The Lagrangian and The Problem of N Connected Pendulums

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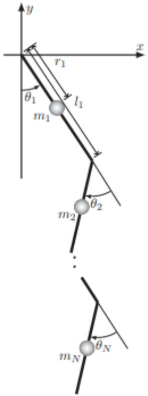
At: SUNY Oswego

April 9, 2021

Derivation: Function for Motion
- Newton's Method on the Double Pendulum
- The Lagrangian Method
  - Double Pendulum
  - The N-Link Pendulum

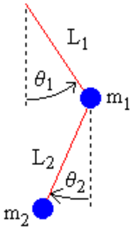
An Analysis
- Chaos Theory
- Fractals

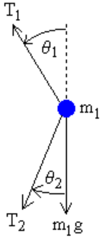
Questions & Bibliography



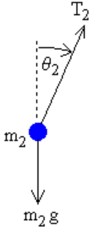
Newton's Method on the Double Pendulum

Newton's Second Law in 2-D: $\vec{F} = m\vec{a} = m\ddot{x}_i\hat{x} + m\ddot{y}_i\hat{y}$





upper mass



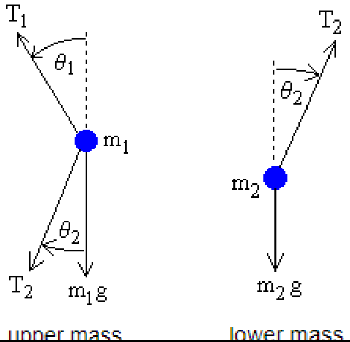
lower mass

$$m_1\ddot{x}_1 = -T_1\sin\theta_1 + T_2\sin\theta_2 \quad (1)$$

$$m_1\ddot{y}_1 = T_1\cos\theta_1 - T_2\cos\theta_2 - m_1 g \quad (2)$$

$$m_2\ddot{x}_2 = -T_2\sin\theta_2 \quad (3)$$

$$m_2\ddot{y}_2 = T_2\cos\theta_2 - m_2 g \quad (4)$$



$$m_1\ddot{x}_1 = -T_1 \sin\theta_1 + T_2 \sin\theta_2 \quad (1)$$

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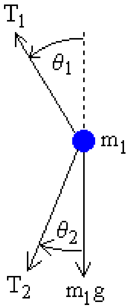
$$m_2\ddot{y}_2 = T_2 \cos\theta_2 - m_2 g \quad (4)$$

$$x_1 = L_1 \sin\theta_1 \quad (5)$$

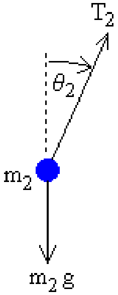
$$y_1 = -L_1 \cos\theta_1 \quad (6)$$

$$x_2 = x_1 + L_2 \sin\theta_2 \quad (7)$$

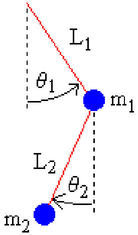
$$y_2 = y_1 - L_2 \cos\theta_2 \quad (8)$$



upper mass



lower mass



$$\sin(\theta_1)(m_1 y_1'' + m_2 y_2'' + m_2 g + m_1 g) = -\cos(\theta_2)(m_1 x_1'' + m_2 x_2'')$$

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$$\theta_1'' = \frac{-g\,(2\,m_1 + m_2)\sin\theta_1 - m_2\,g\sin(\theta_1 - 2\,\theta_2) - 2\sin(\theta_1 - \theta_2)\,m_2\,(\theta_2'^2\,L_2 + \theta_1'^2\,L_1\cos(\theta_1 - \theta_2))}{L_1\,(2\,m_1 + m_2 - m_2\cos(2\,\theta_1 - 2\,\theta_2))}$$

$$\theta_2'' = \frac{2\sin(\theta_1 - \theta_2)\,(\theta_1'^2\,L_1\,(m_1 + m_2) + g(m_1 + m_2)\cos\theta_1 + \theta_2'^2\,L_2\,m_2\cos(\theta_1 - \theta_2))}{L_2\,(2\,m_1 + m_2 - m_2\cos(2\,\theta_1 - 2\,\theta_2))}$$

$$\theta_1{}' = \omega_1$$

$$\theta_2{}' = \omega_2$$

$$\theta_1' = \omega_1$$

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$$\omega_2' = \frac{2\sin(\theta_1 - \theta_2)\,(\omega_1{}^2\,L_1\,(m_1 + m_2) + g(m_1 + m_2)\cos\theta_1 + \omega_2{}^2\,L_2\,m_2\cos(\theta_1 - \theta_2))}{L_2\,(2\,m_1 + m_2 - m_2\cos(2\,\theta_1 - 2\,\theta_2))}$$

The Lagrangian Method on the Double Pendulum

$$U = -mgh \qquad T = \frac{1}{2}mv^2 \qquad v^2 = (L\dot{\theta})^2$$

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$$KE = \frac{1}{2}m_1 L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)].$$

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$$L = KE - PE \qquad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0.$$

$$\frac{\partial L}{\partial \dot\theta_1} = m_1 l_1^2 \dot\theta_1 + m_2 l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2 \cos(\theta_1 - \theta_2)\,, \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = (m_1+m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot\theta_2 \sin(\theta_1 - \theta_2)\left(\dot\theta_1 - \dot\theta_2\right), \tag{10}$$

$$\frac{\partial L}{\partial \theta_1} = -l_1 g (m_1 + m_2) \sin\theta_1 - m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2)\,. \tag{11}$$

$$\frac{\partial L}{\partial \dot\theta_2} = m_2 l_2^2 \dot\theta_2 + m_2 l_1 l_2 \dot\theta_1 \cos(\theta_1 - \theta_2)\,, \tag{12}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot\theta_1 \sin(\theta_1 - \theta_2)\left(\dot\theta_1 - \dot\theta_2\right), \tag{13}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - l_2 m_2 g \sin\theta_2. \tag{14}$$

