THE NUMBERS BEHIND THE NOTES

By

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As I sat in church one Sunday afternoon, the sound of the drummer tapping the snare drum caught my attention. I turned to watch him play and was captivated by the beats and their patterns. After a short time, my attention shifted to the bass player who sat in front of the drummer. There too I became entranced, studying each pluck and movement of the strings. I thought to myself, oh how I wish I could manipulate sounds as they do. You see, I have always loved music. While I never learned to play an instrument, I was blessed with the ability to sing and dance, thereby developing a personal connection with the sounds around me. To me, music serves as a form of expression, exposing both the heart of the composer/writer and invoking an emotional response in the listener. But what exactly invokes that response? Before lyrics even come into play, what makes one arrangement sound serene and peaceful, while another sounds chaotic and yet another sounds cheerful? As you think about these questions, some obvious answers may come to mind such as the melody, the tempo or the intensity at which a note is played. While all of these are correct, I set out to explore an even deeper basis: Perhaps, this effect can be attributed to The Numbers behind the Notes.

In this paper, I will provide a brief overview of early Mathematics leading up to the discovery and study of Math as it relates to Music. I will also cover how we receive and interpret sound and the mathematical components that exist within music. It is my hope that through this paper, readers will gain a new understanding of and appreciation for Mathematics and the role it plays in our everyday lives.

I. Early Mathematics

Mathematics is defined as the abstract science of number, quantity and space (1a). It is a vast subject that has been studied both for pure knowledge sake (pure mathematics) and to develop a better understanding of the world around us (applied mathematics). The earliest evidence of basic counting principles exists in the form of African tally sticks, where notches were carved into bones dating as far back as 35,000 - 20,000 BCE (2a). The evolution of Mathematics however, is believed to have begun as early as 3,000 BCE in Ancient Egypt (2b). The Ancient Egyptians are credited with the introduction of the earliest base 10 number system where written symbols were assigned to the numbers one (repeated for two through nine), ten, one hundred, one thousand, ten thousand, one hundred thousand and one million.

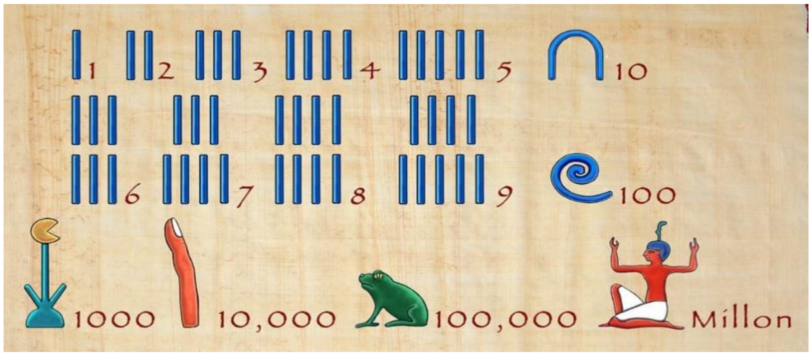


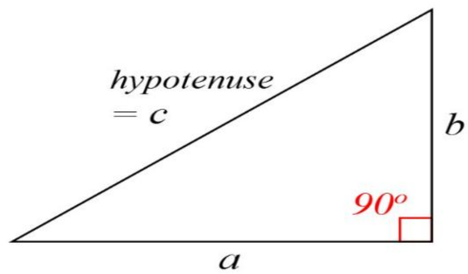
Fig. 1 Ancient Egyptian Hieroglyphic Numerals

The Rhind Papyrus, a mathematical text from around 1650 BCE, offers insight to other advancements of Egyptian Mathematics including but not limited to multiplication, division, fractions, arithmetic, geometry and linear equations.

The Sumerian and Babylonian civilizations of Mesopotamia are also credited with early discoveries in Mathematics from about 2,600 BCE onward (2c). They too developed their own numeric system using base 60 which is still used today in our measurement of time: 60 seconds to a minute, 60 minutes to an hour. Unlike the Egyptian number system however, the Sumerians and Babylonians implemented a true place-value system (as seen in today's decimal system), which listed larger numbers in the left column. This factor greatly simplified the process of writing large numbers. Additional developments made by these civilizations include basic arithmetic, square roots, cube roots, algebra, solutions to early quadratic equations and an intricate system of metrology.

