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CHARACTERIZATIONS OF BERGMAN SPACES AND BLOCH SPACE IN THE UNIT BALL OF $\mathbf{C}^n$

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ABSTRACT. In this paper we prove that, in the unit ball B of $\mathbf{C}^n$, a holomorphic function f is in the Bergman space $L_a^p(B)$, $0 < p < \infty$, if and only if

$$\int_B |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^{n+1}\, d\lambda(z) < \infty,$$

where $\widetilde{\nabla}$ and λ denote the invariant gradient and invariant measure on B, respectively. Further, we give some characterizations of Bloch functions in the unit ball B, including an exponential decay characterization of Bloch functions. We also give the analogous results for $\mathrm{BMOA}(\partial B)$ functions in the unit ball.

1. INTRODUCTION

Let $A(B)$ denote the class of holomorphic functions in the unit ball of $\mathbf{C}^n$. For $0 < p < \infty$, the Bergman spaces $L_a^p(B)$, the Hardy spaces $H^p(B)$ and the Bloch space $\mathscr{B}(B)$ on the unit ball B are defined respectively as

$$L_a^p(B) = \left\{ f : f \in A(B),\ \|f\|_{L_a^p}^p = \int_B |f(z)|^p\, dm(z) < \infty \right\},$$

$$H^p(B) = \left\{ f : f \in A(B),\ \|f\|_{H^p}^p = \sup_{0<r<1} \int_{\partial B} |f(r\xi)|^p\, d\sigma(\xi) < \infty \right\}$$

and

$$\mathscr{B}(B) = \left\{ f : f \in A(B),\ \|f\|_{\mathscr{B}} = \sup_{z\in B} Q_f(z) < \infty \right\},$$

where Q_f was defined by R. Timoney in [9], dm is the normalized Lebesgue measure on B, and $d\sigma$ is the normalized Lebesgue measure on the boundary ∂B of B.

In [8], M. Stoll proved that a holomorphic function f on B is in $H^p(B)$, $0 < p < \infty$, if and only if

$$\int_B |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^n\, d\lambda(z) < \infty,$$

where $\widetilde{\nabla}$ denotes the invariant gradient and λ the invariant measure on B. Furthermore, if $f \in H^p(B)$, then

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$$\lim_{r\to 1}(1-r^2)^n \int_{B_r} |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}\, d\lambda(z) = 0,$$

where $B_r = \{z \in B : |z| < r\}$.

These results were first given by S. Yamashita in [11] and [12] in the unit disk of $\mathbf{C}$. In [13], the results for Bergman spaces similar to that of Yamashita's were given on the unit disk D.

The main purpose of this paper is to obtain the analogous result for functions in the Bergman spaces L_a^p on the unit ball B of $\mathbf{C}^n$. Furthermore, some new characterizations of Bloch space $\mathscr{B}(B)$, including an exponential decay type characterization, are given too. The main results of this paper, which are also similar to that of [13] in case $n = 1$, are as follows:

Theorem 1. *A holomorphic function f is in $L_a^p(B)$, $0 < p < \infty$, if and only if*

$$\int_B |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1-|z|^2)^{n+1}\, d\lambda(z) < \infty.$$

Furthermore, if $f \in L_a^p(B)$, then

$$\lim_{r\to 1}(1-r^2)^{n+1} \int_{B_r} |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}\, d\lambda(z) = 0.$$

Theorem 2. *Let $n > 1$, $p \geq 2$; then the following quantities are equivalent:*

(i) $\|f\|_{\mathscr{B}}^p$,

(ii) $J_2 = \sup_{a\in B} \int_B |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} \cdot (1-|\varphi_a(z)|^2)^{n+1} |\varphi_a(z)|^{-2n+2}\, d\lambda(z)$,

(iii) $J_3 = \sup_{a\in B} \int_B |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} [G(z,a)]^{1+\frac{1}{n}}\, d\lambda(z)$,

where φ_a denotes the involutive automorphism of B satisfying $\varphi_a(0) = a$, $\varphi_a(a) = 0$, $\varphi_a(\varphi_a(z)) = z$, and $G(z,a)$ denotes the Green's function of B.

Theorem 3. *Let $n > 1$; then a holomorphic function $f \in \mathscr{B}(B)$ if and only if for every $a \in B$ and every $t > 0$ there are positive constants K and β, such that*

$$\int_{E_{a,t}} |\widetilde{\nabla} f(z)|^2 [G(z,a)]^{1+\frac{1}{n}}\, d\lambda(z) \leq K e^{-\beta t},$$

where $E_{a,t} = \{z \in B : |f(z) - f(a)| > t\}$. When $f \in \mathscr{B}$, $K = K_0\|f\|_{\mathscr{B}}^2$, $\beta = C/\|f\|_{\mathscr{B}}$, where K_0 and C are constants depending only on n.

