

Risk Sensitive Optimal Synchronization of Coupled Stochastic Neural Networks with Chaotic Phenomena

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Abstract—This paper presents a new theoretical design of how an optimal synchronization is achieved for stochastic coupled neural networks with respect to a risk sensitive optimality criterion. The approach is rigorously developed by using the Hamilton-Jacobi-Bellman equation, Lyapunov technique, and inverse optimality, to obtain a risk sensitive state feedback controller, which guarantees that the chaotic drive network synchronizes with the chaotic response network influenced by uncertain noise signals, with an eye on a given risk sensitivity parameter. Finally, a numerical example is given to demonstrate the effectiveness of the proposed approach.

Keywords—Coupled Stochastic Neural Networks, Chaotic Synchronization, Risk Sensitive Optimal Control, Hamilton-Jacobi-Bellman Equation

I. INTRODUCTION

Currently, the study of chaotic synchronization using artificial neural networks has become an important issue particularly in the area of complex networks, see, for example, [1-9], and reference therein. It is known that the noise is an unavoidable factor that should be taken into consideration during the implementation of artificial neural networks. Therefore, in the past few years, the study of stochastic neural networks has started to attract the attention from the research community [10-12]. In the society of control engineering, there is a strong motivation for designing optimal systems because such systems automatically have many desirable properties, such as, stability, robustness, reduced sensitivity, etc. [13]. Furthermore, latest research results have shown that the methodology of optimal control appears to be a very encouraging method in the area of modeling biologically-inspired neural networks. [14-15]. There are many strong evidences to support the statement that biological movements are optimal. Even though, the exact cost function that is being optimized in a particular mission is not always clear. It has been shown by recent research that risk sensitive optimal control is an efficient methodology to the control of stochastic nonlinear systems, with a focus on

noise disturbance attenuation. However, to the author's knowledge, no publication presents the results about using risk sensitive optimal control for chaotic synchronization of coupled stochastic neural networks.

Motivated by the discussions above, the main goal of this paper is to present how a risk sensitive optimal control is achieved for chaotic synchronization of coupled stochastic neural networks. Therefore, the chaotic response network influenced by uncertain noise signals can be guaranteed to synchronize with the chaotic drive network. The rest of the paper is organized as follows. In Section II, we present the problem formulation and mathematical preliminaries. In Section III, we detail the theoretical results. In Section IV, we demonstrate the performance of our design with a numerical example. Finally, the conclusion of the paper is given in Section V.

II. PROBLEM FORMULATION

In this paper, we consider the following chaotic delayed neural networks as the drive system:

$$dx_i(t) = \left(-\lambda_i x_i(t) + \sum_{j=1}^n w_{ij}^1 \tilde{f}_j(x_j(t)) + \sum_{j=1}^n w_{ij}^2 \tilde{g}_j(x_j(t-\tau)) + I_i \right) dt \quad (1)$$

where $i = 1, 2, \dots, n$. Mathematically, this can be described by the following matrix-vector compact form:

$$dx(t) = \left(-Ax(t) + W_1 \tilde{f}(x(t)) + W_2 \tilde{g}(x(t-\tau)) + I \right) dt \quad (2)$$

where $x(t) \in R^n$ is the state of the time-delay neural network, $I \in R^n$ is the input, $A = \text{diag}(\lambda_1, \dots, \lambda_1, \dots, \lambda_n) \in R^{n \times n}$, both $\tilde{f}(x(t)) = [\tilde{f}_1(x_1(t)), \dots, \tilde{f}_j(x_j(t)), \dots, \tilde{f}_n(x_n(t))]^T \in R^n$ and $\tilde{g}(x(t-\tau)) = [\tilde{g}_1(x_1(t-\tau)), \dots, \tilde{g}_j(x_j(t-\tau)), \dots, \tilde{g}_n(x_n(t-\tau))]^T \in R^n$ represent the activation functions of neurons, in addition,

III. CONTROLLER DESIGN

both $\tilde{f}_j(x_j(t))$ and $\tilde{g}_j(x_j(t - \tau))$ are sigmoidal functions that are scalar ones, $W_1 \in R^{n \times n}$ and $W_2 \in R^{n \times n}$ are weight matrices, $\tau \in R^+$ is the time delay.

