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Essential norm estimates of weighted composition operators between Bergman spaces on strongly pseudoconvex domains

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Abstract

We give estimates of the essential norms of weighted composition operators acting between Bergman spaces on strongly pseudoconvex domains. We also characterize boundedness and compactness of these operators.

1. *Introduction*

The study of composition operators has a long and rich history. In a seminal paper, Shapiro [**19**] expressed the essential norm of composition operators on the Hardy space H^2 on the open unit disk D in terms of the Nevanlinna counting function. In the same paper, he also gave estimates of the essential norms of composition operators on the weighted Bergman spaces A^2_α on D in terms of the generalized Nevanlinna counting functions. For the Bergman space A^2 and the weighted Bergman space A^2_1, Poggi–Corradini computed the essential norms exactly in [**16**]. The criteria for compactness of these operators then follow immediately. Riedl [**18**] characterized boundedness and compactness of composition operators between Hardy spaces H^p and H^q and Smith [**20**] between weighted Bergman spaces A^p_α and A^q_β, $p \leqslant q$, on the unit disk, in terms of the Nevanlinna and generalized Nevanlinna counting functions. The compactness criteria for the case $q < p$ were done by Jarchow [**9**] for the Hardy spaces and by Smith and Yang [**21**] for the weighted Bergman spaces. Essential norm estimates of composition operators from H^p to H^q $(q < p)$ on the unit ball in $\mathbb{C}^n$ were recently obtained by Gorkin and MacCluer in [**8**].

On the other hand, weighted composition operators have been studied only recently. In this context we mention works by Contreras and Hernandez–Diaz [**3**, **4**], Ohno–Stroethoff–Zhao [**15**], and the present authors [**5**, **6**]. These papers treat weighted composition operators on various spaces of analytic functions on the disk. For other domains, in particular for domains in $\mathbb{C}^n$, the theory is still in its infancy.

Let $\varphi : \Omega \to \Omega$ be an analytic map, and let u be an analytic function on Ω. We define the weighted composition operator on the space of analytic functions f on Ω as follows: $(uC_\varphi)f(z) = u(z)f(\varphi(z))$. In this paper we give essential norm estimates for uC_φ from

$A^p(\Omega)$ to $A^q(\Omega)$, $1 < p \leqslant q < \infty$, where Ω is a smooth, bounded strongly pseudoconvex domain. We also characterize boundedness and compactness of these operators in terms of certain integral operators that reduce to the generalized Berezin transform in the case $p = q$. Our results are new even for the unit ball in $\mathbb{C}^n$. We believe our results could be extended to other types of domains in $\mathbb{C}^n$, but we leave it for future projects.

Our work has been inspired by the paper of Li [**13**], and it is a continuation of our previous paper [**6**], where we studied weighted composition operators between different weighted Bergman spaces on the unit disk. In [**13**], he studied composition operators acting on $A^2(\Omega)$. By substituting $u = 1$ in our results, we obtain characterizations of bounded and compact composition operators from $A^p(\Omega)$ to $A^q(\Omega)$, $1 < p \leqslant q < \infty$, that extend Li's results. Similarly the essential norm estimates of composition operators that improve integrability are completely new.

2. *Boundedness*

Suppose that Ω is a smoothly bounded domain in $\mathbb{C}^n$, i.e. there is a C^∞, real-valued function ρ defined on a neighborhood of $\overline{\Omega}$ such that $\Omega = \{z : \rho(z) < 0\}$ and $d\rho \neq 0$ when $\rho = 0$. A smoothly bounded domain Ω is strongly pseudoconvex if the Levi form

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) t_j \bar{t}_k > 0$$

for all $p \in \partial\Omega$ and all nonzero vectors $t \in \mathbb{C}^n$ satisfying $\partial\rho(p)(t) = 0$.

For $0 < p < \infty$, let $L^p(\Omega)$ denote the usual Lebesgue space over Ω with respect to the Lebesgue volume measure dv on $\mathbb{R}^{2n}$, and let $A^p(\Omega)$ be the Bergman space over Ω, which consists of analytic functions in $L^p(\Omega)$. Let $P : L^2(\Omega) \to A^2(\Omega)$ be the Bergman projection with the reproducing kernel $K(z, w)$, or the Bergman kernel. Let $k_z(w) = K(z, z)^{-1/2} K(z, w)$ be the normalized Bergman kernel, which is a unit vector in $A^2(\Omega)$.

