit circle at.

10. Continue this process until you have a point corresponding to every angle measure on the number line.

11. Connect your points.

12. Does this plot look like any function that you are familiar with?

13. In the context of circular motion, what does the first point that you plotted represent? On your graph, label this point, in the context of the problem.

14. What does this function represent in the context of circular motion?

15. How would the pattern of change of the sine function appear in a plot of (t,sint) values?

16. Now think, what does the point (90,1) represent on the graph?
17. What do all points in between (0,0) and (90,1) represent on the graph?

18. Think back to past problems, what did the cosine ratio calculation help us to find?

19. Repeat steps 1-3 on the second sheet of paper.
20. How was the value of the x-coordinate seen in the wheel where we discussed the pattern of change in Investigation 2, #1a? How does this relate to what you should be measuring in this problem?

21. With this in mind, repeat steps 6-11.
22. Does this plot look like any function that you are familiar with?

23. What does this function represent in the context of circular motion?

24. Explain what the point (0,1) represents in the context of circular motion. Label this on your graph.

25. Compare this graph to the graph of the sine curve. What are some differences that you notice?
26. Why does it make sense that the cosine function starts at (0,1) but the sine curve starts at (0,0)?

27. How would the pattern of change of the cosine function appear in a plot of (t, cost) values?

28. If the point were to go around the circle again, how would the x-coordinate and y-coordinate change?
   Why?

29. How will this second revolution be represented in the graphs of these coordinate functions?

30. We call this type of graph sinusoidal, and the pattern that it shows is cyclic.
31. Imagine if you were to wrap the wheel with the string in a clockwise direction. How would this be seen in a graph? Test this out if you need help.
32. Think of some specific points that you may find on a unit circle, for example:
   a) (1,0)
   b) (0,1)
   c) (-1,0)
   d) (0,-1)

   Explain how these points seen on the circle can be seen on the graphs.

33. Look at both graphs. Notice the point at 45 degrees, or $\frac{\pi}{4}$ radians. What do you notice about this point on your graph? **Explain** and **extend** your thinking here.

34. Do you notice any other special points?

35. **Summarize** your findings. How do the graphs of the trigonometric functions sine and cosine relate to the motion around a unit circle? Be prepared to share out in a full class discussion.
How has this been used in the Classroom?

The author of this project uses the Core-Plus Mathematics textbook and seeks to use purposeful questioning to push student’s learning and problem solving. The content presented in the curriculum above has been placed in a real world context of riding a Ferris wheel to make the concepts of circular motion tangible to students. Other contexts that could also work would be flying around the world in an airplane, or with a space shuttle, tracking specific points. Other ideas include a hamster on a wheel, or even a pebble stuck in a tire.

Even though students may not know they interact with sine and cosine waves in the real world every day, the concepts presented in this curriculum project can improve their understanding of the concepts of waves. When students are posed with questions, rather than being shown what to do, their problem solving skills can increase dramatically. In this unit, students have the opportunity to think through and problem solve mathematically and grapple with why the mathematics makes sense conceptually. Through the investigative approach to the questions, and the prompting questions that students are asked by their teacher, students are challenged to think deeply about the mathematical content they are challenged to learn. Through teachers providing meaningful guided questioning, students can become better problem solvers, which is a skill students can use throughout their lives.

The author of this curriculum project strongly believes that if students understand why they need to learn mathematics then they will be more successful in doing so. Part of the teachers work is to get students to buy into the idea that mathematical knowledge needed to support meaningful problem solving is important.

This curriculum project was used in a High School in Western New York. After the culminating project was generated, many teachers implemented it in their classroom. One veteran teacher claimed that this was a great hands-on activity that allowed all learners to be a part of the activity; whether this meant they were holding the string around the wheel, or sharing out their ideas on what the markings on the string represented. Teachers also thought that the visual and act of transferring the vertical and horizontal
distances onto the graphs to develop the sine and cosine curves reinforced the idea of how the trigonometric functions models the x- and y- positions along the wheel. Students expressed that they had a better understanding of the functions after the activity than they did by using the graphing calculator. They also shared they now had a way of remembering each function’s patterns, without the use of their calculator.

The author of this project found that the ability for the students to create the sine and cosine curves on their own was imperative to their understanding. This activity made it very clear that the cosine curve was measuring the horizontal distances, and thus the x-coordinate, and that the sine curve was measuring the vertical distances, and thus the y-coordinate. The culminating project took additional class time. However, the author believes that it is worth taking the time to make thinking visible, and to push students to have deep, conceptual understanding of the material.