In Section 2, we first give some notations. Theorem 1 is proved in Section 3. Theorems 2 and 3 are proved in Section 4. In Section 5, we give some characterizations of $\mathrm{BMOA}(\partial B)$ which are similar to Theorems 2 and 3.

2. Notations

For each $a \in B$, let $\varphi_a(z)$ denote the involutive automorphism of B as given in [6] by W. Rudin. Let $\nabla f(z) = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ denote the complex gradient of f and $Rf = \sum_{j=1}^n z_j(\partial f/\partial z_j)$ the radial derivative of f. Let

$$d\lambda(z) = \frac{n+1}{(1-|z|^2)^{n+1}}\, dm(z);$$

then $d\lambda$ is the invariant volume measure corresponding to the Bergmen metric on B; that is,

$$\int_B f(z)\,d\lambda(z) = \int_B f \circ \psi(z)\,d\lambda(z)$$

for each $f \in L^1(d\lambda)$ and all $\psi \in \mathscr{M}$, the group of Möbius transformations of B.

For $f \in C^2(\Omega)$, Ω an open subset of B, define

$$\widetilde{\Delta} f(z) = \frac{1}{n+1}\Delta(f \circ \varphi_z)(0),$$

as in [1],

$$\widetilde{\Delta} f(z) = \frac{4}{n+1}(1-|z|^2)\sum_{i,j=1}^{n}(\delta_{ij} - z_i\bar{z}_j)\frac{\partial^2 f(z)}{\partial z_i \partial \bar{z}_j}.$$

The operator $\widetilde{\Delta}$ is invariant under $\mathscr{M}$; that is, $\widetilde{\Delta}(f \circ \psi) = (\widetilde{\Delta} f) \circ \psi$ for all $\psi \in \mathscr{M}$. See [6, Section 4.1] for details. Let $\widetilde{\nabla}$ denote the invariant gradient on B. Then

$$(\widetilde{\nabla} f(z), \widetilde{\nabla} g(z)) = \frac{4}{n+1}(1-|z|^2)\mathscr{R}\left[\sum_{i,j=1}^{n}(\delta_{ij} - z_i\bar{z}_j)\frac{\partial f}{\partial z_i}\frac{\partial g}{\partial \bar{z}_j}\right].$$

If f is holomorphic on B, it is given in [8] that

$$\widetilde{\Delta}|f|^2 = |\widetilde{\nabla} f|^2 = \frac{4}{n+1}(1-|z|^2)(|\nabla f|^2 - |Rf|^2).$$

Throughout this paper, C and C_j are constants depending only on the dimension n. M is a finite number, and $M(r)$ is a finite number for a fixed $r \in (0, 1]$. C is not necessarily the same in each appearance, nor are C_j, M, $M(r)$.

For convenience, $A(f, r) \sim B(f, r)$ means that there exist constants N_1, N_2, C_1 and C_2, so that

$$N_1 + C_1 A(f, r) \le B(f, r) \le N_2 + C_2 A(f, r),$$

where N_1, N_2 may depend on f, but they are finite quantities for a fixed function f.

By [10], the invariant Green's function on B is given by $G(z, a) = g(\varphi_a(z))$, where

$$g(z) = \frac{n+1}{2n}\int_{|z|}^{1}(1-t^2)^{n-1}t^{-2n+1}\,dt.$$

Here we state the Green's formula for an invariant Laplacian (see [7, (92.5)]). If Ω is an open subset of B, $\overline{\Omega} \subset B$, whose boundary is good enough (in our application, Ω will be an annulus) and if u, v are real-valued functions such that $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then

$$\int_\Omega (u\widetilde{\Delta} v - v\widetilde{\Delta} u)\,d\bar{\tau} = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial \bar{n}} - v\frac{\partial u}{\partial \bar{n}}\right)d\bar{\sigma},$$

where $\bar{\tau}$ and $\bar{\sigma}$ are the volume element on B and surface area element on $\partial\Omega$ determined by the Bergman metric, and $\frac{\partial}{\partial \bar{n}}$ denotes the outward normal differentiation along $\partial\Omega$ with respect to the Bergman metric. It is known (cf. [1]) that the volume element $\tilde{\tau}$ is given by

$$d\tilde{\tau}(z) = \frac{\omega_n (n+1)^n}{2n(1-|z|^2)^{n+1}}\, dm(z),$$

where ω_n denotes the Euclidean surface area of ∂B and the surface area element $\bar{\sigma}_r$ on ∂B_r is given by

$$d\bar{\sigma}_r(r\xi) = \frac{\omega_n (n+1)^{n-\frac{1}{2}} r^{2n-1}}{(1-r^2)^n}\, d\sigma(\xi).$$

