Sudoku: A Bridge to Proof Writing

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Abstract

This project involved creating curriculum materials aimed at teaching students how to validate proofs. The curriculum incorporates the Proof Framework and the Toulmin argument structure. It is common for students to have an unfamiliarity with proof structures and struggle to effectively validate proofs. This study used a pre/post-test design. The analysis of the pre-test supports the prevailing lack of understanding among students. Post-test results demonstrated improvements in both vocabulary usage and the application of proof frameworks. These findings suggest that integrating Proof Frameworks and validation techniques into the discrete mathematics curriculum can be an effective strategy to address the challenges associated with students' proof validation abilities.

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I. Acknowledgements

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II. Introduction

At SUNY Oswego, MAT 215 marks the beginning of the journey into the realm of proofs. I designed a curriculum focused on enhancing students' grasp of proofs and their ability to validate them. This curriculum was implemented in two sections of MAT 215 in Fall 2023. The assessment of the curriculum was conducted through pre- and post-tests, alongside feedback gathered via a survey.

The challenge of mastering mathematical proofs is a well-known one. Many students lack familiarity with the format or structure of a proof, frequently mistaking a single example as sufficient proof (Ko, 2010). This issue extends beyond proofs to counter examples.

Less frequently, students are tasked with reviewing proofs and justifying their validity. The initial hurdle lies in comprehending the proofs themselves, with many students struggling to decipher their contents (Mejia-Ramos & Weber, 2014). If students cannot understand what they are reading, validating the reasoning within the proof becomes an even greater challenge.

What significance do proofs hold for undergraduate mathematics students? Typically introduced in courses like Oswego's MAT 215 and high school geometry, proofs lay the groundwork for understanding logic—the cornerstone of mathematical proofs. Through hands-on creation, students encounter various proof types, setting the stage for upper-division courses. MAT 215 is often is students' first exposure to formal proof writing, but is not their last. Proofs serve as the bedrock upon which the entire structure of mathematics rests. They are an indispensable mechanism for validating the statements that form the basis of mathematical exploration. In designing my curriculum, my aim was to foster an understanding of proof validation. Mastering this skill enables students to delve deeper into the intricacies of proof writing and develop a comprehensive understanding of mathematical concepts. This solid foundation propels them forward in their undergraduate mathematics journey, ensuring enduring success.

III. Literature Review

Proofs and their Purpose

Throughout the mathematical community, the definition of a proof has been disputed and varied. The Greeks' created our current idea of proof (Harel & Sowder, 1998); However, the Greeks need for rigorous reasoning hindered them from being able to use certain mathematical concepts. In contrast, the Babylonians chose not to use any notion of modern-day proof (Harel & Sowder, 1998). Thus, our current process and notion of proof is somewhere between the two civilizations with several iterations. While there may be known origins of the premise and concept of proof, there is not a clear definition of a mathematical proof. Weber (2014) defined proof as "an argument that possesses one or more desirable properties" including convincing oneself, convincing the mathematical community, or using deductive reasoning (Weber, 2014). However, there is not an agreed upon set of properties that must be met by all proofs. It is difficult to produce one correct and full proof since there is no universal definition. There may be missing warrants in a proof, which some may see as a gap in the proof and others may see as a needed omission.

In order to conjoin the lack of unity, there have been efforts by Weber to find ways to better define and teach what a proof really contains. His model includes a cluster concept. A cluster concept is defined as "a number of cognitive models combined to form a complex cluster that is psychologically more basic than the models taken individually" (Weber, 2014). Weber's cluster model states that a proof must have a convincing argument that a mathematician would claim is true, use deductive arguments that leave no room for refutations, be transparent in the sense that a mathematician can fill in any gaps, contain a clear argument that shows why the theorem or statement is true, must be in a form that conforms to the mathematical societal norms, and finally approved by the mathematical community (Weber, 2014).

Proofs serve several functions within the mathematical community. Deductive reasoning is used throughout mathematics and mathematics education, but what is the function of proofs using deductive reasoning? The obvious answer might be to verify statements. This is true and a huge part of the function; however, as described by Micheal De Villers (1990), there are four other functions. Since there are several functions, it is unclear how a single, unified definition could be developed for proofs. Thus, the cluster concept is useful since it can capture all uses while not sacrificing one function over another.

Another function of proofs is explanation. This gives the community and the mathematician insight into why the statement or theorem is true. Thus a function of proof is answering the question "why does this statement hold true?" The next described function is systematization. Systematization can help uncover circularity in reasoning, simplify theorems and statements into axioms and definitions, help provide a "bird's-eye view" of the theorem or statements within the mathematical community. Proofs may help us see the application of statements within and outside of the mathematics community. The next role of proof is discovery. There is a common notion that the statement or theorem is created then the proof. However, many theorems were created by deductive reasoning. Examples of this can be some theorems in Euclidean Geometry. Many of the theorems would have likely never been thought of without the use of other proofs and deductive reasoning. Finally, the last function is communication. Within the mathematical community proofs are a form of discourse (Villiers, 1990). These proofs may not always be agreed upon, but they offer ways to share knowledge between mathematicians.

Previously discussed were the roles in the mathematical community, but what role do proofs play for undergraduate mathematics students? Typically, students are introduced to proofs through an introduction-to-proof class, like Oswego's MAT 215 course. This class introduces the logic that is the foundation of proofs. Then, through the hands-on creation of proofs, the different types of proofs are introduced. This class creates a foundation for upper-division proof writing courses. Proofs at this level help create a deeper understanding of theorems and topics they have always known. Reading and writing proofs share a common thread in their ability to sharpen critical thinking skills, yet students often face challenges in navigating the complexities.

Students Struggles with Proofs

Students are frequently asked to prove something in their undergraduate and graduate-level math classes. However, it is not clear that students know what qualifies as a proof or disproof. Ko (2010) found that 15% of students accepted both a counterexample and proof of the same statement (Ko, 2010). Less than 50% of students were able to construct a proof that would be considered coherent or generalizable (Ko, 2010), which is also supported by Moore (1994) (Moore, 1994). Often, students believe that proof by example is a valid format of proof. Students also believe that a counterexample of a theorem is an outlier and does not affect the truth of the statement (Ko, 2010). However, a statement or theorem cannot be true when there is a valid counterexample. Thus, these studies show that students may not have a clear understanding of what a proof includes or implies for a theorem. Ko and Knuth (2013) (Ko & Knuth, 2012) summarize these difficulties in two categories: line-to-line errors and structural errors. Line-by-line errors involve misusing or the lack thereof of warrants. Structural errors involve the mistakes created when setting up proofs.

Students may also struggle with an authoritative proof style. This means that they initially assume the theorem is true since it was presented by a credible mathematics source (Harel & Sowder, 1998; Mejia-Ramos & Weber, 2014). Beyond their ability to create a valid proof or counterexample, students lack the ability to use a consistent empirical proof scheme (Weber, 2009).

Furthermore, students are asked to read proofs throughout their education. Students are typically given information through proofs during lectures or may be asked to read proofs from their textbooks. This poses struggles for students similar to constructing proofs. Reading proofs is not often asked of them, but understanding the proof is asked of students.

There are suggestions that students benefit from reading proofs; however, students typically end up confused since they do not understand what they are reading (Mejia-Ramos & Weber, 2014). One of the most frequent struggles with reading proofs is accepting invalid proofs as valid. Students tend to focus on the details of calculations in a proof instead of looking at the warrants for the calculations. In a study done by Ramos and Weber (2014), students focused more on figuring out how the proof went from one statement to the next instead of understanding the proof. This indicates that in this study, they focused on validation rather than understanding. Many times, while reading proofs students are given valid proofs. In lectures or handouts, it is uncommon for professors to purposefully give incorrect proofs. Sometimes textbooks implement questions of grading proofs, but this does not occur often enough. Without giving students both sides of proofs, valid and invalid, students struggle to tell the difference.