**Conclusion**

It is the hope of the author that this curriculum project can be used in classrooms to help students to become better problem solvers, and for students to understand the concepts behind circular motion with a deeper understanding. The New York State Common Core Standards for Trigonometric Functions have been taken and used for an inquiry based approach to enhance lessons with the specific intent of deep student understanding when learning about Trigonometry in Algebra II. This has been done through specific questioning strategies and techniques to promote student to student discourse as well as student to teacher discourse.

This curriculum project contained lessons, followed synchronously with specific questions to benefit student understanding. It also contained a summative activity to improve student understanding. The end goal of this curriculum is how questioning in deep inquiry based learning can get students to truly think about the reasoning and background behind the mathematics, and thus, buy in to the learning
process. In turn, this process will strengthen the learners’ understanding of the mathematical content, but also their problem skills that they will use throughout their lives.
References


Appendix

Culminating Project Answer Key:

Name:____________________________________ Date:_____________

Circular Motion

Purpose:

This activity allows you to explore the relationship between the unit circle, angle measures, and trigonometric functions.

Materials:

- Two 8 ½ by 14 inch piece of paper with a horizontal line drawn through the middle
- Circle
- String, cut to the length of the circumference of the circle
- Marker

Activity:

1. **Consider** the pattern of change that you would expect to see in the cosine and sine functions. Get out the wheels from #1, Investigation 2 to access discoveries from past work. **Discuss** these patterns with your group.
2. Begin by **wrapping** the given unit circle with the string, **marking** off each segment from the circle, on to the string with the marker.
3. **Lay** this string flat on the straight line given and record on the line where you see marks on the string.
4. **What are you creating as you unwind the string from the circle?** What do the dashes represent in the context of circular motion?
   - *I am creating a number line that represents how the position of the point has changed based on the angle of rotation.*
   - *Each dash represents another rotation of 30 degrees, or $\frac{\pi}{6}$ radians."

5. **What is the independent variable in the context of a point moving around a circle?** So where could we see this number line in a coordinate plane of a graph?
   - *Angle of rotation*
   - *The x axis of a coordinate plane*

6. **In what units would it make sense to label this axis?**
   - *Degrees or radians because these units measure angles.*

7. **Label** this number line in both degrees and radians. *This is a great chance for you to practice the strategy you liked best for converting radians and degrees!"
8. Once the number line has been created, take the paper with the number line on it, and at the first angle measure, **fold** it.
9. Line the angle measure on the number line up with the x axis and place a dot on the fold where the paper intersects the unit circle at.
10. Continue this process until you have a **point corresponding to every angle measure** on the number line.
11. **Connect** your points.
12. Does this plot look like any function that you are familiar with?

   *The sine curve.*

13. In the context of circular motion, what does the first point that you plotted represent? On your graph, label this point, in the context of the problem.

   *The point (0,1) represents that at 0 degrees or 0 radians, the value of the y-coordinate or the vertical distance is 0.*

14. What does this function represent in the context of circular motion?

   *The motion of the y-coordinate around the circle, or the lengths of the vertical distances as the central angle of rotation changes.*

15. How would the pattern of change of the sine function appear in a plot of (t,sint) values?

   **Same patterns that the students found in Investigation 2, #1**

   *Quadrant 1:* As angle of rotation increases at a constant rate of 1, the y-coordinate increases at a decreasing rate.

   *Quadrant 2:* As angle of rotation increases at a constant rate of 1, the y-coordinate decreases at an increasing rate.

   *Quadrant 3:* As angle of rotation increases at a constant rate of 1, the y-coordinate decreases at a decreasing rate.

   *Quadrant 1:* As angle of rotation increases at a constant rate of 1, the y-coordinate increases at an increasing rate.

16. Now think, what does the point (90,1) represent on the graph?

   *At 90 degrees, the y-coordinate is 1.*
17. What do all points in between (0,0) and (90,1) represent on the graph?

*Points on the circle in the first quadrant.*

18. **Think back** to past problems, what did the cosine ratio calculation help us to find?

\[ \cos \theta = \frac{x}{r} \]

*Thus, we could find the x coordinate by multiplying the radius by the cosine of theta.*

19. **Repeat** steps 1-3 on the second sheet of paper.

20. How was the value of the x-coordinate seen in the wheel where we discussed the pattern of change in Investigation 2, #1a? How does this relate to what you should be measuring in this problem?