Our first result is a characterization of bounded weighted composition operators between different Bergman spaces.

THEOREM 2·1. *Let $0 < p \leqslant q < \infty$, let $\varphi : \Omega \to \Omega$ be an analytic map, and let u be an analytic function on Ω. Then uC_φ is bounded from $A^p(\Omega)$ into $A^q(\Omega)$ if and only if*

$$\sup_{z \in \Omega} \int_\Omega |u(w)|^q |k_z(\varphi(w))|^{2q/p} \, dv(w) < \infty.$$

In order to prove this result, we need to use Carleson measures. For $f \in A^p(\Omega)$, let

$$\|f\|_p = \left(\int_\Omega |f(z)|^p \, dv(z) \right)^{1/p}$$

be the norm of f in $A^p(\Omega)$. If X is a Banach space of analytic functions on Ω with norm $\|f\|_X$, then for $0 < q < \infty$, a positive measure μ on Ω is called (X, q)-Carleson measure if there is a constant $C > 0$, independent of f, such that for any $f \in X$,

$$\int_\Omega |f(z)|^q \, d\mu(z) \leqslant C \|f\|_X^q.$$

In this case, we also denote

$$\|\mu\| = \sup_{f\in X, \|f\|_X\leqslant 1} \left(\int_\Omega |f(z)|^q \, d\mu(z)\right)^{1/q}.$$

The next theorem characterizes $(A^p(\Omega), q)$-Carleson measures for $0 < p \leqslant q < \infty$. The characterization of $(A^p(D), q)$-Carleson measure, where D is the open unit disk, was obtained by several authors. For the history of this result, we refer the reader to [**14**].

THEOREM 2·2. *Let $0 < p \leqslant q < \infty$. A positive measure μ on Ω is an $(A^p(\Omega), q)$-Carleson measure if and only if*

$$\sup_{z\in\Omega} \int_\Omega |k_z(w)|^{2q/p} \, d\mu(w) < \infty. \tag{2·1}$$

Proof. Let μ be an $(A^p(\Omega), q)$-Carleson measure. For any $z \in \Omega$, let $f_z = k_z^{2/p}$. Then f_z is a unit vector in $A^p(\Omega)$. Thus for any $z \in \Omega$

$$\int_\Omega |f_z(w)|^q \, d\mu(w) \leqslant C\|f_z\|_p^q = C.$$

Taking supremum over $z \in \Omega$ we get (2·1).

Conversely, let (2·1) be true. For any point $z \in \Omega$, let $r(z)$ be the distance from z to the boundary $\partial\Omega$. Let $d(z, w)$ be the quasimetric on $\partial\Omega \times \partial\Omega$ (see, for example, [**1**] or [**12**]). It is known that there is $\varepsilon > 0$ such that for each $z \in \Omega$ near the boundary of Ω, if we let $B_\varepsilon(z) = \{w \in \Omega : |r(w) - r(z)| \leqslant \varepsilon r(z), d(\pi(w), \pi(z)) < \varepsilon r(z)\}$, where $\pi(z)$ is a smooth projection from a neighborhood U of Ω to $\partial\Omega$ such that $\pi(p) = p$ for any $p \in \partial\Omega$ and $\pi^{-1}(p)$ is a smooth curve in U which intersects $\partial\Omega$ transversally at p, then we have

$$v(B_\varepsilon(z))^{-1} \approx K(z,z) \approx |K(z,w)| \approx K(w,w) \approx |k_z(w)|^2 \approx |k_w(z)|^2, \quad w \in B_\varepsilon(z). \tag{2·2}$$

(See, [**13**, p. 1306]); noticing a misprint there.) Let $f \in A^p(\Omega)$. Then $|f(w)|^p$ is plurisubharmonic. By the mean value inequality and (2·2)

$$\begin{aligned} |f(z)|^p &\leqslant \frac{C}{v(B_\varepsilon(z))} \int_{B_\varepsilon(z)} |f(w)|^p \, dv(w) \\ &\leqslant C \int_{B_\varepsilon(z)} |f(w)|^p |k_w(z)|^2 \, dv(w) \\ &\leqslant C \int_\Omega |f(w)|^p |k_w(z)|^2 \, dv(w). \end{aligned}$$