Based on the concept of drive-response chaotic systems, the corresponding response system of (2) is given as follows:

$$dy(t) = \left(-Ay(t) + W_1 \tilde{f}(y(t)) + W_2 \tilde{g}(y(t - \tau)) + I \right) dt + u(t) dt + d\Psi \quad (3)$$

where $u(t) \in R^n$ is the control signal and $\Psi \in R^n$ is an n -dimensional independent standard Wiener process.

Our design is to develop a risk sensitive optimal control $u(t)$ to guarantee that the chaotic response system influenced by uncertain noise signals will be synchronized with the chaotic drive system.

Let us subtract (2) from (3), which yield the following stochastic error system:

$$de(t) = \left(-Ae(t) + W_1 f(e(t)) + W_2 g(e(t - \tau)) \right) dt + u(t) dt + d\Psi \quad (4)$$

where $e(t) = y(t) - x(t)$ is the error state, $f(e(t)) = \tilde{f}(y(t)) - \tilde{f}(x(t)) = \tilde{f}(x(t) + e(t)) - \tilde{f}(x(t))$ and $g(e(t - \tau)) = \tilde{g}(y(t - \tau)) - \tilde{g}(x(t - \tau)) = \tilde{g}(x(t - \tau) + e(t - \tau)) - \tilde{g}(x(t - \tau))$.

Associated with system (4), let us introduce a risk sensitive cost functional

$$J = \limsup_{T \rightarrow \infty} \frac{2}{\theta T} \ln E \left[\exp \left(\frac{\theta}{2} \int_0^T (q(e(t)) + u^T r(e(t)) u) dt \right) \right] \quad (5)$$

where $\theta > 0$ is the risk sensitive parameter, and both $q(e(t))$ and $r(e(t))$ are nonnegative definite continuous functions.

Our goal is to design $u^* \in U$ so as to achieve global inverse optimality, i.e. to attain $J^* = \inf_{u \in U} J(u)$.

Before we proceed to the next section, let us introduce the following notations.

Notations: The norm $\|x\|$ of a vector x is the Euclidean norm. If A is a matrix, then $\|A\|$ denotes the Frobenius matrix norm, defined as $\|A\| = (T_r\{A^T A\})^{1/2}$, where $T_r\{\cdot\}$ denotes the trace of a matrix.

Remark 2.1: The model (2) represents a very general neural network model that includes the popular Hopfield neural networks, the paradigm of cellular neural networks, the bio-directional associative memory networks, memristor-based neural networks, and several other neural networks frequently employed in the literature.

Consider the following type of general stochastic nonlinear system based on the theory of risk sensitive stochastic control ([16] and [17]),

$$dx = (F(x) + G(x)u)dt + H(x)dw \quad (6)$$

Suppose that there exists a C^1 positive semi-definite function $V(x)$, which satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$J^* = \frac{\partial V}{\partial t} + V_x(x)F(x) - \frac{1}{4}V_{xx}(x)G(x)r^{-1}(x)G^T(x)V_x^T(x) + \frac{\theta}{4} \left| H^T(x)V_x^T(x) \right|^2 + q(x) + \frac{1}{2}Tr \left[V_{xx}(x)H(x)H^T(x) \right] \quad (7)$$

Then, the following feedback control

$$u^*(x) = -\frac{1}{2}r^{-1}(x)G^T(x)V_x^T(x) \quad (8)$$

is the optimal stabilizing control, which minimizes the cost functional

$$J = \limsup_{T \rightarrow \infty} \frac{2}{\theta T} \ln E \left[\exp \left(\frac{\theta}{2} \int_0^T (q(x) + u^T r(x)u) dt \right) \right] \quad (9)$$

where $q(x) \geq 0$ and $r(x) > 0$ for all x and J^* is the infimum of (9) over all state-feedback policies.