3. Proof of theorem 1

To prove Theorem 1, for $\varepsilon > 0$, let

$$v_\varepsilon(z) = (|f(z)|^2 + \varepsilon)^{p/2}, \qquad 0 < p < \infty;$$

then $v_\varepsilon \in C^\infty$. Since $\widetilde{\Delta} g = 0$ on $B\backslash\{0\}$ and $g = g(r)$ on $\partial B_r = \{z \in B : |z| = r\}$, using Green's formula with $u = g - g(r)$, $v = v_\varepsilon$ and $\Omega = \{z \in B : \delta < |z| < r\}$, we can conclude

$$\int_{\delta<|z|<r} (g - g(r))\widetilde{\Delta} v_\varepsilon\, d\bar{\tau}$$
$$= -\int_{\partial B_r} v_\varepsilon \frac{\partial g}{\partial \bar{n}}\, d\bar{\sigma}_r - \left[(g(\delta) - g(r)) \int_{\partial B_\delta} \frac{\partial v_\varepsilon}{\partial \bar{n}}\, d\bar{\sigma} - \int_{\partial B_\delta} v_\varepsilon \frac{\partial g}{\partial \bar{n}}\, d\bar{\sigma}_\delta\right].$$

Because $\frac{\partial v_\varepsilon}{\partial \bar{n}}$ is bounded on ∂B_δ, $g(\delta)\delta^{2n-1} \to 0$ $(\delta \to 0)$ and g is integrable near 0 $(\delta \neq 0)$, taking the limit $\delta \to 0$, we get

$$\text{(1)} \qquad \int_{B_r} (g - g(r))\widetilde{\Delta} v_\varepsilon\, d\lambda = \int_{\partial B} v_\varepsilon(r\xi)\, d\sigma(\xi) - v_\varepsilon(0).$$

Let

$$f_p^\#(z) = \frac{p^2}{4}|f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2,$$

by [8], for $0 < p < \infty$; when $\varepsilon \to 0$,

$$\widetilde{\Delta} v_\varepsilon(z) \to f_p^\#(z) \quad \text{a.e. on } B.$$

For a fixed r, from (1) and by the monotone convergence theorem, we have

$$\text{(2)} \qquad \begin{aligned} \lim_{\varepsilon\to 0} \int_{B_r} (g - g(r))\widetilde{\Delta} v_\varepsilon\, d\lambda &= \lim_{\varepsilon\to 0} \left(\int_{\partial B} v_\varepsilon(r\xi)\, d\sigma(\xi) - v_\varepsilon(0)\right) \\ &= \int_{\partial B} |f(r\xi)|^p\, d\sigma(\xi) - |f(0)|^p. \end{aligned}$$

By (2) and the Fatou Lemma

$$\begin{aligned} \int_{B_r} (g - g(r)) f_p^\#\, d\lambda &= \int_{B_r} \underline{\lim}_{\varepsilon\to 0} (g - g(r))\widetilde{\Delta} v_\varepsilon\, d\lambda \\ &\le \underline{\lim}_{\varepsilon\to 0} \int_{B_r} (g - g(r))\widetilde{\Delta} v_\varepsilon\, d\lambda \\ &= \int_{\partial B} |f(r\xi)|^p\, d\sigma(\xi) - |f(0)|^p = M(r) < \infty; \end{aligned}$$

thus, $(g - g(r))f_p^\#$ is integrable on B_r with respect to $d\lambda$.

As $0 < p < 2$, for a fixed r, by [8], $\widetilde{\Delta} v_\varepsilon(z) \le \frac{2}{p} f_p^\#(z)$, a.e. on B, and thus

$$(g - g(r))\widetilde{\Delta} v_\varepsilon \le (g - g(r))\frac{2}{p} f_p^\#, \quad \text{a.e. on } B.$$

As $p \ge 2$, for $\varepsilon \in (0, 1]$

$$\widetilde{\Delta} v_\varepsilon \le M(r) < \infty, \quad \text{on } B_r,$$

and thus

$$(g - g(r))\widetilde{\Delta} v_\varepsilon \le M(r)(g - g(r))$$

(here $g - g(r)$ is obviously integrable on B_r).

Using the dominated convergence theorem with both $0 < p < 2$ and $p \ge 2$ and from (2), we get

$$\begin{aligned}\int_{B_r} (g - g(r)) f_p^\# \, d\lambda &= \lim_{\varepsilon \to 0} \int_{B_r} (g - g(r))\widetilde{\Delta} v_\varepsilon \, d\lambda \\ &= \int_{\partial B} |f(r\xi)|^p \, d\sigma(\xi) - |f(0)|^p.\end{aligned} \tag{3}$$

