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INFINITY AND THE DOUBLE LANGUAGE OF MATHEMATICS

by

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I

According to Plato in the Republic (Bk. VII. 527a), the language of the mathematicians is ridiculous. They speak most laughably (μάλα γελοίως), though they are quite unaware of the absurdity inherent in their speech. Nor does anyone else (Plato excepted) appear to have found in their language any notable source of uproarious mirth. Are we to say then that only some twisted sense of humor in the old metaphysician can account for this idiosyncratic reaction? I shall try to explain what Plato found so funny in the mathematical language, and in a few fairly short steps we shall be drawn into some of the more perplexing questions in the philosophy of mathematics of our own day, thereby providing me with a suitable occasion to bring my thoughts on infinity up to date.

Plato himself explains the absurdity of mathematical language in as straightforward a fashion as one could wish. Although the entire discipline (πᾶν τὸ μάθημα) is undertaken for the sake of cognition (γνώσεως ἕνεκα) it is for the sake of something very different—action (πράξεως ἕνεκα)—that all the actual words (πάντας τοὺς λόγους) of the mathematicians seem to be fashioned. For all their prattle (φθεγγομενόι) is of squaring and applying and adding, that sort of thing, as if they were engaged in some kind of practical activity (ὡς γάρ πράττοντες) when in fact the science of geometry is the cognition of something unchanging, something that always is (τοῦ ἀεὶ ὄντος). There is then a radical incongruity amounting to flagrant absurdity in an enterprise devoted to the sublime contemplation of the eternal expressing itself in the vulgar, operational language of mechanics going about their work. From these remarks one might readily suppose that Plato is recommending a rational reconstruction of mathematical language, in order to bring it more nearly into line with its underlying intent. Not so. The mathematicians are not to be blamed for their ridiculous mode of speech: they are compelled (ἀναγκαίως) to speak in their coarse, inadequate way, doubtless because they are mere mathematicians and not philosophers privileged to exercise dialectic on the highest level of the Divided Line.

The distinction between theory and practice is familar enough, especially since Aristole in whom the dichotomy between theoretical and practical wisdom is firmly entrenched. But if pure mathematics is obviously to be understood as high theory, the actual words of the mathematicians Plato finds to be drawn rather from the jargon of practice, and it is the absurd conflation of the two—theory and practice—that strikes him as peculiarly droll. Owing to a radical transformation of mathematical language in modern times, the primary data which quite reduced Plato to the broadest of smiles if not positive guffaws (cf. Rep. 606 c) is no longer as readily available to us as it was in antiquity; and it will then be helpful to consult some actual pages of ancient mathematics if we are to succeed in representing to ourselves what Plato was getting at, with all the vividness that it deserves. Merely to cast one's eye over a page of twentieth century mathematics will not suffice. Quite the contrary. Plato's remarks will only seem mystifyingly opaque, they will have that thick wooliness that gives philosophy a bad name when they should be relished for their sharp bite. Let us then crack open our Euclid if not quite to the first page of his Elements where he lists the definitions required for the work of Bk.I then immediately after where he lists his five postulates. These should bring home to us with a shock of recognition the direct import of Plato's remarks. For of the five postulates it will be seen that all but one are shot through with the language of action. Euclid writes (in T.L. Heath's translation):

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, *if produced indefinitely*, meet on that side on which are the angles less than two right angles. (my italics).

The first postulate may be taken as representative. It assures us that between any two points a straight line can be drawn. Here then is something which can be done, an operation that can be performed—presumably by the mathematician himself; and in consequence we are not surprised to learn that Proposition I of Bk. I reads, "On a given finite straight line to construct an equilateral triangle." But that is merely high-grade engineering from Plato's standpoint and altogether at odds with the true nature of geometry as a contemplative science. Even when Euclid, for the first time, in Proposition 4 of Bk. I undertakes to prove a purely theoretical theorem the means he employs are incongruously constructive. Thus one step of his proof reads: "If the triangle ABC be applied to the triangle DEF and if point A be placed on the point D, ...then the point B will also coincide with E...". Taken at face value, the whole notion that one mathematical point might be "placed on" another might well be felt to be wildly preposterous even if one is otherwise unsympathetic to Platonism, and one can scarcely be supposed to believe that it is literally in any one's power actually to draw a mathematical straight line, Euclid himself having defined a line as "breadthless length".