Finally, students tend to believe that understanding proof does not require the construction of subproofs or diagrams (Mejia-Ramos & Weber, 2014). This assumption by students hinders them from fully understanding the proof and even seeing the larger picture and applicability of the proof they are reading.

When asked to validate a proof, students often focus on the incorrect features of a proof. Students tend to focus entirely on definition use, even if a definition is not applicable to the proof. Since students tend to get caught up on the incorrect features of the proofs, they fail to notice or consider the warrants and claims made in the proof. Since they are overlooking the warrants and claims, students do not "[gain] conviction and understanding from proofs presented in their classrooms" (Alcock & Weber, 2013). Students overlook the importance of understanding the underlying principles and different norms governing proof construction.

Differences in Proof Norms

Beyond students' abilities to understand and write valid mathematical proofs, the disagreement on the definition of proofs creates a challenging and ever-changing proof environment for students. Students are already learning a new skill of proof-writing, but they are also forced to consider their direct audience. One direct audience that students may have to consider is their professor. Professors may have different opinions or might grade the same proof differently (Moore, 2016). Thus, students might have to adapt the same proof for each different professor to receive the same or similar grade. This extra step of consideration can make proof-writing tasks more challenging and less enjoyable for students.

When it comes to writing proofs, students also need to focus on their language and writing skills. Both mathematicians and students think it is crucial for students to use typical language when creating proofs (Lew & Mejía-Ramos, 2019). If students do not stick to the rules, they might lose points or get critiqued. Key features from Lew and Ramos (2019) are using the right symbols and notation, getting the punctuation and capitalization down, and being crystal clear about definitions and assumptions. Plus, how formal or informal you are in your writing can also make a difference. By paying attention to these language details, students can make their proofs stronger, clearer, and more convincing. Thus, they must not only adapt their proof based on the audience but also the language they are using.

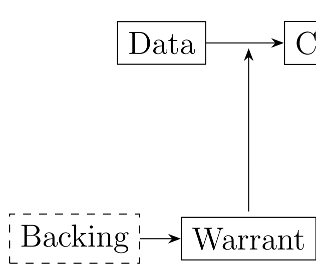
Existing Strategies

There are created methods to help students with proof structure and validation. The first is a proof framework (Selden et al., 2017). The concept of proof frameworks is a way to make proof writing almost automatic for students. Their framework begins by determining the logical structure of the statement. Firstly, the student views the quantifier of the statement, if present. If the quantifier is for all $a \in A$, then the student writes "Let $a \in A$." If it is an existence statement, then the students create such an example. These previous steps conclude the "first-level" of their framework (Selden et al., 2017). Moving onto the "second-level", the student translates the first and last lines using an ”operable interpretation” (Selden et al., 2017). The definition of ”operable interpretation” (Selden et al., 2017) will be elaborated on in the next paragraph. After doing so, the students leave a blank space for their proof. After leaving the space, the students create a statement concluding the conclusion of the statement. This is to give the students a good beginning which keeps the end in sight. This proof structure method would help students reduce their structural mistakes.

In addition to the proof framework, Selden and Selden explain an idea of “operable interpretation” (Selden et al., 2017), which is similar to Bills and Tall’s (1998) idea of operable definitions (Bills & Tall, 1998). This operable interpretation is an adaptation of the formal definition that is easily digested and applied within a proof. For example, ”if one has ‘$q \in f(A)$’, one can say ‘there is $p \in A$ so that $f(p) = q$’” (Selden et al., 2017). With the use of the operable interpretation, the students have a better idea of how and when to apply definitions within a proof (Selden et al., 2017).

After completing a proof, there are logical schemes that will help students check their proofs for validity. The Toulmin argument structure focuses on four main components of the argument: the data, warrants, backing, and claim (Toulmin, 1958). The claim is the assertion that needs justification. Then comes data, which provides justification of the claim. Then, the warrant shows how the data is connected and applicable to the claim. Finally, the backing justifies the warrant and its authority within the specific context. However, the backing is not always necessary. The relationship between each piece is illustrated below in Figure 1.

Figure 1: Toulmin Argument Structure



While the Toulmin argument structure may seem abstract, there is strong applicability to mathematics education. The Toulmin argument structure helps students catch and eliminate line-to-line errors within their proofs. Aberdein (2005) uses a similar diagram as Figure 1, to break down each line of a proof into its data type. He arranged each piece and linked them like the figure above. This allowed for the analyses of proofs and their validity. His approach to break down all pieces of the argument has been deemed rigid, but the premise of line-by-line viewing can be helpful for students. Thus, the Toulmin argument structure may be adopted in a more general method that may help students with their proof writing and validation. In similar fashion, Weber and Alcock (2013) created a framework in which students may validate proofs. They also adopt a line-to-line method of checking each implication made (Weber & Alcock, 2005). In Weber and Alcock's example, they use an analysis proof, but it can be applicable to proofs in all disciplines. This type of checking may be done informally and quickly or more in depth like Aberdein (2005) achieved.

IV. Curriculum Design

While I designed the curriculum, I followed a Backward Design method (Richards, 2013). Following the backward design method of starting from the inside and working out, I created the assessment and then guided the worksheets needed to succeed on the assessment. I focused on four behavioral objectives, as seen in Table 1. Based on the behavioral objectives, I created activities to aid in students' learning.

V. Methodology

Twenty-three students signed the consent forms to participate in the study from two sections of MAT 215. All participants were students at SUNY Oswego. To assess the curriculum, I implemented a pre-test and post-test model. The pre-test was taken in class during week six, and then the lessons were given during week eight. Additionally, a survey was included

Table 1: Behavioral Objectives

	Objectives
1.	Students will be able to create valid counter examples and explain the implications.
2.	Students will be able to produce the assumptions and conclusion for a proof framework.
3.	Students will be able to create a conditional statement from the assumptions and a conclusion found in the proof.
4.	Students will be able to validate given or self-made proofs using the Toulmin argument structure and their knowledge of proof frameworks.

at the end of the semester to assess the usefulness and frequency at which the students used the covered topics.

Only three out of the four lessons were able to be taught. I taught the three lessons for both sections, while Dr. Hanusch supervised. The professor of the course was not present during the lessons. I presented the lessons using the document camera and the projector. Students were engaged during the Sudoku warm-up. The students expressed interest in Sudoku and the proofs regarding Sudoku. When the students had free time, they would work more on the Sudoku worksheet and complete the puzzle. The students were less interactive throughout the other two lessons. They expressed insecurity when answering questions since they were learning new material. Students had not yet learned some material that I had assumed they did, which caused more hesitation from all students. During the worksheets, they were partnered or grouped up so that they could bounce ideas off of each other while practicing their new techniques. In the first section, a student had several questions that I was able to clarify. Other than that student, there were no questions regarding the material. The three lessons were completed and all three worksheets were completed in both sections.

The students were given the post-test two weeks after instruction. This was meant to be completed in class, but due to time constraints, the post-tests were completed at home. I received six post-tests in total, but only five of the post-tests could be paired to their pre-tests. The survey was distributed to the students during the last week of classes. I received five completed surveys throughout the week after they were distributed.

Methods of Analysis

Although there is limited information gained from the post-test, insightful evidence is gained from the pre-tests. The pre-test was designed to assess the student's knowledge of proof validation, counter examples, and writing their proof. I wanted to focus more deeply on the explanations of answers since eight of the ten questions were two parts, and the first part was either answered by a valid/invalid answer. Therefore, the first part of those questions had a 50% chance of being correct, regardless the student's knowledge. I scored the questions by marking them correct, half correct, or incorrect. The correct category was for questions with the correct invalid/valid answer and correct justification. Half correct was a missing or incorrect justification with a correct invalid/valid answer. Finally, incorrect was awarded when both the binary answer and justification were incorrect.