Let

$$\chi_{|z|}(t) = \begin{cases} 1, & t > |z|, \\ 0, & \text{otherwise;} \end{cases}$$

then the left side of (3) is

$$\begin{aligned}&\int_{B_r} (g(z) - g(r)) f_p^\#(z) \, d\lambda(z) \\ &\quad = \int_{B_r} f_p^\#(z) \, d\lambda(z) \left(\frac{n+1}{2n} \int_{|z|}^r \frac{(1-t^2)^{n-1}}{t^{2n-1}} \, dt \right) \\ &\quad = \frac{n+1}{2n} \int_{B_r} f_p^\#(z) \, d\lambda(z) \int_0^r \frac{(1-t^2)^{n-1}}{t^{2n-1}} \chi_{|z|}(t) \, dt \\ &\quad = \frac{n+1}{2n} \int_0^r \frac{(1-t^2)^{n-1}}{t^{2n-1}} \, dt \int_{B_t} f_p^\#(z) \, d\lambda(z).\end{aligned} \tag{4}$$

Obviously, the end of (4)

$$\begin{aligned}&\frac{n+1}{2n} \int_0^r \frac{(1-t^2)^{n-1}}{t^{2n-1}} \, dt \int_{B_t} f_p^\#(z) \, d\lambda(z) \\ &\quad \ge \frac{n+1}{2n} r^{-2n+1} \int_0^r (1-t^2)^{n-1} \, dt \int_{B_t} f_p^\#(z) \, d\lambda(z).\end{aligned} \tag{5}$$

On the other hand, for $0 < r < 1$, there exists a positive integer k, so that $1/2^k < r \le 1/2^{k-1}$; then

$$
\begin{aligned}
&\frac{n+1}{2n}\int_0^r \frac{(1-t^2)^{n-1}}{t^{2n-1}}\left(\int_{B_t} f_p^{\#}(z)\,d\lambda(z)\right)dt \\
&\quad = \frac{n+1}{2n}\left(\int_0^{\frac{1}{2^k}} + \int_{\frac{1}{2^k}}^r\right)\frac{(1-t^2)^{n-1}}{t^{2n-1}}\left(\int_{B_t} f_p^{\#}(z)\,d\lambda(z)\right)dt \\
&\quad \le \frac{n+1}{2n}\int_0^{\frac{1}{2}} \frac{(1-t^2)^{n-1}}{t^{2n-1}}\left(\int_{B_t} f_p^{\#}(z)\,d\lambda(z)\right)dt \\
(6)\qquad &\quad + \frac{n+1}{2n}2^{k(2n-1)}\left(\frac{2^{-(k-1)}}{r}\right)^{2n-1}\int_0^r (1-t^2)^{n-1}\left(\int_{B_t} f_p^{\#}(z)\,d\lambda(z)\right)dt \\
&\quad = \frac{n+1}{2n}\int_0^{\frac{1}{2}} \frac{(1-t^2)^{n-1}}{t^{2n-1}}\left(\int_{B_t} f_p^{\#}(z)\,d\lambda(z)\right)dt \\
&\quad + \frac{n+1}{2n}2^{2n-1}r^{-2n+1}\int_0^r (1-t^2)^{n-1}\left(\int_{B_t} f_p^{\#}(z)\,d\lambda(z)\right)dt \\
&\quad = I_1 + I_2.
\end{aligned}
$$

By (4) and (3),

$$
\begin{aligned}
I_1 &= \int_{B_{\frac{1}{2}}} (g(z) - g(\tfrac{1}{2}))f_p^{\#}(z)\,d\lambda(z) \\
&= \int_{\partial B} |f(\tfrac{1}{2}\xi)|^p\,d\sigma(\xi) - |f(0)|^p < \infty.
\end{aligned}
$$

Thus, it follows from (4), (5) and (6) that

$$
\int_{B_r}(g(z)-g(r))f_p^{\#}(z)\,d\lambda(z) \sim r^{-2n+1}\int_0^r (1-t^2)^{n-1}\left(\int_{B_t} f_p^{\#}(z)\,d\lambda(z)\right)dt.
$$

Moreover, by (3), we have

$$
(7)\qquad \int_{\partial B}|f(r\xi)|^p\,d\sigma(\xi) \sim r^{-2n+1}\int_0^r (1-t^2)^{n-1}\left(\int_{B_t} f_p^{\#}(z)\,d\lambda(z)\right)dt.
$$

Hence

$$
\begin{aligned}
\|f\|_{L_a^p}^p &= 2n\int_0^1 r^{2n-1}\,dr\int_{\partial B}|f(r\xi)|^p\,d\sigma(\xi) \\
&\sim \int_0^1 dr\left(\int_0^r (1-t^2)^{n-1}\,dt\int_{B_t} f_p^{\#}(z)\,d\lambda(z)\right) \\
&= \int_0^1 (1-t^2)^{n-1}\,dt\int_{B_t} f_p^{\#}(z)\,d\lambda(z)\int_t^1 dr \\
&\sim \int_0^1 (1-t^2)^{n}\,dt\int_{B_t} f_p^{\#}(z)\,d\lambda(z) \\
&= \int_B f_p^{\#}(z)\,d\lambda(z)\int_{|z|}^1 (1-t^2)^n\,dt \\
&\sim \int_B (1-|z|^2)^{n+1} f_p^{\#}(z)\,d\lambda(z) \\
&\sim \int_B |\widetilde{\nabla} f(z)|^2|f(z)|^{p-2}(1-|z|^2)^{n+1}\,d\lambda(z),
\end{aligned}
$$

and thus

$$f \in L_a^p(B) \Leftrightarrow \int_B |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^{n+1}\,d\lambda(z) < \infty.$$

That is the main part of Theorem 1.