That something pretty 'funny', i.e. peculiar, is going on in Euclid's language, can scarcely be denied; and once we have been alerted by Plato to the anomalous presence of the fourth postulate among the others—it alone is suitably enshrined in the language of pure being (of what is eternally the case)—we are likely now to marvel at Plato's forbearance in being prepared to indulge Euclid in his absurdities where previously we were perhaps inclined to enjoy a smile of our own at the philosopher's expense, for having the temerity to adopt a high-and-mighty attitude toward the mathematicians, very much *de haut en bas*. Indeed no one today will be more eager to side with Plato as against Euclid—if it came down to a crunch—than the modern mathematician himself who will be quick to draw our attention to the following two sentences:

S_1 Between any two points a straight line can be drawn.

S_2 Between any two points there exists a straight line.

Although the two sentences may be said to be functionally equivalent from the standpoint of the working mathematician, it is of course the second which is characteristic of modem mathematics in its official formulations, and it is here that the heart's desire of the old metaphysician can be seen to be amply fulfilled. Whether Plato can actually be credited with having succeeded in prodding the mathematicians into mending their ways if only by some kind of remote control across the centuries, is a question that may be left to the historian to investigate. It is, however, clear that while S_1 is expressed in the language of becoming (to use Plato's idiom) S_2 is austerely cast in the language of being, and if the one mode of speech is appropriate enough for practice the other is surely demanded by theory.

At this point someone of an anti-metaphysical bent might perhaps be forgiven for choosing to sum up the narrowly 'factual' content of my remarks as follows. Historically, there are two languages of mathematics, the one constructive, the other non-constructive, where the former is characteristic of ancient as the latter is of modern mathematics. And he might even feel that there is no substantive difference between them, on the ground that one can translate back and forth from the one to the other without any essential loss of meaning. That this sentiment presupposes some unclarified notion of substance and essence of its own, I notice merely in passing. More important for our purposes, there are two theses here that while (broadly speaking) true enough in their own way require serious correction. The first is historical, to the effect that the distinction between the constructive and non-constructive idioms in mathematics is roughly to be assigned to ancient and modern times respectively. The second may be stated cautiously as a symmetry thesis, to the effect that corresponding to each non-constructive statement in mathematics there is a constructive one and vice versa. The historical thesis is soon seen to require emendation when one considers the following two statements.

S_3 Two sets have the same number of members if and only if they can be put in one-to-one correspondence with each other.

S_4 Two sets have the same number of members if and only if there exists a one-to-one correspondence between the members of the one and the members of the other.

Set theory being a peculiarly recent development, S_3 and S_4 are alike in being thoroughly up-to-date, and the constructive language of S_3 will be found to come tripping off the tongue of the contemporary mathematician at least as readily as the non-constructive, platonistic idiom of S_4.

There remains, however, a striking difference in the mathematician's use of S_3 and S_4. The former belongs to his shirt-sleeves language, to the informal idiom of the workshop, the latter to the dressed-up language of the printed page. When the mathematician is prepared to deposit his results in the archives of science it is of course the high dignity of the platonistic language of pure being that he will choose to affect. When he is engaged in the earlier, preliminary thinking through of his theorems he will often be found to lapse happily into the unbuttoned freedom of action language in his informal discussions with his colleagues. Thus the ancient idiom of Euclid may be said to reassert itself even in set theory, living a kind of underground life that rarely surfaces on the printed page of canonical mathematics, and once again Plato seems to be vindicated when it comes to the more authoritative presentation of mathematical results. Only on one minor point would Plato seem to be decisively refuted: he was persuaded that some kind of necessity compelled the mathematician to express his truths of pure being in the language of becoming. No such necessity would seem to be evident today if one is content with the official, dressed-up version of his science. But not everyone is content with that version, and in bringing out the deep sources of dissatisfaction we are brought into the thick of recent controversy.