Then

$$|f(z)|^q \leqslant C \left(\int_\Omega |f(w)|^p |k_w(z)|^2 \, dv(w)\right)^{q/p}.$$

Thus by Minkowski's inequality for integrals (note that $q/p \geqslant 1$),

$$\begin{aligned} \int_\Omega |f(z)|^q \, d\mu(z) &\leqslant C \int_\Omega \left(\int_\Omega |f(w)|^p |k_w(z)|^2 \, dv(w)\right)^{q/p} d\mu(z) \\ &\leqslant C\left(\int_\Omega \left(\int_\Omega |k_w(z)|^{2q/p} \, d\mu(z)\right)^{p/q} |f(w)|^p \, dv(w)\right)^{q/p} \\ &\leqslant C\|f\|_p^q \sup_{w\in\Omega} \int_\Omega |k_w(z)|^{2q/p} \, d\mu(z). \end{aligned}$$

Thus (2·1) implies that μ is an $(A^p(\Omega), q)$-Carleson measure.

Remark. From the proof we can easily see that

$$\|\mu\|^q \approx \sup_{z\in\Omega} \int_\Omega |k_z(w)|^{2q/p}\, d\mu(w).$$

Proof of Theorem 2·1. By definition, uC_φ is bounded from $A^p(\Omega)$ into $A^q(\Omega)$ if and only if for any $f \in A^p(\Omega)$,

$$\|(uC_\varphi)f\|_q^q \leqslant C\|f\|_p^q,$$

that is

$$\int_\Omega |u(z)|^q |f(\varphi(z))|^q\, dv(z) \leqslant C\|f\|_p^q. \tag{2·3}$$

Letting $w = \varphi(z)$ we get

$$\int_\Omega |f(w)|^q\, d\mu_u(w) \leqslant C\|f\|_p^q,$$

where $\mu_u = \nu_u \circ \varphi^{-1}$ and $d\nu_u(z) = |u(z)|^q\, dv(z)$. But this means μ_u is an $(A^p(\Omega), q)$-Carleson measure. By Theorem 2·2, this is equivalent to

$$\sup_{a\in\Omega} \int_\Omega |k_a(w)|^{2q/p}\, d\mu_u(w) < \infty.$$

Changing the variable back to z we get the condition in the theorem.

3. *Essential norms*

In this section we give estimates of the essential norms of weighted composition operators from $A^p(\Omega)$ into $A^q(\Omega)$ with $1 < p \leqslant q < \infty$, where Ω is a bounded, smoothly bounded strongly pseudoconvex domain. Recall that if $T : X \to Y$ is a bounded operator between two Banach spaces X and Y, then the essential norm of T, denoted by $\|T\|_e$, is defined as the distance from T to the set of compact operators from X to Y. The following theorem is the main result of our paper.

THEOREM 3·1. *Let $1 < p \leqslant q < \infty$. Let Ω be a smoothly bounded strongly pseudoconvex domain. Suppose uC_φ is bounded from $A^p(\Omega)$ into $A^q(\Omega)$. Then there is a constant $C \geqslant 1$, independent of u and φ, such that*

$$\limsup_{z\to\partial\Omega} \int_\Omega |u(w)|^q |k_z(\varphi(w))|^{2q/p}\, dv(w)$$
$$\leqslant \|uC_\varphi\|_e^q \leqslant C \limsup_{z\to\partial\Omega} \int_\Omega |u(w)|^q |k_z(\varphi(w))|^{2q/p}\, dv(w).$$

In order to prove this theorem, we need the following lemmas. For $r > 0$, let $\Omega_r = \{z \in \Omega : r(z) \geqslant r\}$.