When $f \in L_a^p(B)$, by the above result,

$$\begin{aligned}\int_0^1 \int_{B_t} &|\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^n\,d\lambda(z)\,dt \\ &= \int_B \int_{|z|}^1 |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^n\,dt\,d\lambda(z) \\ &\leq \int_B |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^{n+1}\,d\lambda(z) < \infty,\end{aligned}$$

and thus

$$\lim_{r\to 1} \int_r^1 \int_{B_t} |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^n\,d\lambda(z)\,dt = 0. \tag{8}$$

Furthermore

$$\begin{aligned}(1-r^2)^{n+1} &\int_{B_r} |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}\,d\lambda(z) \\ &\leq 2(1-r)\int_{B_r} |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^n\,d\lambda(z) \\ &\leq 2\int_r^1 \int_{B_t} |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^n\,d\lambda(z)\,dt.\end{aligned} \tag{9}$$

By (8) and (9), we conclude

$$\lim_{r\to 1}(1-r^2)^{n+1}\int_{B_r} |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}\,d\lambda(z) = 0.$$

That is the last part of Theorem 1.

Remark 1. From (7) we can get another proof of Theorem 1 of [8]. In fact, letting $r \to 1$ in (7), we get

$$\begin{aligned}\|f\|_{H^p}^p &\sim \int_0^1 (1-t^2)^{n-1}\,dt \int_{B_t} f_p^{\#}(z)\,d\lambda(z) \\ &= \int_B f_p^{\#}(z)\,d\lambda(z) \int_{|z|}^1 (1-t^2)^{n-1}\,dt \\ &\sim \int_B |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^n\,d\lambda(z).\end{aligned}$$

Remark 2. Theorem 2 in [8] can be concluded from (3). Taking the limit $r \to 1$ on two sides of (3), using the monotone convergence theorem, we get

$$\|f\|_{H^p}^p = |f(0)|^p + \frac{p^2}{4}\int_B |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2} g(z)\,d\lambda(z). \tag{10}$$

This is equivalent to the result of Theorem 2 of [8].

4. Characterizations of Bloch space in the unit ball

Lemma 1. *Let $n \geq 2$ be an integer; then there are constants C_1 and C_2 such that, for all $z \in B \backslash \{0\}$,*

$$C_1(1-|z|^2)^n|z|^{-2(n-1)} \leq g(z) \leq C_2(1-|z|^2)^n|z|^{-2(n-1)},$$

where

$$g(z) = \frac{n+1}{2n}\int_{|z|}^1 r^{-2n+1}(1-r^2)^{n-1}\,dr.$$

Proof. It is easy to see that

$$\lim_{|z|\to 1}\frac{g(z)}{(1-|z|^2)^n|z|^{-2(n-1)}} = \frac{n+1}{4n^2}, \tag{11}$$

and

$$\lim_{|z|\to 0}\frac{g(z)}{(1-|z|^2)^n|z|^{-2(n-1)}} = \frac{n+1}{4n(n-1)}. \tag{12}$$

The result of Lemma 1 comes by the continuity of $g(z)$, (11) and (12).

Proof of Theorem 2. Replacing f in (3) by $f_a = f\circ\varphi_a(\cdot) - f\circ\varphi_a(0)$, we get