II.

It is the question of infinity that forces us to examine afresh the double language of mathematics. For it can be shown that some of the farther reaches of the infinite (if I may so express myself) can only be grasped by us—assuming again that this manual conceit of 'grasping' is not too absurd—by means of what are called non-constructive methods in mathematics and that if one limits himself to the constructive idioms he must relinquish many of the results that are characteristic of the science in its standard, dressed up formulations. The symmetry thesis can thus be shown to be quite false. To mention but one example: in standard mathematics there are 'proved' to be sets (e.g., inaccessible sets of irrationals) which have the same number of members but of which it cannot be said that they can be put in one-to-one correspondence with each other. That there exists a one-to-one correspondence between the members of those obscure sets, will of course be insisted upon in some high platonistic sense of 'exists'. One may indeed feel that such irreducibly non-constructive locutions that cannot be recast in the constructive idiom must somehow outrun our intuitions, being much too metaphysical to be accepted by anyone of a sober empiricist turn of mind.

One may then retreat to what I have called the shirt-sleeves idiom of the mathematician, characteristic of S_1 and S_3, and choose to dismiss what I have styled his dressed up language, displayed in S_2 and S_4, as being merely a stuffed shirt mode of expression. Thus one can blink one's eyes incredulously at the notion that between any two points there exists a straight line. Is that really true? Does there exist a straight line connecting the center of the earth and that of the moon? Should we say at least that there is an imaginary line connecting them? But an imaginary line is no more a line than an imaginary cat is a cat. Well, you will perhaps say that in principle at least a very thin wire, as thin as you please, might be extended all the way from the earth's center to the moon's. So be it. But then you are in effect taking S_1 to be more authentic and authoritative than S_2. Indeed S_2 then seems to be no more than a high-falutin, even pretentious way of asserting S_1. And one may feel in very much the same way in regard to S_3 and S_4. Is it really true in the case of any two equinumerous sets that there exists (where? in the mind of God?) a one-to-one correspondence between them? And again one may feel moved to posit an imaginary line connecting each member with its assigned counterpart. It is then only a short step to insisting that the hard cash value of S_4 is really to be found in S_3.

If these suggestions strike a responsive chord in your heart you may well be attracted to a Kantian philosophy of mathematics. Kant was as impressed as Plato by the constructive idiom of classical mathematics but where Plato took it to be a deficiency, albeit an unavoidable one, Kant came to view it as the peculiar glory of science. It was what distinguished mathematics as a source of genuine cognition from metaphysics which he felt was ultimately spurious. Although metaphysics and mathematics were alike in one decisive respect—in being purely a priori, non-empirical discipline of the rational faculty—metaphysics was merely an empty game played with pure concepts that could generate no more than fairly trivial analytic propositions. It was precisely by means of constructions that the mathematician enjoyed a privileged position, enabling him to synthesize his concepts in the a priori medium of a construction-guided intuition. And this constructive activity of mathematics is seen to be only a special if exalted case of the constructive activity of the human mind as such in all its genuinely cognitive employments. The objective world then is itself constructed by us from the raw data of the senses, and the objects that result from our constructive conceptualizations are always theoretical entities that being invincibly mind-dependent must not be mistaken for real things in themselves.

The two languages of mathematics are thus found to correspond to Kantian and Platonistic modes of thought respectively, the one being expressed in the constructive idiom, the other being enshrined in the non-constructive formulas. And if only the symmetry thesis were true the working mathematician at least could choose to stand above the battle, speaking the language of Kant in his everyday work clothes and that of Plato in his Sunday best. How the question of the infinite, which Kant not surprisingly takes to lie beyond the finite capacity of the human mind, should bring about the breakdown of the symmetry thesis, I wish to explain in some vivid detail. But it is not of course in mathematics that the puzzles connected with infinity first arise. Perhaps the earliest encounter that anyone has with them occurs in context of what may be described as scarcely more than a joke that can hardly fail to delight even the dullest child in kindergarten. I am, however, almost embarrassed to mention it in adult company, as it may be felt to be much too childish, literally, for so sublime a theme. Begging your indulgence, then, the joke (if it is a joke) is simply this. Which came first, the chicken or the egg? What we have here, actually, is a high-grade paradox, and though the infinite as such is not mentioned it lurks powerfully in the background and generates the puzzle. For if one could somehow accept the infinite as being fully intelligible he could escape between the horns of the dilemma by acquiescing in what the mathematician calls a $*_{w}$ sequence, chicken being preceded by egg being preceded by chicken being preceded by egg, and so forth ad infinitum (represented thus: ..., e, c, e, c, e, c,).