LEMMA 3·2. *Let $0 < p \leqslant q < \infty$. Let Ω be a strongly pseudoconvex domain. Suppose a positive measure μ on Ω is an $(A^p(\Omega), q)$-Carleson measure. Then $\mu_r = \mu|_{\Omega\setminus\Omega_r}$ is also an $(A^p(\Omega), q)$-Carleson measure. Moreover, there is an absolute constant $C > 0$ and $\varepsilon > 0$ such that*

$$\|\mu_r\|^q \leqslant C \sup_{z\in\Omega\setminus\Omega_{(1+\varepsilon)r}} \int_\Omega |k_z(w)|^{2q/p}\, d\mu(w).$$

Proof. Let $B_\varepsilon(\omega)$ be the domain defined in the proof of Theorem 2·2. Let $D_\varepsilon(z) = \{w \in \Omega : z \in B_\varepsilon(w)\}$. Let $w \in \Omega \setminus \Omega_r$. Then $r(w) < r$. It is obvious that, when $z \in B_\varepsilon(w)$, $r(z) \leqslant r(w) + \varepsilon r(w) = (1+\varepsilon)r(w) < (1+\varepsilon)r$, or $z \in \Omega \setminus \Omega_{(1+\varepsilon)r}$. Applying the mean value inequality, Minkowski's inequality for integrals and (2·2) we get

$$\begin{aligned}
\|\mu_r\|^q &= \sup_{f\in A^p(\Omega),\|f\|_p\leqslant 1} \left(\int_\Omega |f(w)|^q \, d\mu_r(w)\right) \\
&\leqslant C \sup_{f\in A^p(\Omega),\|f\|_p\leqslant 1} \int_\Omega \left(\frac{1}{v(B_\varepsilon(w))}\int_{B_\varepsilon(w)} |f(z)|^p \, dv(z)\right)^{q/p} d\mu_r(w) \\
&\leqslant C \sup_{f\in A^p(\Omega),\|f\|_p\leqslant 1} \int_\Omega \left(\int_{B_\varepsilon(w)} |f(z)|^p |k_z(w)|^2 \, dv(z)\right)^{q/p} d\mu_r(w) \\
&\leqslant C \sup_{f\in A^p(\Omega),\|f\|_p\leqslant 1} \left(\int_\Omega \left(\int_{D_\varepsilon(z)} |k_z(w)|^{2q/p} \, d\mu_r(w)\right)^{p/q} |f(z)|^p \, dv(z)\right)^{q/p} \\
&\leqslant C \sup_{f\in A^p(\Omega),\|f\|_p\leqslant 1} \sup_{z\in\Omega\setminus\Omega_{(1+\varepsilon)r}} \left(\int_\Omega |k_z(w)|^{2q/p} \, d\mu_r(w)\right)\left(\int_\Omega |f(z)|^p \, dv(z)\right)^{q/p} \\
&\leqslant C \sup_{z\in\Omega\setminus\Omega_{(1+\varepsilon)r}} \int_\Omega |k_z(w)|^{2q/p} \, d\mu(w).
\end{aligned}$$

The proof is complete.

LEMMA 3·3. *For any compact subset E on Ω, let $\chi_E = \chi(E)$ be the characteristic function on E. Then the multiplication operator $M_{\chi_E} f = \chi_E f$ is compact from $A^p(\Omega)$ to $L^p(\Omega)$.*

Proof. An easy modification of the proof of lemma 1·4·1 in [**11**, p. 50] shows that, for any compact subset E of Ω, there is a constant $C_E > 0$, depending on E and n, such that for $f \in A^p(\Omega)$,

$$\sup_{z\in E} |f(z)| \leqslant C_E \|f\|_p. \tag{3·1}$$

Let $\{f_m\}$ be a sequence in $A^p(\Omega)$ such that $\|f_m\|_p \leqslant 1$. Then for any $z \in E$,

$$|f_m(z)| \leqslant C_E \|f_m\|_p \leqslant C_E.$$

Thus $\{f_m\}$ is a normal family on Ω, and hence there is a subsequence, still denoted by $\{f_m\}$, which converges uniformly on compact subsets to an analytic function f.

Now for a fixed compact subset E and any $z \in \Omega$, we have

$$|\chi_E(z) f_m(z) - \chi_E(z) f(z)|^p \leqslant 2^p (C_E^p + \chi_E(z)|f(z)|^p),$$

and the right term is integrable on Ω. By Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}
&\lim_{m\to\infty} \int_\Omega |\chi_E(z) f_m(z) - \chi_E(z) f(z)|^p \, dv(z) \\
&= \int_\Omega \lim_{m\to\infty} |\chi_E(z) f_m(z) - \chi_E(z) f(z)|^p \, dv(z) \\
&= 0.
\end{aligned}$$

Thus $M_{\chi_E} f = \chi_E f$ is compact from $A^p(\Omega)$ to $L^p(\Omega)$.