$$\begin{aligned}&\frac{p^2}{4}\int_{B_r}|\tilde{\nabla} f_a(w)|^2|f_a(w)|^{p-2}(g(w)-g(r))\,d\lambda(w)\\&\quad=\int_{\partial B}|f_a(r\zeta)|^p\,d\sigma(\zeta).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{4}{p^2}\|f_a\|^p_{L^p_a} &= \frac{4}{p^2}\int_B|f_a(z)|^p\,dm(z) = \frac{8n}{p^2}\int_0^1 r^{2n-1}\,dr\int_{\partial B}|f_a(r\zeta)|^p\,d\sigma(\zeta)\\
&= 2n\int_0^1 r^{2n-1}\,dr\int_{B_r}|\tilde{\nabla} f_a(w)|^2|f_a(w)|^{p-2}(g(w)-g(r))\,d\lambda(w)\\
&\leq 2n\int_0^1 dr\int_{B_r}|\tilde{\nabla} f_a(w)|^2|f_a(w)|^{p-2}(g(w)-g(r))\,d\lambda(w)\\
&= (n+1)\int_0^1 dr\int_0^r\frac{(1-t^2)^{n-1}}{t^{2n-1}}\,dt\int_{B_t}|\tilde{\nabla} f_a(w)|^2|f_a(w)|^{p-2}\,d\lambda(w)\\
&= (n+1)\int_0^1\frac{(1-t^2)^{n-1}}{t^{2n-1}}\,dt\int_t^1 dr\int_{B_t}|\tilde{\nabla} f_a(w)|^2|f_a(w)|^{p-2}\,d\lambda(w)\\
&\leq (n+1)\int_0^1\frac{(1-t^2)^{n}}{t^{2n-1}}\,dt\int_{B_t}|\tilde{\nabla} f_a(w)|^2|f_a(w)|^{p-2}\,d\lambda(w)\\
&= (n+1)\int_B|\tilde{\nabla} f_a(w)|^2|f_a(w)|^{p-2}\,d\lambda(w)\int_{|w|}^1\frac{(1-t^2)^{n}}{t^{2n-1}}\,dt\\
&\leq 2n\int_B|\tilde{\nabla} f_a(w)|^2|f_a(w)|^{p-2}(1-|w|^2)g(w)\,d\lambda(w)\\
&\leq C\int_B|\tilde{\nabla} f_a(w)|^2|f_a(w)|^{p-2}(1-|w|^2)^{n+1}|w|^{-2n+2}\,d\lambda(w).\end{aligned}$$

The last inequality is given by Lemma 1. Letting $\varphi_a(w) = z$, we can find

$$\|f_a\|_{L_a^p}^p \le C \int_B |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} \cdot (1 - |\varphi_a(z)|^2)^{n+1} |\varphi_a(z)|^{-2n+2} \, d\lambda(z).$$

Thus we have

$$\sup_{a \in B} \|f_a\|_{L_a^p}^p \le C J_2.$$

By Theorem 4.7 in [9],

$$\|f\|_{\mathscr{B}} \le C \|f\|_X,$$

where $\|f\|_X = \sup_{z \in B} |\nabla f(z)|(1 - |z|^2)$. By the lemma in [5].

$$\|f\|_X \le c \sup_{a \in B} \|f_a\|_{L_a^p}.$$

Therefore

$$\|f\|_{\mathscr{B}}^P \le C J_2. \tag{13}$$

Because $|\varphi_a(z)| < 1$ for $z, a \in B$, we know

$$|\varphi_a(z)|^{-2n+2} \le |\varphi_a(z)|^{-2(n - \frac{1}{n})}.$$

By Lemma 1 and $G(z, a) = g(\varphi_a(z))$,

$$(1 - |\varphi_a(z)|^2)^{n+1} |\varphi_a(z)|^{-2(n - \frac{1}{n})} \le C(G(z, a))^{1 + \frac{1}{n}}.$$

Hence

$$\begin{aligned} J_2 &\le \sup_{a \in B} \int_B |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} \\ &\qquad \cdot (1 - |\varphi_a(z)|^2)^{n+1} |\varphi_a(z)|^{-2(n - \frac{1}{n})} \, d\lambda(z) \\ &\le C \sup_{a \in B} \int_B |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} (G(z, a))^{1 + \frac{1}{n}} \, d\lambda(z) \\ &= C J_3. \end{aligned} \tag{14}$$

Now let $\|f\|_{\mathscr{B}} < \infty$. By Theorem 4.7 in [9],

$$|\nabla f(z)|(1 - |z|^2) \le C_1 \|f\|_{\mathscr{B}}.$$

Thus by Lemma 2.2 in [2],

$$|\nabla_T f(z)|(1 - |z|^2)^{\frac{1}{2}} \le C_2 \|f\|_{\mathscr{B}},$$

where $\nabla_T f$ is the complex tangential gradient of f. Hence by the proof of Theorem 2.4 in [2],

$$\begin{aligned} |\widetilde{\nabla} f(z)|^2 &= \widetilde{\Delta} |f|^2(z) \\ &\le 4(1 - |z|^2)^2 |\nabla f(z)|^2 + 4(1 - |z|^2) |\nabla_T f(z)|^2 \\ &\le C \|f\|_{\mathscr{B}}^2. \end{aligned}$$

From this and Lemma 1,

$$
\begin{aligned}
J_3(a) &= \int_B |\widetilde{\nabla} f(z)|^2 |f(z)-f(a)|^{p-2} (G(z,a))^{1+\frac{1}{n}}\, d\lambda(z) \\
&\le C\|f\|_{\mathscr{B}}^2 \int_B |f\circ\varphi_a(w) - f\circ\varphi_a(0)|^{p-2} (g(w))^{1+\frac{1}{n}} \\
&\qquad\qquad \cdot (1-|w|^2)^{-n-1}\, dm(w) \\
&\le C\|f\|_{\mathscr{B}}^2 \int_B |f_a(w)|^{p-2} |w|^{\frac{-2(n^2-1)}{n}}\, dm(w).
\end{aligned} \tag{15}
$$