That any child should be expected to recoil in some kind of primitive, conceptual horror from such a prospect, is by no means implausible when one finds the foremost philosopher of our age recoiling in much the same way. In the recently published translation of his Philosophical Remarks dating back to 1930 Wittgenstein entertains the following conceit on p. 166.

> Let's imagine a man whose life goes back for an infinite time and who says to us: 'I'm just writing down the last digit of pi, and it's a 3.' Every day of his life he has written down a digit, without ever having begun; he has just finished. This seems utter nonsense, and a reductio ad absurdum of the concept of an infinite totality.

No wonder, then, that the poor child should be stumped by his riddle. Like Wittgenstein he rejects—better: he is encouraged to reject— the prospect of an infinite totality, but where is he now to turn in his puzzlement? For it is not merely the concept of an infinite totality that is being ruled out; tactily, the biblical account in Genesis is also being dismissed, which would answer the riddle by assuring us that it is a chicken, directly created by God, that came first. Only the release provided by laughter remains to the child as a way out, and as he grows up, proceeding from youth on to old age, he will probably never realize that the highest point in his intellectual development occurred at that moment, in his childhood, when he had his closest brush with the ultimate, and though he will doubtless regale his children and grandchildren with the familiar riddle he will be quite oblivious of the metaphysical import with which the occasion is charged.

Natural science in the specific form of biology, physical cosmology, religion and mathematics are seen to be all involved, some overtly, others fairly covertly, in the child's riddle, and it is by no means clear that we in our sophistication are in any position to patronize him. Thus one may wonder how Wittgenstein himself having rejected infinite totalities would respond to the riddle. Would he agree that the physical universe must have come into existence at some finite time in the past? Although he never discusses the question I think that we can safely say, simply as a point of scholarship, that he would have been characteristically brusque in dismissing the suggestion. Like Kant before him, Wittgenstein was committed to holding that both alternatives had to be ruled out as pseudo-hypotheses, both the suggestion that the physical world has always been in existence throughout an infinite past as well as the alternative that it had a beginning in time. That one or the other must certainly obtain, would seem to follow almost directly from the Law of the Excluded Middle but once one concludes with Kant that no possible experience could serve to verify (or even falsify) either hypothesis anyone of empiricist convictions may well feel forced to suspend the rule of tertium non datur in the present case. That was Kant's policy in regard to all such metaphysical issues which (by definition) transcend the empirical, and it was Brouwer above all, steeped in his Kant, who in our own time ruthlessly applied this teaching to mathematics itself.

Wittgenstein is said to have attended Brouwer's lectures in Vienna in 1928, and the remark I have quoted certainly connects with Brouwer's thinking in the most intimate way. For Brouwer denied that logic entitles us to say that we know the following proposition to be true.

S_5 Either 7777 occurs in the decimal expansion of pi or it does not.

In standard mathematics of course S_5 is taken to be self-evidently true, and the working mathematician will have no qualms in using S_5 as a premise in the proof of some theorem. But not Brouwer. Any such putative proof he takes to be fallacious, as resting on the metaphysical error of confusing a mere rule for generating digits as far as you please with some mind-independent entity consisting of the entire decimal expansion of pi as an infinite totality. S_5 is not indeed an exact parallel to the cosomological case, for here we can understand how tomorrow we might inspect the printout of an electronic computer that has been programmed to grind out the first billion digits in the decimal expansion of pi, only to discover (empirically) that the last four digits in the sequence were indeed all 7's. But this deficiency can be easily remedied if we consider

S_6 Either there is a terminal 7 in the decimal expansion of pi or there is not.