Following the advancements of these civilizations came those of the Greek Empire, who began their conquest of Sumer and neighboring territories throughout Mesopotamia around 2,300 BCE (3). By incorporating key components from the civilizations they conquered, the Greeks absorbed many aspects of Egyptian and Babylonian mathematics and overtime began making their own valuable contributions to the study (2d). One of the most influential contributors was famed philosopher and mathematician Pythagoras of Samos, who is believed to have coined the words "Philosophy" meaning "the love of wisdom" and "Mathematics" meaning "that which is learned." Pythagoras is most famously known for the "Pythagorean Theorem" which states that the square of the hypotenuse of a right triangle is equal to the sum of the square of the remaining sides (2e).

Fig. 2 $c^2 = a^2 + b^2$



Long before Pythagoras' discovery, traces of this concept existed in the mathematical texts of ancient Egypt and Babylon as well as early proofs from ancient China. However, Pythagoras was the first to offer a definitive theorem as he and his successors represented one of the most notable contributions of Greek Mathematics: the concept of proof. While previous civilizations used inductive reasoning to establish their truths, the Greeks applied a process of logical steps to prove or disprove all theories. As it was with mathematics, so it was in another area of study considered to be one of three "sister sciences" in Greek education: The study of Music (4). Pythagoras believed that "the study of music and mathematics enabled a person to understand and see the structures of nature (5)." Being a musician as well, it is possible that his most notable musical discovery came as a result of his exploratory nature and experience playing the Lyre, a stringed instrument with the appearance of a small u-shaped harp (1b). Pythagoras' studies led to his discovery of the Harmonic Series and the ratios that exist between harmonious musical tones. In the next few pages, I will begin to explore the various components of sound and the mathematics behind what we hear.

II. What is Sound?

Sound is the ears interpretation of vibrations in the air around us (6). These vibrations or *sound waves* originate from an object and travel through the air into the outer ear, where they trigger a chain reaction. From the outer ear, they flow through the auditory canal into the eardrum, causing the eardrum to vibrate. Three small bones in the middle ear that together make up the piston, then pick up the vibrations and pass them along to the cochlea, a fluid filled tube within the inner ear. A membrane in the cochlea called the oval window receives the vibrations and jolts the fluid within the tube. This movement causes the hair cells that line the basilar membrane to bend, thereby activating impulses in nearby nerve cells. The axons of these cells connect to form the auditory nerve and finally, neural messages are sent from the auditory nerve to the auditory cortex in the brain.

There are two primary factors that affect the way we hear a sound, the first of which is Frequency (7a). The frequency of a sound describes the speed of its vibration and is measured by the number of reoccurring cycles per second. The human ear is able to detect sounds ranging between 20 and 20,000 Hertz (Hz), Hertz being the standard unit of measurement for wave frequencies (8). A frequency of 1 Hz = 1 wave cycle per second. The Pitch of a sound, how high or low the tone is, is directly correlated to its Frequency (7b).

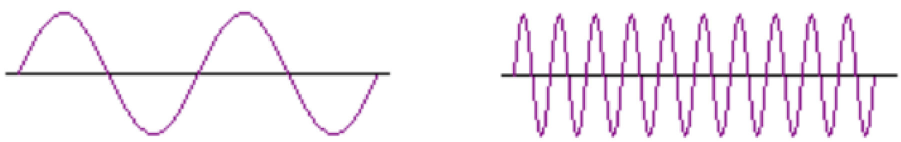


Fig 3. *Low Frequency / Low Pitch* *High Frequency / High Pitch*

The second primary factor is Amplitude, which describes the size of the vibration. Smaller vibrations produce audibly lower sounds and larger vibrations produce louder sounds.