For a decreasing sequence $\{r_k\}$ in $(0, 1)$ such that $r_k \to 0$, define $\Omega_k = \{z \in \Omega : r(z) \geqslant r_k\}$. Then each Ω_k is a compact subset of Ω. Let $\chi_k = \chi(\Omega_k)$ be the characteristic function of Ω_k.

LEMMA 3·4. *Let $1 < p < \infty$. Let $f \in A^p(\Omega)$. Then as $k \to \infty$,*

$$\sup_{f \in A^p(\Omega), \|f\|_p \leqslant 1} |P(f - \chi_k f)(w)| \to 0$$

uniformly on compact subsets of Ω.

Proof. Let $1/p + 1/p' = 1$. Using Hölder's inequality, for any $f \in A^p(\Omega)$ with $\|f\|_p \leqslant 1$, we have

$$\begin{aligned} |P(f - \chi_k f)(w)| &= \left| \int_\Omega [f(\zeta) - \chi_k(\zeta) f(\zeta)] K(w, \zeta)\, dv(\zeta) \right| \\ &= \left| \int_{\Omega \setminus \Omega_{r_k}} f(\zeta) K(w, \zeta)\, dv(\zeta) \right| \\ &\leqslant \left(\int_{\Omega \setminus \Omega_{r_k}} |f(\zeta)|^p\, dv(\zeta) \right)^{1/p} \left(\int_{\Omega \setminus \Omega_{r_k}} |K(w, \zeta)|^{p'}\, dv(\zeta) \right)^{1/p'} \\ &\leqslant \left(\int_{\Omega \setminus \Omega_{r_k}} |K(w, \zeta)|^{p'}\, dv(\zeta) \right)^{1/p'}. \end{aligned}$$

Let E be a compact subset of Ω. Since on a smoothly strongly pseudoconvex domain, the Bergman kernel function $K(w, \zeta)$ has no singularities except on the boundary diagonal $F = \{(\zeta, \zeta) : \zeta \in \partial\Omega\}$ (see **[10]**), we can easily see that $|K(w, \zeta)|$ is uniformly bounded by a constant C_E for all $w \in E$ and $\zeta \in \Omega$. Thus as $k \to \infty$,

$$\sup_{f \in A^p(\Omega), \|f\|_p \leqslant 1} |P(f - \chi_k f)(w)| \leqslant \left(\int_{\Omega \setminus \Omega_{r_k}} |K(w, \zeta)|^{p'}\, dv(\zeta) \right)^{1/p'} \leqslant C_E v(\Omega \setminus \Omega_{r_k}) \to 0$$

uniformly for $w \in E$.

LEMMA 3·5. *Let $1 < p < \infty$. A sequence of functions $\{f_k\}$ in $A^p(\Omega)$ converges to 0 weakly in $A^p(\Omega)$ if and only if f_k is norm bounded and converges to 0 uniformly on compact subsets of Ω.*

Proof. The proof is similar to the case of A^p on a planar domain (see **[2]**) since the usual duality relations hold for Bergman spaces on strongly pseudoconvex domains. Suppose f_k approaches 0 weakly in $A^p(\Omega)$. Then it is clear that $\{f_k\}$ is norm bounded and goes to 0 pointwise. But the norm boundedness of f_k implies that $\{f_k\}$ is a normal family. Thus Arzela-Ascoli Theorem implies that f_k converges to 0 uniformly on compact subsets of Ω.

Conversely, suppose there is a constant $M > 0$ such that $\|f_k\|_p \leqslant M$ for all k, and f_k converges to 0 uniformly on compact subsets of Ω. We can write $\Omega = K \cup (\Omega \setminus K)$ where K is compact and the volume of $\Omega \setminus K$ is small. Let $1/p + 1/p' = 1$. Then for $g \in A^{p'}(\Omega)$, $\int_K f_k \bar{g}\, dv \to 0$ as $k \to \infty$. For $\Omega \setminus K$, using Hölder's inequality we obtain

$$\begin{aligned} \left| \int_{\Omega \setminus K} f_k(z) \overline{g(z)}\, dv(z) \right| &\leqslant \left(\int_{\Omega \setminus K} |f_k(z)|^p\, dv(z) \right)^{1/p} \left(\int_{\Omega \setminus K} |g(z)|^{p'}\, dv(z) \right)^{1/p'} \\ &\leqslant M \left(\int_\Omega \chi_{\Omega \setminus K}(z) |g(z)|^{p'}\, dv(z) \right)^{1/p'}. \end{aligned}$$

Since $g \in A^{p'}$, the integral above can be made small. Thus $f_k \to 0$ weakly in $A^p(\Omega)$.