I grouped each question based on their awarded score and represented these results as shown in Tables 2 and 3 in the results section. I also evaluated the improvement or deterioration between the five matched pre- and post-tests that I could compare. I awarded the student with an I for an improvement in the score, an N for no change in score, or a D for a decrease in the score awarded. I completed this per question for each of the five students. This information is displayed in Table 4 in the results section.

Beyond assigning the score, I also used constant comparative analysis between the pre and post-test. I focused specifically on the five post-tests that I was able to find a matching pre-test to. I performed the analysis by looking for trends and vocabulary usage (Merriam & Tisdell, 2016). I looked through their justifications for their response to see if they included validation vocabulary or signs of using the proof framework method.

VI. Results

After analyzing the pre-test, post-test, and survey data, it is clear that students lack the ability to both validate and explain their proofs with clarity. Students especially struggle with invalid proofs autonomous of the type of mistake made within the proof. This observation is consistent with the research conducted by Weber and Ramos (2014), Ko (2010), Alcock and Weber (2013), and Selden and Selden (2003). The results contribute to the discourse surrounding the pedagogical approaches needed to cultivate proof validation skills among undergraduate students.

Pre-instruction Analysis

While looking at Table 2, there are low correctness percentages on questions one through six. The first question was an invalid proof due to the conclusion and assumption being switched in the proof, which is a common structural error. However, 62% of the students missed that this proof was proving the converse. The fifth question was also an invalid proof due to its misuse of the definition of divisibility. Among the twenty-three students, eleven of them missed the incorrect definition of divisibility, which is a line-by-line error. These two mistakes are emphasized and thoroughly explained within the curriculum since they are common mistakes within proofs at this level (Ko, 2010). Ko's research is supported by the data shown in Table 2, as seen by more than half of the students missing the proof validation questions. Moreover, only 26% of students were able to get the proof validation correct questions two through four. It is also shown that around 40% of students were able to correctly state the validity of the proofs, but did not provide a correct justification. Thus, the pre-test data showed that the students did not have an innate ability to validate and justify given proofs.

The last proof validation question asked the students to validate their own proof. As seen in Table 2, over half of the students did not answer this question. Within these

Table 2: Pre-test Percentages

Pre-Test	Q 1	Q 2	Q 3	Q 4	Q 5	Q 6	Q 7	Q 8	Q 9	Q 10
Incorrect	52.2%	34.8%	13.0%	26.1%	30.4%	4.4%	4.4%	21.7%	21.7%	21.7%
Correct	26.1%	8.7%	26.1%	4.4%	43.5%	60.9%	82.6%	69.6%	65.2%	13.0%
Half-Correct	17.4%	47.8%	39.1%	43.5%	13.0%	26.1%	0.0%	0.0%	0.0%	13.0%
Not Answered	4.4%	8.7%	21.7%	26.1%	13.0%	8.7%	13.0%	8.7%	13.0%	52.2%

Assumption and Conclusion General proof validation Counter example

non-answered responses, some included responses that said "I do not know how to do this." However, this only occurred on two assessments, so why the rest of the students were unable to answer the question is not clear. Additionally, 34% of students were incorrect or partially correct in their validation. Thirteen out of twenty-three proofs were correct but had no justification included with them. Therefore, over half of the students were unable to answer this question, despite creating proof in the previous question. It is reasonable to assume that the students would not know how to answer this question since many of them would have not even heard of proof validation.

As made clear by the pre-test data, the students do not have a natural ability to validate and justify their responses on given or self-made proofs. Therefore, their performance is indicative of the need for students to learn proof validation within this course. In addition to helping students validate their own proofs, research by Kirsten Pfeiffer (2009) has shown that completing proof validation techniques was shown to improve the knowledge students have of mathematical concepts .

In contrast, the students' knowledge of counterexamples was very high. Between the counterexample questions, a minimum of 65% of students were awarded correct for their response. The highest of the counterexample section being 82% of students with a correct score. Thus, the students have a high ability to correctly validate or invalidate and justify their response pertaining to counterexamples. Next, the question that assessed their proof-writing ability, question nine, showed that 65% of students answered correctly. Concluding that the students had strong proof-writing abilities at the time of the pre-test.

Post-instruction Analysis

As previously mentioned, there were only six post-assessment collected. All of the post-tests were complete. Since there were an insufficient number of post-tests, I did not compute any statistical analyses on the post-test data. However, holistically viewing Table 3 in comparison to Table 2, students were able to justify their choice of invalid or valid at a higher rate. This is seen by the overall increase of percentages in the correct row between the two tables.

Table 3: Post-Test Percentages

Post-Test	**Q 1**	**Q 2**	**Q 3**	**Q 4**	**Q 5**	**Q 6**	**Q 7**	**Q 8**	**Q 9**	**Q 10**
Incorrect	50.0%	0.0%	0.0%	16.7%	16.7%	0.0%	0.0%	0.0%	0.0%	0.0%
Correct	50.0%	66.7%	66.7%	66.7%	50.0%	100.0%	100.0%	100.0%	100.0%	66.7%
Half-Correct	0.0%	33.3%	33.3%	16.7%	33.3%	0.0%	0.0%	0.0%	0.0%	33.3%

Assumption and Conclusion   General proof validation   Counter example

I compared five of the post-tests to their pre-tests. Within the five post-tests, all five of them made improvements on two or more questions, as seen in Table 4, with the mean being 4 questions. More specifically, questions one, three, and four had at least three students increase their scores after the lesson. Also, question ten also had three students increase their score. Thus, the students improved their proof validation justification abilities after the lessons were given.

With regards to the vocabulary words regarding proof validation, the only mention that the students received was from the lecture I provided. The main professor of the course did not use the words before or after the lesson I provided. Relating to techniques, the main

Table 4: Pre- and Post-test Comparisons

Q 1	Q 2	Q 3	Q 4	Q 5	Q 6	Q 7	Q 8	Q 9	Q 10	Increase	Decrease
I	N	I	I	N	N	N	N	N	I	4	0
N	D	I	I	N	N	N	N	N	I	3	1
I	I	I	N	N	I	I	N	N	I	6	0
I	N	I	I	I	I	N	N	N	N	5	0
N	N	I	I	D	N	N	N	N	N	2	1

This table compares individual questions from five student's pre and post-test. "I" is used for improvement, "N" is used for no change, and "D" is used for decrease.

Assumption and Conclusion   General proof validation   Counter example

professor did not use a proof framework structure. Thus, the proof framework structure was introduced and used only during my lesson.

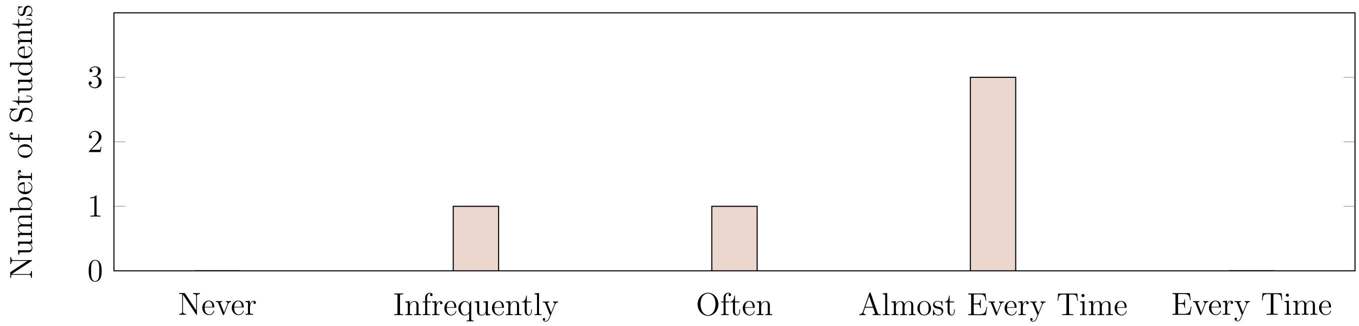
With regards to the vocabulary words regarding proof validation, the only mention that the students received was from the lecture I provided. The main professor of the course did not use the words before or after the lesson I provided. Relating to techniques, the main professor did not use a proof framework structure. Thus, the proof framework structure was introduced and used only during my lesson.

