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BMOA AND Ba SPACES ON COMPACT BORDERED RIEMANN SURFACES*

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In [1], Metzger proposed whether John-Nirenberg's theorem for BMOA in the unit disk can be translated to Riemann surfaces. For the compact bordered Riemann surface R we give an affirmative answer. Also we introduce a special class of Ba spaces on R and then point out a relationship between BMOA(R) and Ba(R).

Let Δ be the unit disk: $\Delta = \{z : |z| < 1\}$. Then [2]

$$\mathrm{BMOA}(\Delta) = \left\{ f,\ f \text{ is analytic in } \Delta,\ \|f\|_* = \sup_{a \in \Delta} \left(\int_{\partial\Delta} |f(e^{i\theta}) - f(a)|\, d\mu_a(\theta) \right) < \infty \right\},$$

where $d\mu_a(\theta) = \dfrac{1 - |a|^2}{|e^{i\theta} - a|^2} \dfrac{d\theta}{2\pi}$.

For BMOA(Δ), we know the following important John-Nirenberg's theorem[2,3].

Theorem A. *Let $f(z)$ be an analytic function in Δ, $f(z)$ belong to* BMOA(Δ) *iff*

$$\mu_a(\{ e^{i\theta} \in \partial\Delta, |f(e^{i\theta}) - f(a)| > t \}) \leqslant K e^{-\beta t}, \tag{2.1}$$

where K and β are two constants, and when $f \in$ BMOA(Δ), *$\beta = c/\|f\|_*$, c is a constant, $\mu_a(E)$ is the harmonic measure of E with respect to Δ.*

In [1], Metzger defined the BMOA(R) in the following way.

Definition 1. Let R be a Riemann surface which possesses a harmonic Green's function, denoted by $g_R(q, q_0)$. Then a holomorphic function on R is said to belong to BMOA(R) if

$$(B_R(F))^2 = \sup_{q_0 \in R} \left\{ \int_R \int |F'(q)|^2 g_R(q, q_0)\, dq\, d\bar{q} \right\} < \infty. \tag{2.2}$$

Theorem 1. *Let R be a compact bordered Riemann surface whose boundary ∂R consists of finite numbers of analytic Jordan curves, $F(q)$ be a holomorphic function on R. Put*

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$$\widetilde{E}_0(t)=\{q\in\partial R, |F(q)-F(q_0)|>t, q_0\in R\}.$$

Then, $F(q)\in$ BMOA(R) if and only if

$$\widetilde{\mu}(\widetilde{E}_0(t))=\sup_{q_0\in R}\left\{\int_{\widetilde{E}_0(t)}\frac{\partial g_R(q,q_0)}{\partial n}ds\right\}\leqslant Ke^{-\beta t}, \tag{2.3}$$

where n is the inner normal at q, K and β are two constants, and when $F(q)\in$BMOA(R), then $\beta=c/B_R(F)$, c is a constant.

To prove Theorem 1, we need the following facts: For a compact bordered Riemann surface R, we know that the universal covering surface for R is conformally equivalent to the unit disk Δ, the group of deck transformations is a Fuchsian group $\Gamma=\{\gamma_n\}$. We denote by Ω and Δ/Γ its fundamental region and the associated Riemann surface respectively. Now if F is a holomorphic function on R, then $f(z)=F\circ\varphi(z)$ is its "pull back", where $\varphi:\Delta\to R$ is a universal covering map (or projective map). It follows immediately that $f(z)$ is an automorphic function with respect to Γ. And we define the "pull back" of BMOA(R) as follows:

$$\text{BMOA}(\Delta/\Gamma)=\text{BMOA}(\Delta)\bigcap\{\text{ automorphic functions with respect to } \Gamma\}.$$

Metzger proved

Theorem B. $f(z)\in$ BMOA(Δ/Γ) *if and only if* $F\in$ BMOA(R).

The following theorem will be used.