LEMMA 3·6. *Let $1 < p < \infty$. Then $k_z^{2/p} \to 0$ weakly in $A^p(\Omega)$ as z approaches $\partial\Omega$.*

Proof. Since Ω is a smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^n$, by the Fefferman's asymptotic expansion for the Bergman kernel K (see [**7**]), we have

$$\lim_{z \to \partial\Omega} K(z, z) = \infty$$

and for any compact set E in Ω, there is a constant C_E so that

$$|K(z, w)| \leqslant C_E, \qquad \text{for all } w \in E, z \in \Omega.$$

This implies that

$$|k_z(w)| = K(z, z)^{-1/2} |K(z, w)| \to 0,$$

as $z \to \partial\Omega$ uniformly for $w \in E$. Moreover,

$$\|k_z^{2/p}\|_{A^p} = 1, \qquad z \in \Omega.$$

Combining these with Lemma 3·5, one has proved Lemma 3·6.

Proof of Theorem 3·1. First, we prove the upper estimate. Since uC_φ is bounded from $A^p(\Omega)$ into $A^q(\Omega)$, applying uC_φ to $f(z) = 1$ we get that $u \in A^q(\Omega)$.

For a decreasing sequence r_k on $(0, 1)$ such that $r_k \to 0$, denote, as before, $\Omega_k = \{z \in \Omega : r(z) \geqslant r_k\}$. Then each Ω_k is a compact subset of Ω. Let $\chi_k = \chi(\Omega_k)$ be the characteristic function of Ω_k. By Lemma 3·3, M_{χ_k} is a compact operator from $A^p(\Omega)$ to $L^p(\Omega)$. Let P be the Bergman projection from $L^2(\Omega)$ to $A^2(\Omega)$. Then it is known [**17**] that P is a bounded operator from $L^p(\Omega)$ to $A^p(\Omega)$ for $1 < p < \infty$. Therefore, for every k, the operator

$$T_k = uC_\varphi P M_{\chi_k}$$

is a compact operator from $A^p(\Omega)$ to $A^q(\Omega)$. Define the measure ν_u by $d\nu_u = |u|^q\, dv$ and let $\mu_u = \nu_u \circ \varphi^{-1}$ be the pull-back measure of ν_u. Thus for any $f \in A^p(\Omega)$, $\|f\|_p \leqslant 1$,

$$\begin{aligned}
\|(uC_\varphi - T_k)f\|_q^q &= \int_\Omega |uC_\varphi f(w) - uC_\varphi P M_{\chi_k} f(w)|^q\, dv(w) \\
&= \int_\Omega |f(\varphi(w)) - P(\chi_k f)(\varphi(w))|^q\, d\nu_u(w) \\
&= \int_\Omega |f(\zeta) - P(\chi_k f)(\zeta)|^q\, d\mu_u(\zeta) \\
&= \int_\Omega |P(f - \chi_k f)(\zeta)|^q\, d\mu_u(\zeta) \\
&= \left(\int_{\Omega_r} + \int_{\Omega \setminus \Omega_r}\right) |P(f - \chi_k f)(\zeta)|^q\, d\mu_u(\zeta) \\
&= I_1 + I_2,
\end{aligned}$$

where $r > 0$ is a fixed number and $\Omega_r = \{z \in \Omega : r(z) \geqslant r\}$. Since Ω_r is a compact subset of Ω, by Lemma 3·4,

$$\sup_{f \in A^p(\Omega), \|f\|_p \leqslant 1} |P(f - \chi_k f)(\zeta)| \to 0$$

uniformly on Ω_r. Since $u \in A^q(\Omega)$, we know that $\mu_u(\Omega) < \infty$, and thus for any $\varepsilon > 0$, there exists a $k_0 > 0$ so that for any $k \geqslant k_0$,