As a point of reference, there were only two mentions of "logical sequence" on the pre-tests. Now, focusing on the vocabulary usage within their post-tests. There were eight mentions of the vocabulary words "warrants," "claims," and "logical sequence." In the survey, there were two mentions of "logical reasoning" and "claims."

Another aspect of proof validation was the usage of proof frameworks. There were fourteen instances of students using the proof framework to check the given proofs. This was evident through their justifications. Within the survey, there were also six justifications that showed evidence of students using proof frameworks. As evidenced by the overall increase in responses in the correct and half-correct rows, the proof framework and vocabulary helped these five students increase their ability to validate proofs.

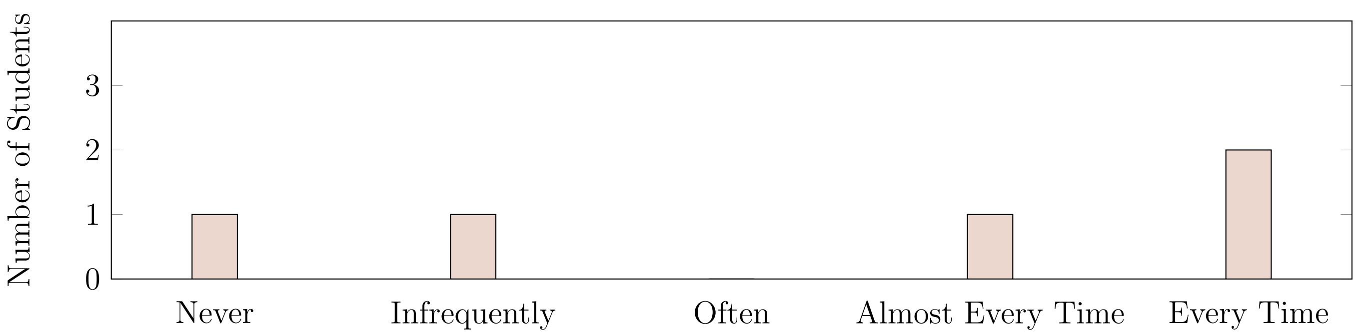
Finally, the last piece of collected data was the survey at the end of the semester.

Figure 2: How often the students validated their own proof after independently writing proofs.



As seen in Figure 2, half of the students who responded stated that they validated their proofs often to almost every time. Secondly, the survey asked how the students went about completing their validation. Over half of the responses stated that they used the same validation techniques that were taught during the lesson. Finally, the survey assessed how often the students used the proof framework taught in the lesson. As seen in Figure 3, three out of five students used the proof framework almost every day. Thus, supporting the claim that the students had long-term retention of the techniques and vocabulary from the lesson.

Figure 3: How often the students used the proof framework for their own proofs.



VII. Discussion

Throughout the results section, it became clear that the students in the two sections of discrete mathematics lacked the innate ability to validate proofs and then justify their decisions. This pertained to both given proofs and proofs they wrote independently. The lack of innate proof validation on the pre-test supports the existence of the curriculum I created, which is consistent with the research base (Ko, 2010; Selden & Selden, 2003; Alcock & Weber, 2013). After the lessons were given, the students showed results of improvement regarding their justification. The improvements were evident through vocabulary and their use of proof frameworks, which were specific to the integrated lessons.

Using the results as a guide to improve the curriculum, it is clear that there is a strong need for proof validation, specifically assumptions and conclusions. The students after the lessons were still missing the assumptions and conclusions question, which implies that there should be more emphasis on it. This is also agreed with by the content and results of Ko (2010).

In contrast, the success of the students regarding the counterexample questions shows that the curriculum could eliminate the counterexample section. This is further supported by the fact that the counterexample section of the lecture was not included in my teaching. This is contradicting Ko (2010).

Regarding the pre-test, time was a significant limitation. Since I integrated the pre-test during a class block, I allotted 30 minutes. However, when the time was up, the students had not yet completed the assessment. Thus, I gave them ten more minutes to finish up. However, this would explain why there are questions that have a high non-answered percentage.

Within the counterexample section, the questions were based on given counterexamples and asking students to validate them. There may have been different results if students were asked to create their own counterexamples and then validate them. Also, another type of question could be a statement that they must prove or disprove, then justify their response. Due to the independent nature of these questions, this may have resulted in more students struggling with the counterexample section.

When integrating the three lessons, I taught them all in one day. This condensed the timeline, which means that there was not enough time to fully explain all of the worksheets. Due to time, I had to have some students leave the Sudoku worksheet so that we could move. Thus, not all lessons were fully explained and taught as hoped. The lessons were also taught by myself, who has very limited experience in teaching discrete mathematics topics.

For the post-test, the students were able to bring the assessment home and complete it at their own leisure. This gave the students access to their notes, textbooks, and the internet. This means that the post-test data is not as clear of an indication of the lessons compared to if the students were to complete the post-test in class. The course also covers very similar topics and proofs, which also makes it difficult to accredit the students' improvements to my lessons alone. However, as seen by the improvements in Table 4, it is clear that the students improved their proof validating abilities. Thus, integrating the curriculum in a discrete mathematics course will only benefit the students with their proof validation.

If this were to be implemented again, I would advise having the assessment be completed in class. The assessment would be shorter, and I would complete the assessment in the second week of classes. Then, the lessons would be spread out throughout the next few weeks to naturally match the timeline of the course. The professor of the course should be the educator to complete the lessons so that it is most effective for the students. After the lessons are completed, the post-test will be taken in class.

VIII. Author's Reflection

Curriculum Reflection

I knew that I wanted a way to help the students in discrete mathematics, and I am so glad that I was able to help even a few students. However, creating a curriculum from scratch was a much bigger task than I had ever imagined. Through my education courses, I had done some lesson planning and creation of objectives, but never something this large. It was a very enlightening process that pushed me out of my comfort zone academically and personally.

The choice of my topic was something I am incredibly happy about. I know that sometimes working on one topic for so long can burn someone out, but I still love the topic. Beyond my persistent love for my topic, I am grateful for the impact my topic had on my future. When I began my capstone, I wanted to teach high school mathematics. Now that I have had the opportunity to do research at the collegiate level, I want to continue my education and pursue a Ph.D. in mathematics education.

The idea of creating a script for whoever was going to teach my lesson was one of the best decisions. Lesson plans give insight into what the creator wants to occur, but now how those should occur. A script gives a clear and directed approach to explaining a curriculum. Creating a script not only helped the transferability of the curriculum but also helped me hone in on the creation of the lessons. Going into the process of creating the curriculum, I had several ideas floating around in my head. However, when I started creating the script, the ideas became tangible. I was able to evaluate the legitimacy of each goal I had and decide which would be most important for me to focus on.

Concerning the creation of the worksheets, the decision of Sudoku for the opening assignment was well received. Although the proofs for the Sudoku worksheet were challenging for some students, I believe that thinking about proofs in a tangible, applicable manner was helpful. I received great engagement and feedback from students during and after the worksheet. During the creation of the worksheet, it made me think about proofs differently. This has been helpful for me in my upper-division courses, so I hope that this was at least equally beneficial for the students in discrete mathematics.