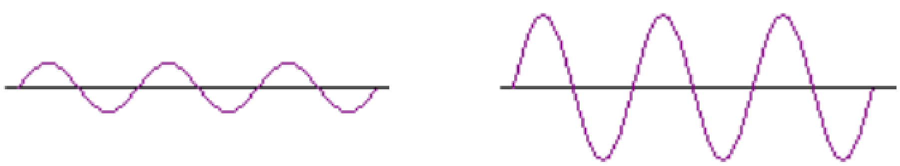


Fig 4. *Low Amplitude* *High Amplitude*

The sound produced by a musical note is defined as “a single tone of definite [clear] pitch made by a musical instrument or the human voice.” Unlike non-musical sounds which tend to have multiple and irregular frequencies, musical notes contain highly regular frequencies which allow for their clarity (7c). An interesting phenomenon occurs when we double the frequency of a note. The second note, played at double the frequency of the first, will always sound similar whereas the other notes between them will sound distinctly different (7d). As such the note played at double the frequency is considered to be the same note as the first, even though it has a higher pitch. The difference in pitch between the lower note and one that is double its frequency is called an Octave which is also the largest interval between two notes on the chromatic scale (7e). The chromatic scale, whose name is derived from the Greek word “chrôma,” is the term used to identify a set of all musical notes. In total there are 12 intervals in one set which are referred to as semi-tones. The best way to visualize this is to observe the layout of piano keys:

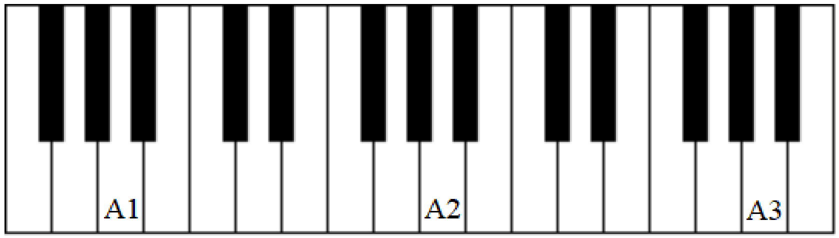


Fig. 5

In the above illustration, A1 represents the fundamental which is the note with the lowest frequency, followed by A2 which is one octave higher and A3 which is two octaves higher. The intervals between the adjacent black and white keys are the semi-tones. One fact about semi-tones that is particularly interesting is their mathematical relationship. In western music, the twelfth root of two is a highly regarded number because we are able to determine the frequency of a note simply by multiplying the twelfth root of two by the frequency of the previous note.

Another compelling relationship is that which exists between harmonious musical tones. As previously mentioned, it was Pythagoras' quest for knowledge and understanding that led him to make a discovery in this area, one that helped shape Music Theory as we know it (5). Pythagoras' work largely surrounded the relationship between string frequencies. He observed that two strings of equal length, thickness and tension produced the same sound when plucked (10). The sounds were considered to be "consonant" as they complimented one another and were pleasing to the ear. However, when the length of one string was changed, a change in pitch occurred. The resulting sounds were considered "dissonant" as they did not complement one another and were displeasing to the ear. Through continued research however, Pythagoras discovered that not all differences in pitch produced a dissonant sound. Specifically, he identified that the key to consonance was found in specific ratios. A few of these ratios which are considered fundamental intervals are as follows:

- A 2:1 ratio known as the Octave was achieved by reducing the length of one string by exactly one-half
- A 3:2 ratio known as the Perfect Fifth was achieved by reducing the length of one string by exactly one-third

- A 4:3 ratio known as the Perfect Fourth was achieved by reducing the length of one string by exactly one quarter

These examples confirm that in its most basic state, the numbers behind the notes affect which musical notes we perceive to be pleasant and which sound unpleasant. As I researched this topic, I gained a better understanding of just how expansive it actually is. There are many additional factors that contribute to the emotion of music and while they are not all listed here, it is my hope that this paper will invoke a new curiosity about a fascinating topic that continues to evolve and expand even in our present day lives. A topic with endless possibilities waiting to be explored. Thank you for your time.

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