To facilitate our analysis, let us first introduce a definition.

Definition 3.1: The activation functions $\tilde{f}_j(x_j(t))$ and $\tilde{g}_j(x_j(t))$ of model (2) have the following properties:

(i) $\tilde{f}_j(0) = 0$ and $\tilde{g}_j(0) = 0$.

(ii) The scalar functions $\tilde{f}_j(x_j(t))$ and $\tilde{g}_j(x_j(t))$ are monotonically increasing and globally Lipschitz continuous with Lipschitz constants $h_j > 0$ and $k_j > 0$, i.e.

$$|\tilde{f}_j(y_i) - \tilde{f}_j(x_i)| \leq h_j |y_i - x_i| \quad (10)$$

and

$$|\tilde{g}_j(y_i) - \tilde{g}_j(x_i)| \leq k_j |y_i - x_i| \quad (11)$$

where $\forall x_i \in R$, $\forall y_i \in R$, and $|\cdot|$ represents the absolute value.

Following the technique of inverse optimality [18] and [19], we first need to find a stabilizing control. Therefore, let us choose a stochastic Lyapunov function V for the stochastic error system of coupled neural networks (4),

$$V = \frac{1}{2}e(t)^T e(t) + \int_{t-\tau}^t \left((W_2g(e(s)))^T (W_2g(e(s))) \right) ds \quad (12)$$

The infinitesimal generator is

$$\begin{aligned} LV = & -e(t)^T A e(t) + e(t)^T W_1 f(e(t)) + e(t)^T W_2 g(e(t-\tau)) + e(t)^T u \\ & + (W_2g(e(t)))^T (W_2g(e(t))) - (W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) \\ & + \frac{1}{2} T_r \left\{ I_{n \times n}^T \frac{\partial^2 V}{\partial x^2} I_{n \times n} \right\} \end{aligned} \quad (13)$$

Let us apply the following Young's Inequality to both the second term $e(t)^T W_1 f(e(t))$ and the third term $e(t)^T W_2 g(e(t-\tau))$ in (13),

$$x^T y \leq \frac{\|x\|^2}{2} + \frac{\|y\|^2}{2} = \frac{1}{2} x^T x + \frac{1}{2} y^T y \quad (14)$$

in which x and y are two vectors.

We obtain

$$\begin{aligned} e(t)^T W_1 f(e(t)) & \leq \frac{1}{2} e(t)^T e(t) + \frac{1}{2} \|W_1 f(e(t))\|^2 \\ & \leq \frac{1}{2} e(t)^T e(t) + \frac{1}{2} \|W_1\|^2 \|f(e(t))\|^2 \end{aligned} \quad (15)$$

And

$$e(t)^T W_2 g(e(t-\tau)) \leq \frac{1}{2} e(t)^T e(t) + \frac{1}{2} (W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) \quad (16)$$

From Definition 3.1, we have

$$|f_j(e_j)| = |\tilde{f}_j(y_j) - \tilde{f}_j(x_j)| \leq h_j |y_j - x_j| = h_j e_j$$

Therefore

$$\|f(e(t))\|^2 \leq (He(t))^T (He(t)) = e(t)^T H^2 e(t) \leq h^2 e(t)^T e(t) \quad (17)$$

where $H = \text{diag}(h_1, h_2, \dots, h_n)$ and $h = \max\{h_j\}$, $j = 1, \dots, n$.