On a more constructive note, there were several things that I would like to improve on. I would like to revise the length of the assessment. One of the biggest contributors to the change of plans with the post-assessment was the time commitment. Since the assessment took too long, the students were tasked with completing it at home. Thus invalidating most of the data collected from the post-test. If the assessment was shorter, then it would be less stressful for a professor to implement the curriculum during class. There are so many topics that must be covered that it is hard to implement extra, non-required materials during class.

Secondly, I would like to adjust the questions on the assessment. Since the length was too long, I would like to take out some questions and give the students more time to focus on their explanations. This might mean that I have two or three questions on validation instead of five. I might also have the students create two proofs and then validate them. This will assess their ability to validate self-written proofs, which is the ultimate goal of the curriculum. I would also like to take most or all of the counter example questions out since the students did so well on their pre-test. This will also cut down on the time that it takes students to complete the assessment.

For the proof validation worksheet, I would have liked to create several versions so that students could work in groups. This would expose students to more proofs and opportunities to experience validation. I could also create specific individual proofs and have a student or a group of students present their validation attempt to the whole class. The decision on how to run this exercise should be determined by the professor. Some classrooms will be more receptive to certain types of active learning, thus the professor needs to decide on what is best for their classroom.

The survey was not intended to be a reliable test of the students' validation abilities, but a feedback system for myself and this capstone. However, I would have liked to receive more feedback from the survey. To receive more feedback, I would have added two questions. The first one being "Did you learn more validation techniques in the course other than the lesson I provided? If so, from where?" The second question would be "Did you find the lesson helpful? In what ways did it help you during the semester?" These additional questions would give students the ability to inform me of their opinion of the usefulness and if I was the only source of information on validation. I was able to speak to the professor and ask if she taught any more validation techniques, but the professor was unable to speak about their learning which occurs outside of the classroom.

Teaching Reflection

The counter example section had to be omitted due to time. Regardless of the omission, I was incredibly grateful to experience this research. I learned so many invaluable skills that will be useful for my continued education, career, and personal life. Also, I am happy to have gained connections with faculty, but mostly excited to have been given the chance to reflect and improve my research.

Between the two periods of teaching the curriculum, there were large amounts of feedback and insights gained. During the first class period, the student exhibited a strong interest in the Sudoku worksheet. Many of them had previously played Sudoku and found this to be an interesting take on a mathematics course. However, despite interest in the worksheet, they seemed to have challenges while writing a non-mathematical proof. Using the arbitrary symbols and the rules of Sudoku was challenging for them. I had hoped for the implementation to occur before they had started their proof unit, but that did not occur. Thus, the students already had some proof knowledge which may have made non-mathematical proofs more difficult. Since they had only just started their proof unit, they did not know the different proof types. This reinforces that the students would have been confused on the several different types of proofs. It would have been more efficient to only include direct proof methods.

In the first section of teaching, there were about 20 students in attendance. During the lesson, there were a few students who had taken up a lot of my attention by asking questions that were on and off-topic. One question was very off-topic, and I was unable to give a clear and concise answer. The number of students and off-topic questions made it more difficult for me to be aware of all students in the classroom. Toward the end of the class time, I realized that there were students who were confused for a while, and I was unable to help due to late notice and time constraints. I would like to focus on balancing my attention on all students next time so that all students have the opportunity to ask questions, if needed. This may mean that I walk around the classroom more and implement more instant feedback methods during the lesson.

Some of the students also seemed hesitant to ask questions and share answers. I believe that my status of being a peer played a role in the student's hesitation. Moreover, the unfamiliar and difficult topic of non-mathematical proofs may have made them more nervous to share their answers. Finally, having Dr. Hanusch take notes in the back of the room may have made them feel perceived, which can increase anxiety when speaking out loud.

In the second period of teaching, the number of students decreased significantly. There were only six students in the second period of teaching. Since there were fewer students and fewer questions overall, then I was able to focus on each individual student more. Since there were fewer students, getting them to answer questions out loud and talk to one another was harder. It took more prodding at the students and was a bit more awkward for everyone involved.

Since the first class period struggled with proof by contradiction, I spent more time on it during this period. The students were able to complete the proofs with less help after my explanation. I spent more time on the explanation, but I was also less nervous the second time around. Thus, the quality of the explanation was also better for the second period.

During the validation section of the teaching, I was able to use more examples I had thought of that were not on the worksheet. The extra examples helped the students apply the knowledge. In the future, I will have examples prepared to discuss before we do the worksheet. We also had the time to go over some of the validation techniques and proofs on their worksheet. This way the students were able to see if their work was correct or not, which allowed them to self-grade and reflect on what they had learned.

Teaching the curriculum for two sections was incredibly beneficial for assessing the curriculum and for my own teaching experience. Involving more opportunities for implementing literary strategies like think, pair, and share would be beneficial. Since students may not know all of the proof techniques, I would include information on proof techniques in the lecture and even on the worksheets for reference.

With regard to my teaching, I needed and still need to remember that this is a new topic for students. I went too fast and assumed that they would pick up non-mathematical proofs quickly. However, they had questions and struggles which I would like to focus more on in the future, so the students gain a deeper understanding. I taught all three lessons in one class period, but I believe that separating the lessons into a few classes would benefit the students. Moreover, I think there should be longevity with these lessons. I taught the students validation techniques right as they began proofs at the beginning of the semester. They were then not asked to use the validation skills until the post-test towards the end of the semester. If the skills were asked to be used for homework or even quizzes, the students would have been able to retain and master validation skills.

Research Reflection

While applying for integration into MAT 215, I gained valuable experience working with the Human Subjects Committee. Throughout the beginning of my capstone, I learned the importance of the Human Subject Committee's role in research. While submitting my application to their committee, I learned the importance of meticulous planning and clear writing. It was easy for me to understand my writing and plan for research since I was the one who created it. However, when the committee, who was detached from the project, read the proposal, it was not always clear. Thus, after my initial application, I made revisions to improve the clarity. Moreover, I learned important lessons on confidential information and retaining information. Maintaining anonymity is an important and difficult task to complete, especially over a whole semester.

Beyond my communication with the Human Subjects Committee, I had several chances to work with professors within the mathematics department. In the initial stages, I had wanted to integrate a control and experimental group. Ideally, there would have been two sections that were going to be the experimental group and one section would be the control group. However, after contacting the control group professor, they declined to participate. The experimental sections professor was unable to commit to multiple days of teaching, so I condensed the pre-test and lesson into an hour and twenty minutes. As the semester went on, the experimental sections professor was unable to integrate the post-assessment during class time. Thus, the post-test was completed outside of class for the student. This was a valuable lesson regarding education research. Sometimes things will not go according to plan, especially when it is not your own classroom. Having no control over the situation helped me learn that changing plans and being flexible is vital in research. My capstone may not have been what I had originally thought, but that is okay. It turned out to be something better and more insightful than I had ever thought it could have been.

Finally, the creation of the formula for an alias is something that needs to be updated so that it is more applicable in a university setting. I chose to use a formula that used their phone number multiplied by their area code. However, this failed to be considered for international students who do not have an zip code or a phone number. When the question came up in class, I told them to create a random number that they would remember. However, one of the post-tests that I received was labeled with a number that did not have a corresponding pre-test. Thus, one student had either misapplied the alias formula or they were the student that had to make up their alias and then forgot it. To mitigate these errors in the future, I would allow them to create their aliases. After they create their alias they are to write it down in some format that can be kept secret from the professor or lecturer. Another professor could keep the paper or online data that stores the aliases.

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Appendices

Appendix 1- Assessment

Validation

In this section, you will be given proofs that are of unknown validity. This means that we do not know if they are complete, valid proofs. Using your knowledge about validating proofs, determine the validity of each proof. Then explain how you were able to make that conclusion.