In this case we know that merely grinding out digits as far as you please can never suffice to determine that there is, or is not, a 7 in pi beyond where there is no other. In the same way merely engaging in archaeological research into the increasingly remote past will never enable us to determine that the world did, or did not, have a beginning in time. For in the one case though we might indeed come upon a terminal 7 in pi and in the other we might succeed in reconstructing a primordial event, in neither case would we know that we had in fact done so. And if we contrast S_7 with S_8 we can see how the symmetry thesis is threatened by a possible counter-example.

S_7 There exists a terminal 7 in the decimal expansion of pi.

S_8 A terminal 7 in pi is effectively constructible, i.e. identifiable by some finite procedure.

According to standard mathematics and standard logic S_7 might well be true even though S_8 should be false. Even in the mind of God there might be no effective procedure for determining in a finite number of steps that some particular 7 in pi is indeed the last to be found, and yet that would in no way militate against the (platonistic) existence of such a 7.

Brouwer's refusal to countenance all such 'noumenal' entities took a very active form indeed in 1913 when as one of the editors of Mathematische Annalen he had the audacity to reject all papers submitted to the Annalen that relied on non-constructive idioms. In the high dudgeon the rest of the editorial board of the journal resigned in protest and then proceeded to re-elect themselves sans Brouwer. The Dutch government resenting this rebuff against their premier mathematician sponsored a new journal of which Brouwer would be in command. From this episode one might easily suppose that Brouwer was a bit of a crank, though in his purely professional work there is irony in his being remembered today for at least one proof that is entirely non-constructive in character, his fixed point theorem in topology, that demonstrates the enormous attractiveness of the standard, platonistic approach. Among mathematicians of any philosophic turn of mind, however, his insistence on constructive procedures has continued to exercise a fascination that remains unabated at the present time, and there is widespread uneasiness regarding the so-called non-constructive sets.

Plato indeed would laugh on hearing the contemporary mathematician describe all of the following sets as being constructible, for all the world as if their very existence depended on his action when if they exist at all it can only be as mind-independent noumenal entities: K_0 (the empty set), K_1 (the set whose only member is K_0), K_2 (the set whose only members are K_0 and K_1), etc. ad infinitum; beyond all of which there is K_w (the set whose only members are K_0, K_1, K_2, K_3,...). If this be the safe ground of construction it is (one might say) very much construction ex nihilo, and one may feel inclined to agree with Poincaré when he said early in the century, "Set theory is a disease from which mathematics will one day recover." The disease (if it is a disease) is today very far advanced, and recovery is nowhere in sight.

Beyond K_w there is in standard mathematics at least the first non-constructible set K_p which is the power set of K_w (the set whose members are all sub-sets of K_w). It can be shown that 'most' of the members of K_p are non-constructible. I am sure that it is K_p that must be kept especially in mind if we are to understand a fateful conversation that took place late in the last century between Dedekind and Cantor. Dedekind said that when he thought of a set he thought—primarily—of a bundle of unknown things. Cantor replied that in his case he thought of something rather different: an abyss. Such innocent sets as the set of Roman emperors whose members are fairly open to view, both mathematicians simply ignored. It was the obscure sets that drew their attention but notice how much more radical was Cantor's figure of speech. A bundle of unknown things can at least be opened; one can look inside. An abyss suggests a bottomless vista that can never be comprehended in one glance, and it is no accident that it was Cantor more than anyone else who envisioned the paradise of the non-constructible sets. "Paradise" is Hilbert's word. "No one," he said, "shall expel us from the paradise into which Cantor has led us," Brouwer doubtless being the spoiler uppermost in his mind. But even Hilbert was intimidated by Brouwer. Distinguishing between two types of statements in mathematics, real statements and ideal statements, he conceded that all of the real statements were constructive in character and that they alone could be said to be strictly speaking true. Taken to be ideal in a fashion reminiscent of Kant, the non-constructive locutions had a merely architectonic function in rounding out the science of mathematics and according to it a systematic completeness that would otherwise be absent. Paradise so construed is scarcely more than a slum.