Then from (16), we gain

$$e(t)^T W_1 f(e(t)) \leq \frac{1}{2} e(t)^T e(t) + \frac{1}{2} \|W_1\|^2 h^2 e(t)^T e(t) = e(t)^T \left(\frac{1 + \|W_1\|^2 h^2}{2} \right) e(t) \quad (18)$$

With respect to the fifth term in (13), we have

$$(W_2g(e(t)))^T (W_2g(e(t))) = \|W_2g(e(t))\|^2 \leq \|W_2\|^2 \|g(e(t))\|^2 \quad (19)$$

From Definition 3.1, we have

$$|g_j(e_j)| = |\tilde{g}_j(y_j) - \tilde{g}_j(x_j)| \leq k_j |y_j - x_j| = k_j e_j$$

Then

$$\|g(e(t))\|^2 \leq (Ke(t))^T (Ke(t)) = e(t)^T K^2 e(t) \leq k^2 e(t)^T e(t) \quad (20)$$

where $K = \text{diag}(k_1, k_2, \dots, k_n)$ and $k = \max\{k_j\}$, $j = 1, \dots, n$.

Therefore, combining (19) and (20) together, we have

$$(W_2g(e(t)))^T (W_2g(e(t))) \leq \|W_2\|^2 \|g(e(t))\|^2 \leq \|W_2\|^2 k^2 e(t)^T e(t) \quad (21)$$

In addition,

$$-e(t)^T A e(t) \leq -\lambda e(t)^T e(t) \quad (22)$$

in which $\lambda = \min\{\lambda_i\}$, $(i = 1, \dots, n)$.

Substitute (16), (18), (21), and (22) into (13), we reach

$$\begin{aligned} LV \leq & -\lambda e(t)^T e(t) + e(t)^T \left(\frac{1 + \|W_1\|^2 h^2}{2} \right) e(t) + \frac{1}{2} e(t)^T e(t) \\ & + \frac{1}{2} (W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) + e(t)^T u + \|W_2\|^2 k^2 e(t)^T e(t) \\ & - (W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) + \frac{1}{2} T_r \left\{ I_{n \times n}^T \frac{\partial^2 V}{\partial x^2} I_{n \times n} \right\} \\ = & -\lambda e(t)^T e(t) + e(t)^T \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2} \right) e(t) + e(t)^T u \\ & - \frac{1}{2} (W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) + \frac{1}{2} T_r \left\{ I_{n \times n}^T \frac{\partial^2 V}{\partial x^2} I_{n \times n} \right\} \end{aligned} \quad (23)$$

Because of $(W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) \geq 0$, we finally achieve

$$\begin{aligned} LV \leq & -\lambda e(t)^T e(t) + e(t)^T \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2} \right) e(t) + e(t)^T u \\ & + \frac{1}{2} T_r \left\{ I_{n \times n}^T \frac{\partial^2 V}{\partial x^2} I_{n \times n} \right\} \end{aligned} \quad (24)$$

Since $\frac{\partial^2 V}{\partial x^2} = I_{n \times n}$,

$$LV \leq -\lambda e(t)^T e(t) + e(t)^T \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2} \right) e(t) + e(t)^T u + \frac{n}{2} \quad (25)$$

Thus, we have the following stabilizing control

$$u = -\left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2}\right) e(t) \quad (26)$$

Therefore, equation (25) becomes

$$LV \leq -\lambda e(t)^T e(t) + \frac{n}{2} \quad (27)$$

We see that LV is negative, that is, $LV < 0$ whenever $\|e(t)\| > \sqrt{\frac{n}{2\lambda}}$. By the definition of stochastic input-to-state stability [20], we conclude that the system (4) achieves stochastic input-to-state stabilization with the control (26), that is, $\lim_{t \rightarrow \infty} e(t) = 0$. Therefore, both system (2) and system (3) are in synchronization, i.e., $\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} x(t)$.