1. Let x and y be integers. If xy is odd, then x and y are odd.

   Proof. Assume $x, y \in \mathbb{Z}$ and x and y are odd. Thus, $x = 2k + 1$ and $y = 2l + 1$ where $k, l \in \mathbb{Z}$. Now,

   $$xy = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1.$$

   Therefore, $xy = 2(2kl + k + l) + 1$ where $(2kl + k + l) \in \mathbb{Z}$ Therefore, by definition of odd, xy is odd. □

2. Let $n \in \mathbb{Z}$. If n is even, then $n + 1$ is odd.

   Proof. Assume n is an even integer. Now, $n = 2k$ where $k \in \mathbb{Z}$ by definition of even. Thus, $n + 1 = 2k + 1$ where $k \in \mathbb{Z}$. Therefore, by definition of odd, $n + 1$ is odd.

   □

3. Let n be an integer and p be prime. If $p|n^2$, then $p|n$.

   Proof. Assume p is a prime and $n \in \mathbb{Z}$. Also, assume $p|n^2$. Then, $p|nn$ by exponent rules. Therefore, $p|n$.

   □

4. Let a and b be two integers. Prove that if $a + b$ is even, then $a - b$ is even.

   Proof. Assume $a, b \in \mathbb{Z}$ and $a + b$ is even. Then, by definition of even, $a + b = 2k$ for some $k \in \mathbb{Z}$. Then, by subtracting b, we get $a = 2k - b$. Now, doing so again, we get $a - b = 2k - 2b$. Thus, $a - b = 2(k - b)$ where $k - b \in \mathbb{Z}$. Therefore, by definition of even, $a - b$ is even. □

5. Let $a, b, c \in \mathbb{Z}$. If $a|b$ and $b|c$, then $a|c$.

   Proof. Assume $a, b, c \in \mathbb{Z}$. Assume $a|b$ and $b|c$. Thus, $a = br$ and $b = ck$ where $r, k \in \mathbb{Z}$ by definition of divisibility. Thus, by substituting b in the equation for a. So, we get $a = (ck)r = c(kr)$ where $kr \in \mathbb{Z}$. Thus, by definition of divisibility, $a|c$.

   □

The next statement is of unknown validity. You are given a proof and a disproof for the same statement. You have to decide which is the valid option for the statement. Once you choose, explain why the other option is invalid.

6. Let a, b, n be integers. If $n|ab$, then $n|a$ or $n|b$.

   Proof. Let $n = 2$, $a = 4$, $b = 3$. Now, $ab = 12$. So, $2|12$ and $2|4$. Therefore, the statement is true. □

   Disproof. Let $n = 6$, $a = 2$, and $b = 3$. Then, $ab = 6$ and $6 \mid 6$. But, $6 \nmid 2$ or $6 \nmid 3$.

Disproofs

Using your knowledge of counter examples, determine if the following counter examples are valid. If the counter example is not valid, explain why.

7. The sum of two composite numbers, n and m, must be a composite number.

   Disproof. Let $n = 11$ and $m = 6$. Then, $n + m = 17$ and 17 is prime. Therefore, the statement is false

8. The sum of two numbers, n and m, is always more than the greater number.

   Disproof. Let $n = -3$ and $m = -3$. Here, $m > n$. But, $n + m = -5$ and $-5 \not> -2$. Therefore, the statement is false.

Proof Writing

9. Prove: If $3|a$ and $3|b$, then $3|(a+b)$.

10. Using the proof validation skills you have developed, discuss the validity of your proof.

Appendix 2- Sudoku Task

Rules of Sudoku

1. Each row must contain 1-9 with no repeated numbers

2. Each column must contain 1-9 with no repeated numbers

3. Each 3x3 square must contain 1-9 with no repeated numbers

Our goal is to find the value of the heart.

7		4	1	♥	2			5
				7	9	2		1
2	1		5	3	8			7
				8	7	5	1	9
	6	8						
	9	7	2	5	4	3		
				4	5	1	7	8
	7	3				6		
	5							3

1. Could the cell contain an 8? Why or Why not?

2. Based on the 3x3 that the heart is in, the value of the heart can belong to what set?

3. However, we see in row 1 that there is already a 4, thus the value must be a __________

4. In part 2, we proved that the following statement:

   If the heart occurred in this specific Sudoku puzzle, then the value of the heart must be _____.

Now, use question 2 as an aide for writing out a proof for the statement above.

7		4	1	6	2			5
			4	7	9	2		1
2	1		5	3	8			7
				8	7	5	1	9
	6	8	▲		♠			
	9	7	2	5	4	3		
			♦	4	5	1	7	8
	7	3				6		
	5							3

Now, let's prove the value of the triangle.

5. The value of the diamond is 3. Use this fact to find the value of the space. Then use the spade to find the value of the triangle.

6. Now, write up the proof of the triangle formally.

Appendix 3- Assumptions and Conclusions

For each statement, underline the hypotheses and conclusion of the statement. Then, write out the assumptions and conclusions as they would appear in a proof.

1. For all $n \in \mathbb{Z}$, if n is odd, then $5n^2+7$ is even.

2. For all $n \in \mathbb{Z}$, if $n > 0$, then $n^2 + 2n + 1$ is composite.

Below is a proof for an unknown statement. Underline the assumptions and conclusions within the proof. After doing so, write the statement that is proved below.

Proof. Assume n is an odd integer. Thus, $n = 2k + 1$ where $k \in \mathbb{Z}$ by the definition of odd. Now, $n + 1 = (2k + 1) + 1 = 2k + 2 = 2(k + 1)$ where $k + 1 \in \mathbb{Z}$. Therefore, by definition of even, $n + 1$ is even. □

Appendix 3- Counter Examples

Counter Example Questions

1. An integer x is positive if and only if x+1 is positive.

2. If $a|(b+c)$, then $a|b$ and $a|c$.

3. Let x and y be integers. If $x^2 = y^2$, then $x = y$.

4. If $x, y, and z$ are integers and x¿y, then $xz > yz$.

5. For a, b, c integers, If $a|bc$, then $a|b$ or $a|c$.

6. For every real number x, if $x2 - 3x - 4 = 0$, then $x \geq 0$.

7. If n is a positive integer, then $n^2 + n + 41$ is prime.

Appendix 4- Validation

For each proof, underline the warrant for each claim. If the warrant is not stated for the claim, then create the warrant for that claim.

1. For all $n \in \mathbb{Z}$, If n is even then n^2 is even.

   Proof. Assume n is an even integer. Then, $n = 2k$ where $k \in \mathbb{Z}$ by definition of even. Now, $n^2 = (2k)^2$ by substitution $= 4k^2$ by exponent properties $= 2(2k^2)$ by associative property of multiplication. But, $2k^2 \in \mathbb{Z}$ by the closure of integers. Thus, by definitions of even numbers, n^2 is an even number. □

2. For all $a, b, c \in \mathbb{Z}$, if $a|b$ and $b|a + c$, then $a|c$.

   Proof. Assume a, b, and c are integers. Assume $a|b$ and $b|b + c$ by hypothesis. Now, $b = ag$ and $a + c = yb$. Thus, $a + c = (ag)y$ by substitution. $= agy$ by associative property. Now, $c = (agy) - a$ by subtraction. $= a(gy - 1)$ Also, $a(gy - 1) \in \mathbb{Z}$. Thus, by definition of division, $a|c$. □

The next proofs may or may not have mistakes. Find as many mistakes and if possible, correct the mistakes. Correcting the mistake might mean re-writing the whole proof, changing the warrant, or changing the assumptions/conclusion. If the proof is correct and there are implicit warrants, fill them in for each claim.