When $p=2$

$$
\begin{aligned}
J_3(a) &\le C\|f\|_{\mathscr{B}}^2 \int_B |w|^{\frac{-2(n^2-1)}{n}}\, dm(w) \\
&= 2nC\|f\|_{\mathscr{B}}^2 \int_0^1 r^{2n-1-\frac{2(n^2-1)}{n}}\, dr \\
&= 2nC\|f\|_{\mathscr{B}}^2 \int_0^1 r^{\frac{2}{n}-1}\, dr = n^2 C\|f\|_{\mathscr{B}}^2.
\end{aligned} \tag{16}
$$

When $p>2$, let $\alpha = \max(n^2+1, \frac{1}{p-2})$; then it is easy to know that

$$
\left(\int_B |w|^{\frac{-2(n^2-1)}{n}\frac{\alpha}{\alpha-1}}\, dm(w)\right)^{1-\frac{1}{\alpha}} = M < \infty.
$$

By the Lemma in [5],

$$
\left(\int_B |f_a(w)|^{(p-2)\alpha}\, dm(w)\right)^{1/\alpha} \le C(\Gamma((p-2)\alpha+1))^{1/\alpha} \|f\|_{\mathscr{B}}^{p-2}.
$$

Thus, by (15), using the Hölder inequality

$$
\begin{aligned}
J_3(a) &\le C\|f\|_{\mathscr{B}}^2 \left(\int_B |f_a(w)|^{(p-2)\alpha}\, dm(w)\right)^{\frac{1}{\alpha}} \\
&\qquad \cdot \left(\int_B |w|^{\frac{-2(n^2-1)}{n}\frac{\alpha}{\alpha-1}}\, dm(w)\right)^{1-\frac{1}{\alpha}} \\
&\le CM(\Gamma((p-2)\alpha+1))^{\frac{1}{\alpha}} \|f\|_{\mathscr{B}}^p.
\end{aligned} \tag{17}
$$

By (16) and (17), for $p \ge 2$,

$$
J_3 = \sup_{a\in B} J_3(a) \le C(\Gamma((p-2)\alpha+1))^{1/\alpha} \|f\|_{\mathscr{B}}^p. \tag{18}
$$

By (13), (14) and (18), the quantities $\|f\|_{\mathscr{B}}^p$, J_2 and J_3 are equivalent. The proof of Theorem 2 is complete.

Remark 3. It is authors' belief that the results of Theorem 2 should hold for all p's, that is, also for $0<p<2$. In this case more delicate techniques seem to be needed.

Proof of Theorem 3. First, let $f\in\mathscr{B}$. For each integer $k>0$, let $\alpha = n^2+1$ in (18), then we get

$$
\begin{aligned}
I_{k+2}(a) &= \int_B |\widetilde{\nabla} f(z)|^2 |f(z)-f(a)|^k (G(z,a))^{1+\frac{1}{n}}\, d\lambda(z) \\
&\le C(\Gamma(k(n^2+1)+1))^{\frac{1}{n^2+1}} \|f\|_{\mathscr{B}}^{k+2}.
\end{aligned}
$$

It is easy to see that $(\Gamma(k(n^2+1)+1))^{\frac{1}{n^2+1}} \le (n^2+1)^k k!$. Hence

$$I_{k+2}(a) \le C(n^2+1)^k k! \|f\|_{\mathscr{B}}^{k+2}.$$

Note that, when $k = 0$, the above inequality is also valid by (16). Taking a constant τ, $0 < \tau < \frac{1}{n^2+1}$, then if we set $\beta = \tau/\|f\|_{\mathscr{B}}$, we get

$$\begin{aligned} e^{\beta t} \int_{E_{a,t}} & |\widetilde{\nabla} f(z)|^2 (G(z,a))^{1+\frac{1}{n}}\, d\lambda(z) \\ &\le \int_{E_{a,t}} |\widetilde{\nabla} f(z)|^2 e^{\beta|f(z)-f(a)|} (G(z,a))^{1+\frac{1}{n}}\, d\lambda(z) \\ &\le \sum_{k=0}^{\infty} \frac{\beta^k}{k!} I_{k+2}(a) \le C \sum_{k=0}^{\infty} \frac{\beta^k}{k!} (n^2+1)^k k! \|f\|_{\mathscr{B}}^{k+2} \\ &= C\|f\|_{\mathscr{B}}^2 \sum_{k=0}^{\infty} ((n^2+1)\tau)^k = K < \infty, \end{aligned}$$

where $K = K_0 \|f\|_{\mathscr{B}}^2$, K_0 is an absolute constant. Hence

$$\int_{E_{a,t}} |\widetilde{\nabla} f(z)|^2 (G(z,a))^{1+\frac{1}{n}}\, d\lambda(z) \le K e^{-\beta t}. \tag{19}$$