Next let us discuss how to construct state-feedback controllers for a meaningful risk sensitive cost functional with which these controllers are optimal. According to the principle of inverse optimality, we now consider the Lyapunov function V as optimal value function and substitute it into HJB (7), which yield the following equation

$$\begin{aligned} & (W_2g(e(t)))^T (W_2g(e(t))) - (W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) - e(t)^T A e(t) \\ & + e(t)^T W_1 f(e(t)) + e(t)^T W_2 g(e(t-\tau)) - \frac{1}{4} e(t)^T r^{-1}(e(t)) e(t) \\ & - \frac{1}{4} \theta e(t)^T e(t) + q(e(t)) = 0 \end{aligned} \quad (28)$$

where the cost of (5) is given by $J^* = Tr[HH^T \Delta]$ with $H = I_{n \times n}$ and $\Delta = \frac{1}{2} I_{n \times n}$.

Consider a new control which is a modification of (26)

$$\begin{aligned} u &= -c \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2} + \frac{\theta}{2c} \right) e(t) \\ &= -\frac{1}{2} c \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{\theta}{c} \right) e(t) \end{aligned} \quad (29)$$

where $c > 2$ is a constant.

We then choose the function $r(e(t))$ as

$$r(e(t)) = c^{-1} \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{\theta}{c} \right)^{-1} \quad (30)$$

and from (28) the function $q(e(t))$ is given by

$$\begin{aligned} q(e(t)) &= e(t)^T A e(t) + \frac{c}{4} \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 \right) e(t)^T e(t) \\ &+ (W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) - (W_2g(e(t)))^T (W_2g(e(t))) \\ &- e(t)^T W_1 f(e(t)) - e(t)^T W_2 g(e(t-\tau)) \end{aligned} \quad (31)$$

We now have the following theorem.

Theorem: For the error dynamic system described by (4), there exists a positive-definite function $q(e(t))$ given by (31), and a strictly positive function $r(e(t))$ given by (30), such that the feedback control law

$$u = u^* = -\frac{1}{2} r^{-1}(e(t)) e(t) \quad (32)$$

achieves both stochastic input-to-state stabilization and inverse optimality with respect to a meaningful cost functional

$$J = \limsup_{T \rightarrow \infty} \frac{2}{\theta T} \ln E \left[\exp \left(\frac{\theta}{2} \int_0^T (q(e(t)) + u^T r(e(t)) u) dt \right) \right] \quad (33)$$

Therefore, both system (2) and system (3) are in synchronization.

Proof:

Step 1: By considering the stochastic Lyapunov function (12), the infinitesimal generator is

$$\begin{aligned} LV &= -e(t)^T A e(t) + e(t)^T W_1 f(e(t)) + e(t)^T W_2 g(e(t-\tau)) + e(t)^T u \\ &+ (W_2g(e(t)))^T (W_2g(e(t))) - (W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) \\ &+ \frac{1}{2} Tr \left\{ I_{n \times n}^T \frac{\partial^2 V}{\partial x^2} I_{n \times n} \right\} \end{aligned} \quad (34)$$

Substitute the control law (32) into LV (34) yields

$$\begin{aligned} LV &= -e(t)^T A e(t) + e(t)^T W_1 f(e(t)) + e(t)^T W_2 g(e(t-\tau)) \\ &- \frac{1}{2} c \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{\theta}{c} \right) e(t)^T e(t) \\ &+ (W_2g(e(t)))^T (W_2g(e(t))) - (W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) + \frac{n}{2} \end{aligned} \quad (35)$$

Substitute (16), (18), (21), and (22) into (35), we obtain

$$\begin{aligned} LV &\leq -\lambda e(t)^T e(t) - \left(\frac{c-1}{2} \right) \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 \right) e(t)^T e(t) \\ &- \frac{\theta}{2} e(t)^T e(t) - \frac{1}{2} (W_2g(e(t-\tau)))^T (W_2g(e(t-\tau))) + \frac{n}{2} \\ &\leq -\lambda e(t)^T e(t) + \frac{n}{2}. \end{aligned} \quad (36)$$

Therefore, $LV \leq 0$ whenever $\|e(t)\| \geq \sqrt{\frac{n}{2\lambda}}$.