1. There is no integer that is both even and odd.

   Proof. Assume there is an integer that is both even and odd, say x Let $x = 2p$ and $x = 2k + 1$ where $p, q \in \mathbb{Z}$ Thus, then $2p = 2k + 1$ Then, moving over $2k$, we get $2p - 2k = 1$ Now, we get $p - k = \frac{1}{2}$, which can not occur. Thus, no integer is both even and odd. □

2. If $a|x$ and $a|y$, then $a|(bx + cy)$ for all $a, b, c \in \mathbb{Z}$.

   Proof. Let $x = as$ and $y = at$ for some $s, t \in \mathbb{Z}$ by definition of divisibility Then, $bx + cy = abs + ayt$ by substitution. Now, $bx + cy = a(bs + yt)$ by factoring. Thus, $a|abs + ayt$ □

Appendix 5- Introduction Script

There might be some students who do not know the rules of the game. They will be included on the worksheet, but showing an example there on the worksheet might be useful. The New York Times offers a puzzle online and there are plenty of websites. However, the hope is that there will be enough students that know the game to explain it to each other if needed.

This worksheet is meant to take time. This is meant to be a shared example to refer back to. This is meant for the student to discover proof writing without giving formal notation that can be confusing. Inquiry-based activity can help students make more meaning of the lesson. Having them work in pairs can help them bounce ideas off of each other while exploring the idea of proof writing.

Question 1

This is a hypothesis question. There is a proposed solution, and students have to see if it is possible. They are to use the rules of the game to check the hypothesis. This is to get them thinking about the possible answers and using the logic they need to determine the value of the cell. The value is not possible. When asked to explain why not, students should explain that the value breaks one of the rules of sudoku.

Question 2

The students now have a few leading questions to help them find the value of the heart. They are to use the rules of the game to answer the questions. The warrants are given, but they have to fill in the claim. The problem is quite simple and only has two lines. This is so students can get used to using logic to deduce the value of the heart. This will help students write out the formal proof in question 3.

Question 3

The students then are asked to write up a proof of the value of the heart. Students are to use question 2 as an aide for the middle sections of the proof. This might result in some students being "in the weeds" or struggling since proofs will be new to them. However, we want to encourage them to keep preserving through the problem. If the students struggle, give a brief mention of the elements in a proof (assumptions/ middle portion/ conclusion). Here the students have to figure out what these elements are based on question 2. The middle portion is the leading questions from question 2, so the main parts that students are reliable for are the assumptions and conclusions. The statement is given to the students, so they are able to use that as help in the proof.

Question 4

Here, students will be asked to find the value of the triangle. They are given the value of one cell, the diamond, and they are to use that value in order to find the value of a spade. This requires them to connect multiple conditional statements and write up a proof. This proof will then be used throughout the lessons as a reference for certain notations, frameworks, and help to clear up confusion. This will act as a shared experience for the class, and hopefully be memorable for the students. This will be difficult since they are to connect multiple statements within one proof. This uses the laws of syllogism. The students should be reminded of the elements of a proof. If they focus on each individual piece and then connect the pieces together, it might help the students create the proof.

In order to find the last conclusion, the students must use the process of elimination. This might pose some challenges for the students. This can be hard to write up in a proof format. The students might benefit from finding the values of the spade and triangle first and then writing up the proof.

Appendix 6- Assumptions/Conclusions Script

Logistics

The students will come to class and the notes will be given lecture style either through a document camera, chalkboard, or whiteboard. The examples will be given out on a handout so that students can avoid taking time to write out the proofs. This also allows for collection if needed for participation grade or data analysis. The examples are meant to be quick and collaborative. When picking who shares, try to present the incomplete or incorrect responses so that there might be chances of whole class discussions about the example.

Notes

The notes will be done lecture style and they will copy the notes down into notebooks. The notes can use supplemental examples if the students present with multiple questions. The examples on the handout provided should help clarify but there may be a need for in-the-moment examples.

These notes connect the ideas of logic and statements into proofs. Students will prove these statements they have previously studied. The statements the noted will focus on are universal statements.

Then, lay out proofs in a general format of A1, A2, ... An. This follows the law of syllogism. In this explanation, A1 is the hypothesis of the statement and An is the conclusion. that the students have from the statement. A2, A3,... An-1 are statements backed up by warrants that help you get to the conclusion. This can be explained with the help of the proof framework (see side note) to help students see the format of proofs that the professor is looking for.

Side Note: The proof framework is a technique for students to get started writing a proof. The technique consists of assuming the hypothesis and concluding the conclusion of the statement and leaving a space in the middle for there to be connections made. The student then unpacks the meaning of the conclusion. This can entail the students looking up definitions or theorems in order to make sense of the conclusion. This process can be repeated several times for students to get a handle on the proof and make it easier to find a plan of action for completing the proof. More of this framework can be found in Proof Frameworks- A Way to Get Started by Selden and Selden.

Make the connection to the assumptions and conclusions from the sudoku worksheet. In question 4 of the worksheet, students created a proof that used multiple claims. Go back, and use the key if need be, to label the proof with the appropriate A1, A2, ... An in order to clarify the notation. This will be reflected in the key of the sudoku worksheet. This is so students can make connections from an abstract format of A1, A2, ... An to a concrete example.

To make assumptions, you must assume the hypotheses. There should be a mention between the two and their relationship. There is a hypothesis in a statement, and then you must assume the hypothesis for it to be an assumption.

The warrants are explanations for something being true. The "because" part of a statement. This should be mentioned to avoid confusion in vocabulary.

The assumptions are the conditions set on the elements for the conclusion to be true. These come from assuming the hypothesis of a statement. Below is an example of the hypothesis underlined and then the matching assumption.

For all integers,n, 2n is even $\rightarrow$ *Assume n is an integer*

Include conclusions are the conclusion of the statement. They are also the characteristics we are saying are true based on the assumptions made. Below, the conclusion in the statement is underlined and there is an example of the conclusion in a proof.

For all integers,n, 2n is even $\rightarrow$ *Thus, 2n is even.*

Example 1

This example will be included in the worksheet that goes along with the lecture notes. For this example, give the students about one or two minutes to think on their own, then have them pair with one or two students or even their table if small enough to discuss for a minute or two. I would mention that someone will be called on to share their answer. If they are unsure, then a friend can help out, or the problem can be worked through together.

"For all n , if n is odd, then 5n2+7 is even." Give this statement to the students and have them find the hypotheses and conclusion from the statement. Then, have the students formally write out the assumptions and conclusion as they would appear in a proof. The assumptions may be one or two statements. The students are not to complete the proof but get practice finding the information needed for the assumptions and conclusion.

The information that should be assumed is underlined and the conclusion is highlighted.

Anticipated student responses:

- Students may forget to include n in the assumptions.
- Students may forget to include n is odd in the assumptions.
- Students may put n is odd in the conclusions.
- Students might struggle to find the conclusion.

Example 2

This example will be included on the handout for the students. I would give the students one or two minutes to complete this on their own and then one or two minutes to share in partners or small groups for another minute or two. I would mention in the beginning that someone will be called on to share their answer.

Give the students the statement “For all n , if $n > 0$, then $n^2 + 2n + 1$ is composite.” Have the students identify the hypothesis and conclusions of the statement. Then, they will be required to formally right out the assumptions and conclusion as if they would appear in a proof. I.e student should write something similar to “let us assume n such that $n > 0$” and then the ending statement “Therefore, $n^2 + 2n + 1$ is composite.” They should recognize where these two statements would be placed in proof.

Anticipated student responses:

- Students might struggle with the language of assuming.
- Students might struggle to use the language of conclusions.
- Students might forget to assume both conditions of n.

Example 3

The full proof will be given on the handout for students. They will be given about two minutes to read the proof and do the assigned tasks, then they will compare to a partner to see if they have differing answers. I would give about two minutes for this as well. If there seems to be disagreement and helpful conversation, I would give some extra time. I would then have them share their responses. If you see incorrect responses after they talked in pairs, I would have them share first.