Conversely, let f satisfy (19); then

$$\int_0^\infty dt \int_{E_{a,t}} |\widetilde{\nabla} f(z)|^2 (G(z,a))^{1+\frac{1}{n}}\, d\lambda(z) \le K \int_0^\infty e^{-\beta t}\, dt = \frac{K}{\beta} < \infty.$$

But

$$\begin{aligned} \int_0^\infty dt \int_{E_{a,t}} & |\widetilde{\nabla} f(z)|^2 (G(z,a))^{1+\frac{1}{n}}\, d\lambda(z) \\ &= \int_B |\widetilde{\nabla} f(z)|^2 (G(z,a))^{1+\frac{1}{n}} \left(\int_0^{|f(z)-f(a)|} dt \right) d\lambda(z) \\ &= \int_B |\widetilde{\nabla} f(z)|^2 |f(z)-f(a)| (G(z,a))^{1+\frac{1}{n}}\, d\lambda(z). \end{aligned}$$

So we get

$$\sup_{a\in B} \int_B |\widetilde{\nabla} f(z)|^2 |f(z)-f(a)| (G(z,a))^{1+\frac{1}{n}}\, d\lambda(z) \le \frac{K}{\beta} < \infty.$$

By Theorem 2 with $p = 3$, we know that $f \in \mathscr{B}(B)$. The proof is complete.

5. Characterizations of BMOA in the unit ball

Let $f \in H^1(B)$, the Hardy space in the unit ball of $\mathbf{C}^n$. We say that $f \in \mathrm{BMOA}(\partial B)$ if its radial limit function f^* is a function of bounded mean oscillations on ∂B with respect to nonisotropic balls generated by the nonisotropic metric $\rho(\zeta, \eta) = |1 - \langle \zeta, \eta \rangle|^{1/2}$ on ∂B. See [3] for details.

Let $f_a = f \circ \varphi_a(\cdot) - f \circ \varphi_a(0)$. In [4], Ouyang proved that a holomorphic function $f \in \mathrm{BMOA}(\partial B)$ if and only if

$$\sup_{a\in B} \|f_a\|_{H^p}^p < \infty. \tag{20}$$

Furthermore, he proved that if $f \in \mathrm{BMOA}(\partial B)$, then

$$\sup_{a\in B} \|f_a\|_{H^p}^p \le \frac{K\Gamma(p+1)}{C^p}\|f\|_{**}^p < \infty \tag{21}$$

where

$$\|f\|_{**} = \sup_{a\in B}\|f_a\|_{H^1}.$$

Now replacing f in (10) with f_a, we get

$$\begin{aligned}\|f_a\|_{H^p}^p &= \frac{p^2}{4}\int_B |\widetilde{\nabla} f_a(w)|^2 |f\circ\varphi_a(w) - f\circ\varphi_a(0)|^{p-2} g(w)\, d\lambda(w)\\ &= \frac{p^2}{4}\int_B |\widetilde{\nabla} f(z)|^2 |f(z)-f(a)|^{p-2} G(z,a)\, d\lambda(z).\end{aligned} \tag{22}$$

By (20), (21) and (22), we get the following

Proposition 1. *For $0 < p < \infty$, a holomorphic function $f \in \mathrm{BMOA}(\partial B)$ if and only if*

$$\sup_{a\in B}\int_B |\widetilde{\nabla} f(z)|^2 |f(z)-f(a)|^{p-2} G(z,a)\, d\lambda(z) < \infty.$$

Moreover, if $f \in \mathrm{BMOA}(\partial B)$, we have

$$\sup_{a\in B}\int_B |\widetilde{\nabla} f(z)|^2 |f(z)-f(a)|^{p-2} G(z,a)\, d\lambda(z) \le \frac{K\Gamma(p+1)}{C^p}\|f\|_{**}^p. \tag{23}$$

Remark 4. When $p = 2$, the above result was proved by J. S. Choa and B. R. Choe (see [1, Theorem A]).

Using (23) and a similar method of the proof of Theorem 3, we can obtain an exponential decay characterization of $\mathrm{BMOA}(\partial B)$ as follows.

Theorem 4. *A holomorphic function $f \in \mathrm{BMOA}(\partial B)$ if and only if for every $a \in B$ and every $t > 0$,*

$$\int_{E_{a,t}} |\widetilde{\nabla} f(z)|^2 (G(z,a))\, d\lambda(z) \le K e^{-\beta t}$$

where $E_{a,t} = \{z \in B : |f(z) - f(a)| > t\}$, and $K, \beta > 0$ are constants. When $f \in \mathrm{BMOA}(\partial B)$, $K = K_0\|f\|_{**}^2$, $\beta = C/\|f\|_{**}$, where K_0 and C are absolute constants.

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