By definition [20], we know that the system (4) achieves stochastic input-to-state stabilization by the control law (32), that is, $\lim_{t \rightarrow \infty} e(t) = 0$. Therefore, both system (2) and system

(3) are in synchronization, i.e., $\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} x(t)$.

Step 2: Let us consider $q(e(t))$ and $r(e(t))$.

By (31)

$$\begin{aligned} q(e(t)) &= e(t)^T A e(t) + \frac{c}{4} \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 \right) e(t)^T e(t) \\ &\quad + (W_2 g(e(t-\tau)))^T (W_2 g(e(t-\tau))) - (W_2 g(e(t)))^T (W_2 g(e(t))) \\ &\quad - e(t)^T W_1 f(e(t)) - e(t)^T W_2 g(e(t-\tau)) \end{aligned} \quad (37)$$

Substitute (16), (18), (21), and (22) into (37) above, we have

$$\begin{aligned} q(e(t)) &\geq e(t)^T A e(t) + \left(\frac{c-2}{2} \right) \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2} \right) e(t)^T e(t) \\ &\quad + \frac{1}{2} (W_2 g(e(t-\tau)))^T (W_2 g(e(t-\tau))) \\ &\geq e(t)^T A e(t) + \left(\frac{c-2}{2} \right) \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2} \right) e(t)^T e(t) \\ &\geq 0 \end{aligned} \quad (38)$$

Then, $q(e(t))$ is positive definite and radially unbounded.

By (30),

$$r(e(t)) = c^{-1} \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{\theta}{c} \right)^{-1} \quad (39)$$

It is obvious that $r(e(t)) > 0$.

Through the preceding derivation, the Lyapunov function V given by (12) satisfies the HJB equation (7). Therefore, V is the value function for the risk sensitive cost functional ([15] and [16]). This completes the proof.

IV. NUMERICAL EXAMPLE

In this section, we will give an example to verify the theoretical analysis and demonstrate the effectiveness of the proposed approach.

Example: The chaotic drive system of a time-delay neural network is given as

$$\begin{aligned} \begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2.11 & -0.12 \\ -5.09 & 3.18 \end{bmatrix} \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix} \\ &\quad + \begin{bmatrix} -1.59 & -0.11 \\ -0.19 & -2.47 \end{bmatrix} \begin{bmatrix} \tanh(x_1(t-\tau)) \\ \tanh(x_2(t-\tau)) \end{bmatrix} dt \end{aligned}$$

where $x_1(0) = 0.5$, $x_2(0) = -2$, $\lambda = -1$, $W_1 = \begin{bmatrix} 2.11 & -0.12 \\ -5.09 & 3.18 \end{bmatrix}$,

$W_2 = \begin{bmatrix} -1.59 & -0.11 \\ -0.19 & -2.47 \end{bmatrix}$, activation functions $f_j(x_j) = g_j(x_j) = \tanh(x_j)$, ($j = 1, 2$), and $\tau = 1$.

The corresponding chaotic response system is given as

$$\begin{aligned} \begin{bmatrix} dy_1(t) \\ dy_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 2.11 & -0.12 \\ -5.09 & 3.18 \end{bmatrix} \begin{bmatrix} \tanh(y_1(t)) \\ \tanh(y_2(t)) \end{bmatrix} \\ &\quad + \begin{bmatrix} -1.59 & -0.11 \\ -0.19 & -2.47 \end{bmatrix} \begin{bmatrix} \tanh(y_1(t-\tau)) \\ \tanh(y_2(t-\tau)) \end{bmatrix} dt + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dt + \begin{bmatrix} d\Psi_1 \\ d\Psi_2 \end{bmatrix} \end{aligned}$$

where $y_1(0) = 0.5$, $y_2(0) = -2$, and Ψ_1, Ψ_2 are white noise (uniformly random) with the magnitude of $|\Psi_j| = 3$ ($j = 1, 2$).