III.

My remarks up to this point having been merely descriptive of the war between Plato and Kant in mathematics, I want now to offer some suggestions of a more constructive kind (pun intended). Although I am convinced that Brouwer is radically mistaken in many respects, I remain to be persuaded that full-blown Cantorian set theory is finally tenable. The position I shall propose is intended as a halfway house between the two, Brouwer and Cantor, Kant and Plato, and it may be designated as an infinitistic constructivism which accepts the intelligibility of the actual infinite, thereby rejecting the qualms of Kant and Brouwer on this score, but at the same time yields to their insistence that all mathematical locutions must be finally cashable in constructive terms, thereby declining to go all the way with Plato and Cantor. The lower reaches of the infinite which are closed to Brouwer shall be opened up to us but the farther reaches where Cantor is entirely at home will remain tantalizingly beyond our grasp.

We may begin by once more turning away from mathematics, in order to consider the following proposition.

S_9 Mt. Everest has always been in existence, throughout the infinite past.

S_9 is false. We know that it is false. But if we were to encounter some shepherd tending his flocks in the foothills of the mountain who took S_9 to be true, owing perhaps to an ancestral kind of sacred awe, are there any purely conceptual considerations that we might adduce to prove that he was mistaken? I do not believe that there are any such. Empirical considerations of the most massive, decisive sort drawn from geology, physics, etc. can indeed be mobilized in support of our own conviction but that is another matter. In fact they lead me to conclude (in the absence of countervailing considerations drawn from logic) that S_9 is a perfectly acceptable empirical hypothesis which if not verifiable by any possible sensory evidence can certainly be falsified by it. But S_9 logically entails

S_{10} The physical universe has always been in existence, throughout the infinite past.

Granted that S_{10} is neither empirically verifiable nor empirically falsifiable, it is simply incoherent to allow one's antimetaphysical prejudices to spurn S_{10} even while one accepts S_9, however grudgingly, as empirically meaningful. For if it is intelligible to suppose that S_9 might be true one has already—in effect—reconciled oneself to the intelligibility of S_{10}.

The case at hand is of the first importance. Kant believed that a cordon sanitaire might be thrown around the empirical realm, sealing it off effectively from the metaphysical. If only he had meditated on the child's puzzle I am sure that he would have seen how impossible was his program. Generalizing in the most approved scientific fashion from familiar barnyard evidence, the rural child at least infers that every chicken comes from a chicken egg and that every chicken egg comes from a chicken. These barnyard truths logically entail the Antithesis of Kant's First Antinomy regarding what he called "the cosmological ideas", namely that the world has always been in existence, for an infinite time. (One has only to add to them the equally innocent, certainly empirical premises that (1) there is at least one chicken and that (2) no chicken is hatched in less than a billionth of a second.) If there is admittedly a certain extravagance attaching to S_9 there is surely none whatever when it comes to the barnyard truths, and it is precisely their everyday banality that renders the child's puzzle so philosophically profound, in showing us what even Kant failed to grasp—that even barnyard empiricism (as we may style it) logically entails the metaphysical infinite. The child indeed is confronted with an antinomy of his own at least as deep as any in Kant. Empirical and conceptual considerations are found by him to be perplexingly at odds with one another, the former entailing, the latter resisting an infinite totality.

How the Child's Antinomy is to be resolved, should now be evident. We must learn to recognize the intelligibility of a $*_w$ sequence of chicken and egg, and once we come to feel at ease with the actual infinite the philosophical consequences prove to be far-reaching indeed. Let me briefly touch on one standard topic, Hume's celebrated problem of induction, in order to indicate how acceptance of the actual infinite can radically transform one's thinking in familiar territory, drawing on my paper "Induction and Infinity" in Ratio, December 1971. If it makes sense to suppose that a mountain might always have been in existence, the same must hold for a person. Assume now that there is some god who has always been in existence throughout the infinite past. Each and every day—past, present and future—the god asks the following question early in the evening, "Has the sun always risen every day throughout the infinite past?" And on any occasion when the answer to his question is yes the god appeals to the principle of induction and concludes, "Therefore the sun will rise tomorrow." According to the strict terms of my fable it is quite possible that the god is never given an opportunity to predict the sun's rising (e.g., if the sun never rises). But let us suppose that there is at least one occasion when he is afforded such an opportunity. It then follows that on infinitely many occasions he makes that same prediction. And of those infinitely many predictions that he makes it is logically impossible for more than one to be false: all the rest must be true. The principle of induction has thus been vindicated, if not for us, at least for any sempiternal being who can count on it as guaranteeing predictive success at a rate of infinity to one.