Theorem C[4]. *Let R be a compact bordered Riemann surface whose Green function $g_R(q,q_0)$ can be lifted to Δ via the projection $\varphi(z)$, i.e. $g_\Gamma(z,a)=g_R(\varphi(z),\varphi(a))$. Then,*

(i) $g_\Gamma(z,a)=\sum_{\gamma_n\in\Gamma}g_\Delta(z,\gamma_n(a))=\sum_{n=0}^{\infty}g_\Delta(z,a_n)$ *where* $a_n=\gamma_n(a)$, $a_0=a$ *and*

$$g_\Delta(z,a)=\log\left|\frac{1-\bar{a}z}{z-a}\right|;$$

(ii) $$\frac{\partial g_\Gamma(z,a)}{\partial\gamma}=\sum_{n=0}^{\infty}\frac{1-|a_n|^2}{|z-a_n|^2},\quad z=e^{i\theta};$$

(iii) $$\int_{\partial\Omega\bigcap\partial\Delta}\frac{\partial g_\Gamma(z,a)}{\partial\gamma}ds=2\pi,\quad z=e^{i\theta},$$

where γ is the inner normal at $z\in\partial\Omega\bigcap\partial\Delta$.

Proof of Theorem 1. We suppose that F belongs to BMOA(R). Let $\varphi:\Delta\to R$ be a projective mapping with $\varphi(a)=q_0$, $a\in\Omega$. Since $\varphi:\Delta\to R$ is one to one, $\varphi:\overline{\Omega}\to R$ is surjective and $\varphi(\partial\Omega\bigcap\partial\Delta)=\partial R$. By the definition, it holds that

$$\int_{\tilde{E_0}(t)} \frac{\partial g_R(q, q_0)}{\partial n}\, ds = \int_{E_0(t)} \frac{\partial g_\Gamma(z,a)}{\partial \gamma}\, ds, \tag{2.4}$$

where $E_0(t) = \{e^{i\theta} \in \partial\Omega \cap \partial\Delta, |f(e^{i\theta}) - f(a)| > t\}$. Putting $\gamma_n(E_0(t)) = E_n(t)$, $\gamma_n \in \Gamma$, we have

$$\bigcup_{n=0}^{\infty} E_n(t) = \{e^{i\theta} \in \partial\Delta, |f(e^{i\theta} - f(a)| > t\} = E(t).$$

Since $g_\Delta(z, \gamma_n, (a)) = g_\Delta(\gamma_n^{-1}(z), a)$, we have $dg_\Delta^*(z, \gamma_n(a)) = dg_\Delta^*(\gamma_n^{-1}(z), a)$, where $g_\Delta^*(z, a)$ is the conjugate function of $g_\Delta(z, a)$. Therefore,

$$\int_{E0(t)} \frac{\partial g_\Delta(z, \gamma_n, (a))}{\partial \gamma}\, ds = \int_{E0(t)} dg_\Delta^*(z, \gamma_n(a)) = \int_{E0(t)} dg_\Delta^*(\gamma_n^{-1}(z), a)$$

$$= \int_{\gamma_n^{-1}(E0(t))} \frac{1 - |a|^2}{(z-a)^2}\, d\theta, \qquad z = e^{i\theta}.$$

According to Theorems A and C(ii), we get

$$\frac{1}{2\pi} \int_{E_0(t)} \frac{\partial g_\Gamma(z,a)}{\partial \gamma}\, ds = \sum_{n=0}^{\infty} \int_{E_0(t)} \frac{1 - |a_n|^2}{|z - a_n|^2} \frac{d\theta}{2\pi} = \int_{\bigcup_{n=0}^{\infty} E_n(t)} \frac{1 - |a|^2}{|z - a|^2} \frac{d\theta}{2\pi}$$

$$= \int_{E(t)} d\mu_a(\theta) \leqslant K e^{-\beta t}, \tag{2.5}$$

where $\beta = c / \|f\|_*$. But we know that[1]

$$\|f\|_* \sim B_\Delta(f) = \sup_{a \in \Delta} \left\{ \iint_\Delta |f'(z)|^2 g_\Delta(z,a)\, dx dy \right\},$$

therefore $\beta = c' / B_\Delta(f)$.