The parameters of the controller (32) are chosen as follows: $c = 20$, $h = k = 1$, and $\theta = 2$.

Figure 1 shows the phase diagrams of drive and response networks. The response network is under the influence of heavy noises ($|\Psi_j| = 3$) but without the control signal. It is obvious that both networks are not in synchronization at all. Figure 2 displays the results of the phase diagrams of drive and response networks under the risk sensitive optimal control signal. One can see that the chaotic drive network synchronizes with the chaotic response network influenced by the aforementioned heavy noises. The control signal is very efficient in synchronization and against the harmful effect of noise signals. Finally, Figure 3 presents the trajectories of error signals. At $t = 50$, the control signal (32) is applied to the response network. Both error signals approach zero immediately, that is, the error dynamical system achieves the stochastic input-to-state stabilization. Therefore, both the chaotic drive network and the chaotic response network are in synchronization after $t = 50$.

V. CONCLUSIONS

This paper has presented a new theoretical design of how an optimal synchronization is achieved for stochastic coupled neural networks with respect to a risk sensitive optimality criterion. The control method developed here is used to guarantee that the chaotic drive network synchronizes with the chaotic response network influenced by uncertain noise signals. The formulation of the risk sensitive state feedback controller is rigorously derived using the Hamilton-Jacobi-Bellman equation, Lyapunov technique, and inverse optimality. Simulation results show that the proposed approach turns out to be very effective in doing chaotic synchronization and against the harmful effect of noise signals. The proposed approach is simple to implement in real application. In addition, it can be easily extended to other chaotic systems, provided that their models can be converted into the model (2). It is our desire that the analytical results in this paper could provide some

useful design insights to accelerate the applications of both risk sensitive optimal control and neural networks.

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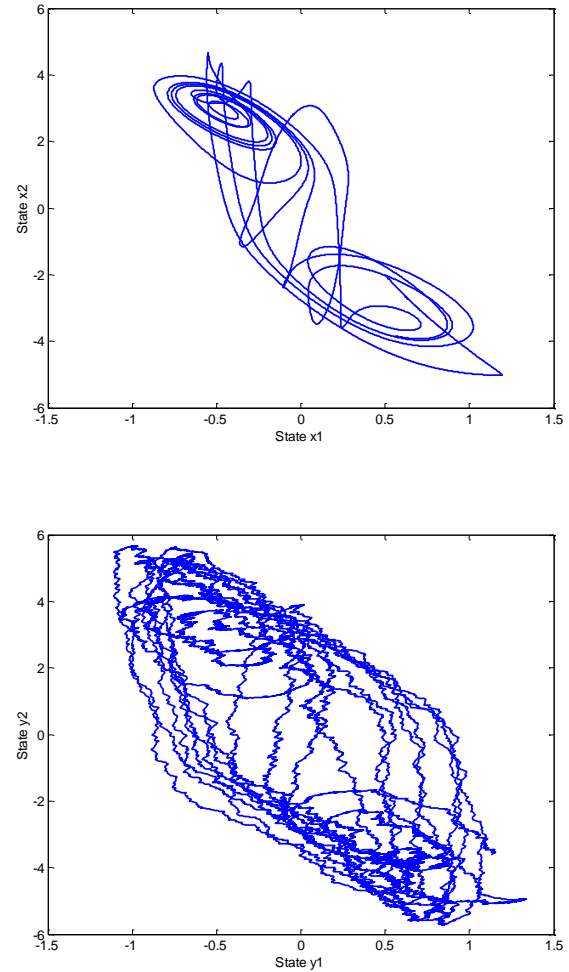


Figure 1: Phase diagrams of drive (top figure) and response (bottom figure) networks under heavy noises ($|\Psi_i|=3, i=1,2$) without the control, i.e., $u_i = 0$ ($i = 1, 2$), both networks are not in synchronization.