Do my own type of pendulum??? multiple masses fixed to a mass or initial stationary position with fixed angle between them

The N-Link Pendulum

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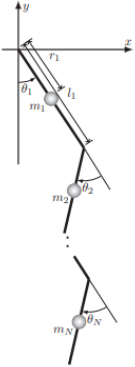
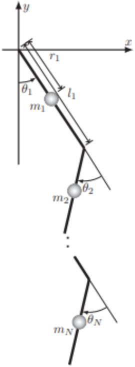


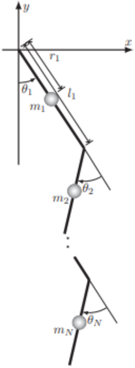
N-Link
Pendulum

The N-Link Pendulum



N-Link
Pendulum

The N-Link Pendulum



N-Link Pendulum

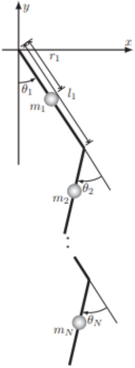
$$K = \sum_{k=1}^{N} \frac{1}{2} m_k \times \left[\left(\sum_{i=1}^{k-1} l_i \cos(\theta_{i,i-1}) \dot{\theta}_{i,i-1} + r_k \cos(\theta_{k,k-1}) \dot{\theta}_{k,k-1} \right)^2 + \left(\sum_{i=1}^{k-1} l_i \sin(\theta_{i,i-1}) \dot{\theta}_{i,i-1} + r_k \sin(\theta_{k,k-1}) \dot{\theta}_{k,k-1} \right)^2 \right]$$

N-Link KE

$$P = -g \sum_{k=1}^{N} m_k \left[\sum_{i=1}^{k-1} l_i \cos(\theta_{i,i-1}) + r_k \cos(\theta_{k,k-1}) \right]$$

N-Link PE

The N-Link Pendulum



N-Link Pendulum

$$K = \sum_{k=1}^{N} \frac{1}{2} m_k \times \left[\left(\sum_{i=1}^{k-1} l_i \cos(\theta_{i,i-1}) \dot{\theta}_{i,i-1} + r_k \cos(\theta_{k,k-1}) \dot{\theta}_{k,k-1} \right)^2 + \left(\sum_{i=1}^{k-1} l_i \sin(\theta_{i,i-1}) \dot{\theta}_{i,i-1} + r_k \sin(\theta_{k,k-1}) \dot{\theta}_{k,k-1} \right)^2 \right]$$

N-Link KE

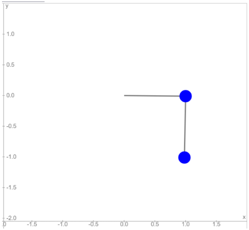
$$P = -g \sum_{k=1}^{N} m_k \left[\sum_{i=1}^{k-1} l_i \cos(\theta_{i,i-1}) + r_k \cos(\theta_{k,k-1}) \right]$$

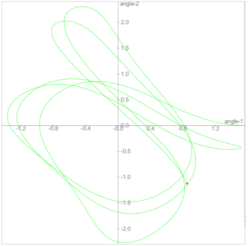
N-Link PE

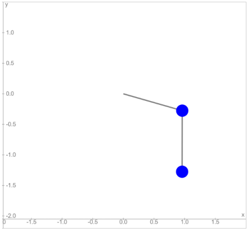
$$PE = -(m_1 + m_2) g L_1 \cos\theta_1 - m_2 g L_2 \cos\theta_2,$$

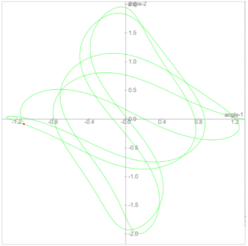
$$KE = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)].$$

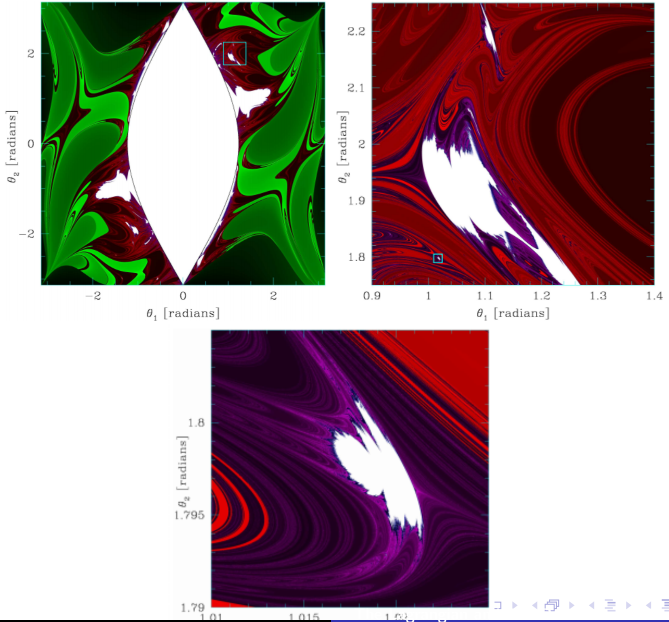
Chaos Theory










2
0
−2
θ2 [radians]
−2
0
2
θ1 [radians]
2.2
2.1
2
1.9
1.8
θ2 [radians]
0.9
1
1.1
1.2
1.3
1.4
θ1 [radians]
1.8
1.795
1.79
θ2 [radians]
1.01
1.015

Questions? I

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Questions? III

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Thanks for coming!