On the other hand, by Theorem C(i)

$$\int_R \int |F'(q)|^2 g_R(q, q_0)\, dq d\bar{q} = \iint_\Omega |f'(z)|^2 g_\Gamma(z,a)\, dx dy$$

$$= \int \int_\Omega |f'(z)|^2 \sum_{n=0}^{\infty} g_\Delta(z, a_n)\, dx dy = \int_\Delta \int |f'(z)|^2 g_\Delta(z,a)\, dx dy. \tag{2.6}$$

Note that the supremum over a in Δ or the supremum over a in Ω is the same, i.e. $B_\Delta(f) = B_R(F)$. Combining (2.6) with (2.5) and (2.4), we obtain (2.3). Conversely, according to the above discussion, if (2.3) holds, then

$$\mu_a(\{e^{i\theta} \in \partial\Delta, |f(e^{i\theta}) - f(a)| > t\} = \frac{1}{2\pi} \int_{\tilde{E}0(t)} \frac{\partial g_R(q, q_0)}{\partial n}\, ds \leqslant K e^{-\beta t}.$$

By Theorem A, $f(z) \in \mathrm{BMOA}(\Delta)$. On the other hand, it is obvious that $f(z)$ is an automorphic function with respect to Γ, and by Theorem B, $f(z) \in \mathrm{BMOA}(\Delta/\Gamma)$, i.e. $F \in \mathrm{BMOA}(R)$.

Similar to the proof in the case of the unit disk and noting Theorem C(iii), we have

Corollary 1. *$F \in \mathrm{BMOA}(R)$ if and only if*

$$\sup_{q_0 \in R} \left(\int_{\partial R} |F(q) - F(q_0)|^p \frac{\partial g_R(q, q_0)}{\partial n} \, ds \right) = M_p < \infty \tag{2.7}$$

and $M_p^{1/p} \sim B_R(F)$. *When* $F \in \mathrm{BMOA}(R)$, $M_p \leqslant \frac{K\Gamma(p+1)}{c^p} (B_R(F))^p$.

In [5], a new function space called a Ba space is introduced. Here we shall give a special class of Ba space on a compact bordered Riemann surface, denoted by $H_{\mathrm{Ba}}(R)$.

First, let us recall the definition of Hardy class on the Riemann surface

$H_p(R) = \{F, F$ is holomorphic functions on R and $|F(q)|^p$ has a harmonic majorant $\}$.

If $H(q)$ is the least harmonic majorant of $|F(q)|^p$, then $\|F\|_{H_p} = |H(q_0)|^{1/p}$, $q_0 \in R$, and $H_p(R)$ is a Banach space with the norm $\|\cdot\|_{H_p}$, $p \geqslant 1$[6]. Given $H_p(R)$, let $H_p(\Delta/\Gamma) = H_p(\Delta) \bigcap \{$automorphic functions with respect to $\Gamma\}$ be its "pull back". It is easy to see that if $H(q)$ is the least harmornic majorant of $|F(q)|^p$, then $h(z) = H \circ \varphi(z)$ is the least harmonic majorant of $|f(z)|^p$, and $h(z)$ is automorphic with repect to Γ and $\|f\|_{H_p} = |h(a)|^{1/p}$.