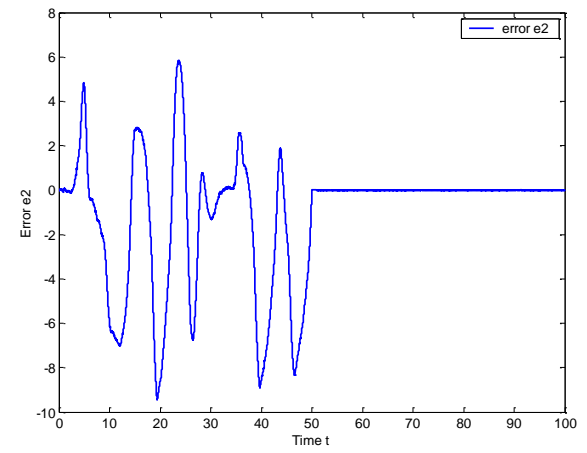
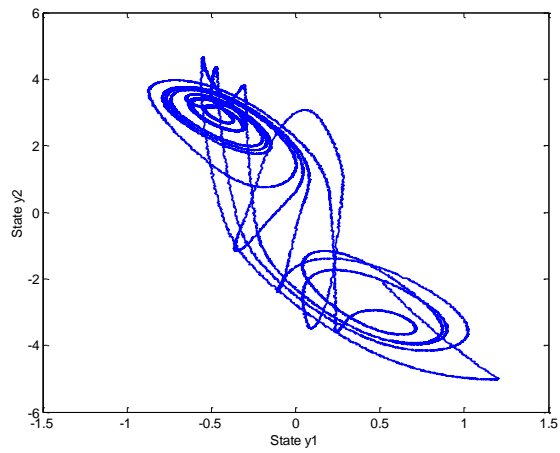
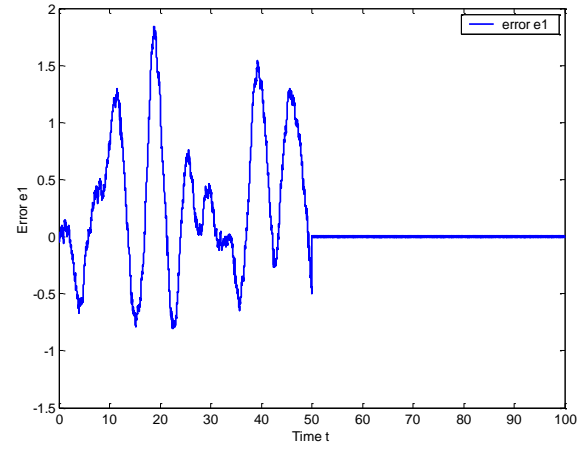
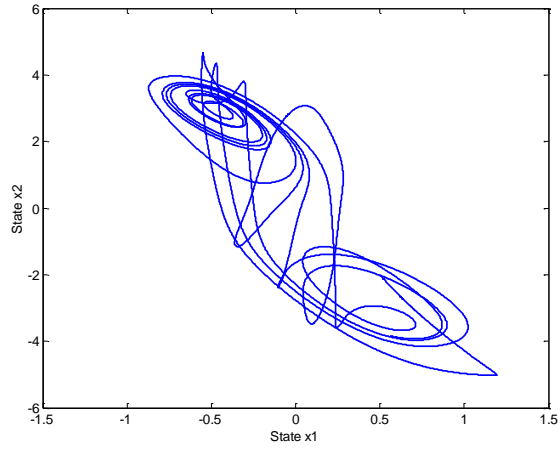


Figure 2: Phase diagrams of drive (top figure) and response (bottom figure with heavy noises $|\Psi_i|=3, i=1,2$) networks under the risk sensitive optimal control, i.e., $u = \text{Equation (32)}$, and the synchronization is achieved.

Figure 3: trajectories of error signals (top figure e_1 and bottom figure e_2) with the control ($u = \text{Equation (32)}$) at $t = 50$).