There you have an example of infinitistic thinking but if you have never encountered such thinking before, never at any rate since your abortive brush with it in connection with the Child's Antinomy, you must not expect to find yourself at home with it simply overnight. Infinitistic thinking requires a discipline very much of its own, and it can only be acquired with some practice. Which is not to say, however, that its relevance to the war between Platonists and Kantians in mathematics is at all difficult to make out. One can be as nominalistic as you choose, refusing to countenance mind-independent abstract entities of any sort, and still feel entirely at ease with infinitistic thinking, thereby breaking at once with both Plato and Kant. In particular, one can welcome Wittgenstein's conceit of the person, doubtless a god, who only today has completed his enterprise, never begun, of writing down the entire decimal expansion of pi, one digit per day. Far from taking the conceit as providing "a reductio ad absurdum of the concept of an infinite totality" you may prefer to join me in my essay Infinity (Oxford: 1964), p. 123 where I take it as showing precisely how acceptance of the actual infinite is by no means incompatible with a constructivist standpoint. Thus we are encouraged to re-examine the constrasting propositions S_7 and S_8, which raised the prospect of a breakdown in the symmetry thesis, in the light of

S_{11} A terminal 7 in pi is infinitistically if not finitistically constructible; i.e., in principle identifiable by some infinitistic procedure.

Having seen that S_7 and S_8 need not be logically equivalent, in that the former might (as far as we know) be true even though the latter be false, we can now insist, from the standpoint of an infinitistic constructivism, that S_7 and S_{11} must have the same truth value: if either is true the other is true, if either is false the other is false. The symmetry thesis is thus restored in the present case, seeing that corresponding to the platonistic idiom of S_7 there is the constructivist locution of S_{11}. We have only to accept the intelligibility of the god reporting to us that in his infinite march through the decimal expansion of pi, back to front, he came upon a 7 for the first time on March 12 in the year 98,486,263 B.C. The god's report being true, there must then be (as we should put it) a last 7 in pi viewing it front to back. Alternatively, the god may report that prior to every 7 he encountered an earlier 7 that loomed into view. Assuming the god to have 'total recall', he must be in a position to report to us either the one way or the other, and the Law of the Excluded Middle retains its full force.

Infinitistic thinking may be conducted in terms of either the small or the large infinite. If it is the large infinite which is exploited in Wittgenstein's conceit the small infinite replaces it in a comparable fancy entertained (and also rejected) by Poincaré. Another great mathematician who agreed with Brouwer and Hilbert, Poincaré was their spokesman when he insisted that "there is no actual infinity", explaining that "when we speak of an infinite collection we understand a collection to which we can add elements increasingly (similar to a subscription list which would never end, waiting for new subscribers)."[1] That Poincaré was mistaken I have no doubt. For on his view we should be entitled to assert—quite apart from all physical evidence—that there are only finitely many stars in the universe and that it cannot possibly be the case that for every star, s_n, there is another star, s_n+1, which is twice as far away from us as s_n. Any such armchair astronomy which feels free—in effect—to posit a last star in the universe on purely aprioristic grounds cannot be taken as scientifically serious.