Typically students are given the statement before the proof, but this example will help students check their own proofs. This can be tied together with the Toulmin argument structure, which will be seen later, to help students validate their own proofs. After going through this example, there can be mention of using this tactic on their own proofs. This tactic can help students feel more confident that the proof they created has the proof framework for the statement they are given.

Give students the proof of “For all integers n, if n is odd, then $n + 1$ is even.” without the statement written out. This should be a simple proof to avoid students focusing on the arithmetic or complexity of the proof. The focus should stay on the assumptions and conclusions made. Ask them to underline the assumptions and highlight the conclusions. Then, they will have to create a statement from their found assumptions and conclusions. Remind them that the assumptions are the assumed hypotheses.

Anticipated student responses:

- I expect there to be similar struggles to the last exercise.
- Students may have trouble finding conclusions and assumptions.

Appendix 7- Counter examples Script

Notes

Typically, students have been given correct statements so far, but that is not always the case. We like to think that everything is right, but not all statements are true.

Reiterate that when given a universal statement, it means that it applies to all elements in the specified set. So, if we were to prove that it does not hold true for all, we need to find a case that it does not work. One case is sufficient. This is a counter example. A counter example satisfies the hypothesis but does not satisfy the conclusion.

A non-math example: If I told you all dogs are brown, you would only need to show me one non-brown dog. The non-brown dog is a counter example.

When you are disproving a universal statement, you are actually proving the negation of that statement. This should be mentioned to students so they can see the relationship.

The negation of a universal statement is an existential statement so only one example would be sufficient proof.

Thus, when we offer a counter-example, we are actually creating the proof for another statement, the negation of the universal.

Example 1

Give the students "For all integers a, b and $a, b0$, if a/b, then b/a". Students will have to copy this down into their notebooks.

After giving them the statement, ask them if they think it is true. If it is, start the proof, but if not, find a counter example. Give them a minute or two, then get a show of hands or a few people to speak if they think it is correct. They can speak to a neighbor if they would like to brainstorm but keep the groups small so that there is more class discussion.

If there are some students who think it is correct, ask them about their proof or their theoretical proof, Try to point out issues they might be running into. This is a good example to show that when you try to prove something incorrect, you will run into logical problems in the proof.

There also might be students who know the statement is false but create incorrect counter examples. You may ask them to see what that "counter example" is actually saying. Remind them that the counter example must abide by the hypothesis and not abide by the conclusion. Or alternatively, it abides by the hypothesis and abides by the negation of the conclusion.

After agreeing on the truth value of the statement, then go over a few counter examples with the students, but keep in mind only one is needed. Ask the students that thought the statement was true to come up with a counter example. Try to highlight the strategy of using known and easy numbers in order to find the counter examples. Keep them simple, it can help to avoid confusion.

Then, after creating a list of possible counter examples, then ask them to find the negation of the given statement. After doing so, ask them if we know the true value of that statement. Why or why not? This may take a few minutes and allow for small discussions.

Students may have trouble finding the negation of the statement. Since this is not the intended point of the example, have them work together or guide them through it more directly so that the time is not spent on finding the negation.

Students are meant to pick up the connection that any correct counter example is actually proof of the negation being true. If there are disagreements, ask the people who are not sure of the truth value or think the truth value is false to speak first. Ask them their thoughts and explanations. Then move on to the people who had resulted in the right answer. Have a student explain the correct answer and why if possible.

Activity 1

Students will be given a piece of paper with three statements on them that are false. The students will be broken into small groups. The groups will then find a space on the board or a piece of paper. There will be at least two groups doing every statement. Thus, the students will check themselves during the bigger discussion. The students should take 5-10 minutes to work together and 10 minutes to go over the statements.

Below are some statements that are not true. Some of these have very obvious counter examples, but some are harder to find. Try to give each group a mixture of easy and difficult statements.

- An integer x is positive if and only if $x+1$ is positive.
- For every real number x, if $x^2 - 3x - 4 = 0$, then $x0$.
- For a, b, c integers, If $a|bc$, then $a|b$ or $a|c$.
- If $a|(b+c)$, then $a|b$ and $a|c$
- If n is a positive integer, then $n^2 + n + 41$ is prime.
- Let x and y be integers. If $x^2 = y^2$, then $x = y$
- If x, y, and z are integers and $x > y$, then $xz > yz$.

The students must determine if the statement is true or false. If the statement is true then they can attempt a proof if they have time. If they believe it is false, then they must find a counter example.

If students think a false statement is true, ask them if they plugged in numbers. If so, which ones did they try? Did they miss an edge case? If they did not plug in any numbers, then advise them to start with that. If they tried some numbers, have them try other numbers: larger numbers, negative numbers, zero, etc.

If there is disagreement, then try to have the incorrect students explain their thinking first. Also, make sure to check that the counter examples actually are valid. Sometimes students may not satisfy the hypothesis when creating a counter example or they do satisfy the conclusion when producing counter examples. These would lead to an invalid counter example.

If there are no incorrect counter examples, ask the class if an incorrect one would work. If they say no, then ask them why it would not work. If they do not know, then ask them what the differences are between their counter example and the invalid counter example. The goal is to have them understand a valid counter example satisfies the hypothesis and does not satisfy the conclusion.

Appendix 8- Validating Proofs Script

Students will be given proofs in a handout as a way to introduce this concept. They will be given three proofs where the first one will have all of the warrants explicitly stated and the second will have a mixture of explicit and implicit proofs. Finally, the last three proofs will be up to the students to determine if it is valid or not. During the lesson, there will be mention of new vocabulary. Since there are no notes for this section, it might be worth giving the students a chance to write the vocabulary in their notebooks. Some vocabulary might have been mentioned previously, but revisiting and directly tying the vocabulary to this method will be beneficial.

Exercise 1

The students will be given the proof on the handout. The professor can start the example by reviewing the assumptions that connect to the hypothesis of the statement, then move on to the Toulmin argument structure (see side note below). The professor can demonstrate by finding the warrant and claim, underlining it, and then checking out loud, "Does this warrant give logical evidence/reason for this claim?" After this, the professor can do one more line for an example and ask the students to finish.

Side note: The Toulmin argument is a model that is used in order to check the validity of an argument. The model, in its most basic form, includes the grounds, warrant, and claim (see picture below). The grounds are the facts or the assumptions in a proof. Then, the warrant is what connects the grounds to the claim. The warrant is the "why is this true" or "because" portion. The claim is typically the conclusion. Breaking down a proof into these parts can help validate a proof. The scheme can be applied to the whole proof or line-by-line. Breaking it down line-by-line can help students validate the "middle part" of the proof. A diagram drawn like the picture below may be drawn to help students understand the concept and help them with the new vocabulary. An example should also help the students. There can also be efforts to make a non-mathematical example, like the picture below, to help the students.

The students should take up to five minutes for the first proof. They are just underlining the warrant for each claim.

The students may have a hard time finding the warrant and they might underlie the claim. Here, prompt the student to remember what warrants it. It is the reason the claim is true. The "because" part of the statement. This can also be tied to the right side column of geometry proofs if the students remember.

The next proof will have some claims with warrants and some may not. The warrants missing are definitions, closure of integers, and algebraic properties.

The students might have difficulty filling in the warrants, so giving them guiding questions or a list of warrant options might help. A guiding question could be along the lines of "What definitions have we been working with recently?" or "What are the algebraic properties?" The goal of this lesson is to help students learn a method to check their own proofs along with being able to produce warrants.

Example 2

On the next three proofs on the worksheet, students must analyze and determine the validity of the proof using the method discussed in exercise 1. This is where the students apply the work from their last two lessons. The first two proofs will be similar to the previous proofs. The last one will be a mix of true and false statements.

It is expected that students will have a hard time finding the validity of the proofs. When working with the students, try to see if they can find the warrants. There might be none, which would show the proof to be false. If there is at least one, then the proof is true. This should be emphasized to the students.