Now we give the definition of Ba space on the Riemann surface as follows. Let $E(z) = \sum_{m=1}^{\infty} a_m z^m$ be an entire function with finite order $0 < \rho < \infty$ and mean type $\sigma < \infty$, and $a_m \geqslant 0$. Let $\{p_m\}$ be a sequence satisfying

$$1 \leqslant p_1 < p_2 < \cdots < p_m \rightarrow \infty$$

and

$$\overline{\lim_{m \to \infty}} \, p_m / m^{\frac{1}{\rho}} = p^* < \infty. \tag{3.1}$$

For $F(q) \in \bigcap_{m=1}^{\infty} H_{p_m}(R)$, set

$$I(F, \alpha) = \sum_{m=1}^{\infty} a_m \|F\|_{H_{p_m}}^m \alpha^m. \tag{3.2}$$

We denote by d_F the radius of convergence of (3.2). Then

$$H_{\mathrm{Ba}}(R) = \{F, F \in \bigcap_{m=1}^{\infty} H_{p_m}(R), \ d_F > 0\},$$

which is a Banach space with the norm $\|\cdot\|_{\mathrm{Ba}}$, defined by

$$\| F \|_{\mathrm{Ba}} = \inf\{ \frac{1}{|\alpha|}, I(F, |\alpha|) \leqslant 1 \}.$$

Putting $||| F ||| = \sup_{q_0 \in R} \{ \| F(q) - F(q_0) \|_{\mathrm{Ba}} \}$, we have

Theorem 2. *There exists a constant c such that*

$$c^{-1} B_R(F) \leqslant ||| F ||| \leqslant c B_R(F). \tag{3.3}$$

Proof. We denote $F(q) - F(q_0)$ by $\hat{F}(q)$. Then by Corollary 1 we have

$$\| \hat{F} \|^{p_m}_{Hp_m} = \int_{\partial R} | F(q) - F(q_0) |^{p_m} \frac{\partial g_R(q, q_0)}{\partial n} ds \leqslant \frac{K\Gamma(p_m + 1)}{C^{p_m}} (B_R(F))^{p_m},$$

and therefore

$$I(\hat{F}, \frac{|\alpha|}{B_R(F)}) = \sum_{m=1}^{\infty} a_m \| \hat{F} \|^m_{Hp_m} (\frac{|\alpha|}{B_R(F)})^m \leqslant \sum_{m=1}^{\infty} A_m |\alpha|^m, \tag{3.4}$$

where $A = aK^{\frac{m}{p_m}} (\Gamma(p+1))^{m/p_m} / c^m$. By Stirling's formula

$$\Gamma(p+1) = p^p e^{-p} \sqrt{2\pi p} (1 + o(1))$$

and the following relation

$$(\rho e \sigma)^{\frac{1}{\rho}} = \overline{\lim_{m \to \infty}} (m^{\frac{1}{\rho}} \sqrt[m]{a_m}),$$

noting the condition (3.1), we can show that $\overline{\lim_{m \to \infty}} \sqrt[m]{A_m} \leqslant p^*(\rho e \sigma)^{1/\rho} / ce$, then the radius of convergence of the series (3.4) $d_F > \alpha_0 = ec / 2p^*(\rho e \sigma)^{1/\rho} > 0$, and therefore

$$I(\hat{F}, \frac{|\alpha_0|}{B_R(F)}) \leqslant K(\rho, \sigma, p^*) < \infty.$$

Similar to the proof in [4], we get

$$||| F ||| \leqslant (\max\{ \frac{1}{\alpha_0}, \frac{1}{\alpha_0} K(\rho, \sigma, p^*) \}) B_R(F).$$

Conversely we take $c_0 = \min_m \{ \frac{1}{\sqrt[m]{a_m}} \} = \frac{1}{\sqrt[m_0]{a_{m_0}}}$, and then

$$1 \geqslant I(\hat{F}, \frac{1}{\| \hat{F} \|_{\mathrm{Ba}}}) \geqslant a_{m_0} \| \hat{F} \|^{m_0}_{Hp_{m_0}} / \| \hat{F} \|^{m_0}_{\mathrm{Ba}}.$$

Thus,

$$\| \hat{F} \|_{Hp_{m_0}} \leqslant c_0 ||| F |||.$$

By Corollary 1 we get

$$B_R(F) \leqslant c ||| F |||.$$

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