$$\sup_{f \in A^p(\Omega), \|f\|_p \leqslant 1} I_1 < \varepsilon.$$

For I_2, let $\mu_{u,r} = \mu_u|_{\Omega\setminus\Omega_r}$. By Lemma 3·2, $\mu_{u,r}$ is an $(A^p(\Omega), q)$-Carleson measure. Since $\|f\|_p \leqslant 1$ we have

$$\begin{aligned} I_2 &= \int_\Omega |P(f - \chi_k f)(\zeta)|^q \, d\mu_{u,r}(\zeta) \\ &\leqslant \|\mu_{u,r}\|^q \|P(f - \chi_k f)\|_p^q \leqslant \|\mu_{u,r}\|^q \|P\|_p^q \|(1 - \chi_k) f\|_p^q \\ &\leqslant \|P\|_p^q \|\mu_{u,r}\|^q. \end{aligned}$$

Combining the estimates for $\|(uC_\varphi - T_k)f\|_q^q$, I_1 and I_2 we get that, for any fixed $0 < r < 1$, whenever $k \geqslant k_0$,

$$\|uC_\varphi - T_k\|^q = \sup_{f\in A^p(\Omega), \|f\|_p\leqslant 1} \|(uC_\varphi - T_k)f\|_q^q \leqslant \varepsilon + \|P\|_p^q \|\mu_{u,r}\|^q.$$

Since ε is arbitrary,

$$\|uC_\varphi\|_e^q \leqslant \inf_k \|uC_\varphi - T_k\|^q \leqslant \|P\|_p^q \|\mu_{u,r}\|^q$$

for any $r > 0$. Letting $r \to 0$ we get

$$\|uC_\varphi\|_e^q \leqslant \|P\|_p^q \limsup_{r\to 0} \|\mu_{u,r}\|^q.$$

By Lemma 3·2 we get

$$\begin{aligned} \|uC_\varphi\|_e^q &\leqslant C\|P\|_p^q \limsup_{z\to\partial\Omega} \int_\Omega |k_z(\zeta)|^{2q/p} \, d\mu_u(\zeta) \\ &= C\|P\|_p^q \limsup_{z\to\partial\Omega} \int_\Omega |u(w)|^q |k_z(\varphi(w))|^{2q/p} \, dv(w), \end{aligned}$$

which gives the desired upper bound.

Next, let us prove the lower estimate. Let $\mathcal{K}$ be any compact operator from $A^p(\Omega)$ to $A^q(\Omega)$. By Lemma 3·6, $k_z^{2/p} \to 0$ weakly in $A^p(\Omega)$ as $z \to \partial\Omega$. So $\|\mathcal{K}k_z^{2/p}\|_q \to 0$ as $z \to \partial\Omega$. Therefore,

$$\begin{aligned} \|uC_\varphi - \mathcal{K}\| &\geqslant \limsup_{z\to\partial\Omega} \left\|(uC_\varphi - \mathcal{K})k_z^{2/p}\right\|_q \\ &\geqslant \limsup_{z\to\partial\Omega} \left(\left\|(uC_\varphi)k_z^{2/p}\right\|_q - \left\|\mathcal{K}k_z^{2/p}\right\|_q\right) \\ &= \limsup_{z\to\partial\Omega} \left\|(uC_\varphi)k_z^{2/p}\right\|_q. \end{aligned}$$

Thus

$$\|uC_\varphi\|_e^q \geqslant \limsup_{z\to\partial\Omega} \left\|(uC_\varphi)k_z^{2/p}\right\|_q^q = \limsup_{z\to\partial\Omega} \int_\Omega |u(w)|^q |k_z(\varphi(w))|^{2q/p} \, dv(w).$$

COROLLARY 3·7. *Let $1 < p \leqslant q < \infty$. Let Ω be a smoothly bounded strongly pseudo-convex domain. Suppose uC_φ is bounded from $A^p(\Omega)$ into $A^q(\Omega)$. Then uC_φ is compact from $A^p(\Omega)$ to $A^q(\Omega)$ if and only if*

$$\limsup_{z\to\partial\Omega} \int_\Omega |u(w)|^q |k_z(\varphi(w))|^{2q/p} \, dv(w) = 0.$$

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