*Now we should be measuring the horizontal distance, or the distance from the point to the y-axis.*

21. With this in mind, repeat steps 6-11.

22. Does this plot look like any function that you are familiar with?

*The cosine function.*

23. What does this function represent in the context of circular motion?

*The motion of the x-coordinate as you move around the circle, or the horizontal distances as the central angle of rotation progresses.*

24. **Explain** what the point (0,1) represents in the context of circular motion. Label this on your graph.

*At 0 degrees, or 0 radians, the x-coordinate is 1.*

25. **Compare** the graph of the sine curve. What are some differences that you notice?

- *The cosine curve starts at the maximum, whereas the sine curve starts at the midline*
- *There are different patterns of change.*
26. Why does it make sense that the cosine function starts at (0,1) but the sine curve starts at (0,0)?

At 0 degrees/ radians, the x-coordinate is 1, but the y-coordinate is 0. The independent variable is the angle of rotation. The dependent variable is the x- or y-coordinate. The cosine curve represents the x-coordinate and the sine curve represents the y-coordinate.

27. How would the pattern of change of the cosine function appear in a plot of (t,cost) values?

**Same patterns that the students found in Investigation 2, #1**

Quadrant 1: As angle of rotation increases at a constant rate of 1, the x-coordinate decreases at an increasing rate.

Quadrant 2: As angle of rotation increases at a constant rate of 1, the x-coordinate decreases at a decreasing rate.

Quadrant 3: As angle of rotation increases at a constant rate of 1, the x-coordinate increases at an increasing rate.

Quadrant 1: As angle of rotation increases at a constant rate of 1, the x-coordinate increases at a decreasing rate.

28. If the point were to go around the circle again, how would the x-coordinate and y-coordinate change? Why?

The points would repeat. This is because the point is travelling around the same circle. The circle is not changing in size.

29. How will this second revolution be represented in the graphs of these coordinate functions?

The graph will continue its pattern.

30. We call this type of graph sinusoidal, and the pattern that it shows is cyclic.

31. Imagine if you were to wrap the wheel with the string in a clockwise direction. How would this be seen in a graph? Test this out if you need help.

The pattern would be continued in the opposite direction. The x-axis would be labeled with negative angle measures.
32. Think of some specific points that you may find on a unit circle, for example:
   a) (1,0)
   b) (0,1)
   c) (-1,0)
   d) (0,-1)

   Explain how these points seen on the circle can be seen on the graphs.
   a.) (1,0): at 0 degrees, the x-coordinate is 1 and the y-coordinate is 0. The x-coordinate can be seen in the cosine graph at (0,1), and the y-coordinate can be seen in the sine graph at (0,0).
   b.) (0,1)): at 90 degrees, the x-coordinate is 0 and the y-coordinate is 1. The x-coordinate can be seen in the cosine graph at (90,0), and the y-coordinate can be seen in the sine graph at (90,1).
   c.) (-1,0): at 180 degrees, the x-coordinate is -1 and the y-coordinate is 0. The x-coordinate can be seen in the cosine graph at (180,-1), and the y-coordinate can be seen in the sine graph at (180,0).
   d.) (0,-1): at 270 degrees, the x-coordinate is 0 and the y-coordinate is -1. The x-coordinate can be seen in the cosine graph at (270,0), and the y-coordinate can be seen in the sine graph at (270,-1).

33. Look at both graphs. Notice the point at 45 degrees, or \( \frac{\pi}{4} \) radians. What do you notice about this point on your graph? Explain and extend your thinking here.
   - Both cosine and sine graphs have the same dependent value of .77 here. This means that both x- and y-coordinates are equal here.
   - If the graphs were on the same set of axis, they would intersect at 45 degrees.

34. Do you notice any other special points?

\[ \cos(30) = \sin(60) \text{ and } \sin(60) = \cos(30). \] This means that the x-coordinate at 30 degrees is equal to the y-coordinate at 60 degrees.

35. Summarize your findings. How do the graphs of the trigonometric functions sine and cosine relate to the motion around a unit circle? Be prepared to share out in a full class discussion.

Students should summarize key points that they found throughout this investigation. These concepts will then be discussed in a full class summary.

- Patterns of change and how the quadrants can be seen on the graph
- What the coordinate points mean
- What the sine and cosine graphs represent
Number Line for the Activity:

- String should be this long, and sting should be the length of the circumference of the wheel below.

Finished Product of the Sine Curve:

- Folds can be seen at each angle marking. Points can be seen on each fold. Here, each fold measure the vertical distance, or the length from the x-axis to the coordinate point on the circle.
Rotating Wheel:

Used in Investigation #2 and the Culminating Project.
Alternate Wheel to use in Culminating Activity:
If teacher wants students to include 45 degree measure in trigonometric functions