All the less surprising, then, that Poincaré did entertain if only to reject the following pro edure expressly designed to determine infinitistically that for every 7 in pi there is another 7 that lies beyond it. Write down the first digit of pi in 1/2 minute, the second digit in the next 1/4 minute, the third digit in the next 1/8 minute, etc. ad infinitum, with the result that after the full minute has elasped the entire decimal expansion of pi—3.141592653...—will be laid out before you. Here we have an infinitistic construction, inspired by Zeno, which has been executed in accordance with the small not the large infinite, but having entertained the fancy Poincaré was prepared as a pragmatist to "refuse to argue on the hypothesis of some infinitely talkative divinity capable of thinking of an infinite number of words in a finite length of time."[2]

More recently, however, it has been acknowledged by W. V. Quine, also a pragmatist, that "having arrived at a view of expressions as finite sequences...the further step to infinite expressions is in no way audacious," though it would be "distinctly a departure from all writings on grammar and most writings on logic...to invoke infinite expressions."[3] If I have chosen to invoke them, albeit in a much more nominalistic spirit than Quine has in mind, it has been solely to narrow the gap between Brouwer and Kant on the one hand and Cantor and Plato on the other. I say 'narrow the gap', not 'close it'. For there remain platonistic theorems of great importance that even the conceits of Poincaré and Wittgenstein fail to translate into the constructive idiom. Thus the power set of the natural numbers, which is equivalent to the set K_p mentioned earlier, proves to be as non-constructible as it ever was by ordinary finitistic means. And the following theorem

S_{12} There is a well-ordering of the real numbers which Zermelo proved on the basis of the desperately controversial because highly non-constructive Axiom of Choice, we are in no position to reformulate as

S_{13} The real numbers can be well-ordered by some (denumerably) infinitistic procedure.

Rather than conclude on a pessimistic note allow me to refer you to my paper "Continuity and the Theory of Measurement," Journal of Philosophy, July 18, 1968 where an admittedly audacious—the word is doubtless required here—suggestion is offered for strengthening our present low-grade infinistic constructivism.[4] If we can succeed in satisfying ourselves of the intelligibility of

S_{14} There is a non-zero interval of time, i, which for any arbitrary rational number r is less than r minutes long.

we shall have penetrated still deeper into the small infinite, laying bare the micro-domain of the actual infinitesimals. For the non-zero interval of time i will extend over less than a millionth of a minute, less also than a billionth of a minute, a trillionth, etc. After being discredited in serious mathematics for almost a century, the actual infinitesimals have recently regained their respectability through the work of Abraham Robinson who has reconstituted the calculus by their means. But Robinson refused to take them with any metaphysical let alone physical seriousness, preferring to regard them as a purely formal device justified only by systematic considerations. Let them, however, be accredited with full ontological import, and a way is thereby opened to us for understanding how the following may be true.

S_{15} The real numbers can be well-ordered by some (nondenumerably) infinitistic procedure.

There being allowed to be one, there must then be nondenumerably many infinitesimal intervals of time in a single minute which can then be used by a god to write down all of the real numbers: he may even succeed (if only by chance) in imposing a well-ordering upon them.

The Continuum Hypothesis itself—especially perplexing after Paul Cohen's proof that its truth or falsity cannot be resolved even in non-constructive set theory as we know it—may finally emerge as provable (or disprovable) by the powerful methods afforded by a high-grade, unrestricted infinitistic constructivism whose field of action has been located in the domain of the actual infinitesimals of time. Finally, one may entertain the hope that in this way the symmetry thesis can be reinstated at last, with every platonistic theorem of Cantorian set theory finding its counterpart in the constructive idiom.

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[1]Henri Poincaré, Mathematics and Science: Last Essays (New York: 1963), p. 47.

[2]Ibid., p. 67. Cf. Charles Chihara, Ontology and the Vicious Circle Principle (Ithaca: 1973), Ch. 4.

[3]W.V. Quine, Philosophy of Logic (Englewood Cliffs: 1970), p. 91. See also Infinitistic Methods, proceedings of a 1959 symposium (Warsaw, 1961).

[4]Criticisms of my paper have been made by Henry Kyburg and Robert Coburn. See Henry Kyburg, "Measurement and Mathematics," Journal of Philosophy, January 30, 1969, pp. 23-42 and Robert Coburn, "Identity and Spatio-temporal Continuity," in Identity and Individuation, ed. M. Munitz, (New York: 1971), pp. 63-64.