Full Textbook WITH Exercises
for
Business Calculus Using Excel
MAT 210
Dutchess Community College
Edited and Compiled by
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&
Maryanne Johnson

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Attributions:

- This textbook was created from the Open Education Resources shown below, as well as combining the *Lecture Guide for Business Calculus* created by Sara Taylor and Maryanne Johnson at Dutchess Community College.

- The textbook material was created by primarily modifying, remixing and adding to the textbook *Business Calculus*, Copyright © 2013 Shana Calaway, Dale Hoffman, David Lippman. That original book is available to read or download free online at [http://www.opentextbookstore.com/buscalc/](http://www.opentextbookstore.com/buscalc/).

- Some components in Unit 1 were also mixed in from Precalculus by Jay Abramsom. Download that entire book for free online at [https://openstax.org/details/books/precalculus](https://openstax.org/details/books/precalculus).

About This Book:

- This book is meant to support the Business Calculus Course as taught at Dutchess Community College.

- There is no extraneous material in the textbook. Everything included is an important component of our course.

Additional Resources:

- We have compiled sets of Essential Practice Exercises in lumenohm to accompany every section of the book. These exercises are exactly what they say – Essential to Practice with in order to become fluent with many of the core concepts covered in the textbook and course. You will find links to these exercise sets in the Course Management System used by your teacher (Either Blackboard or lumenohm).

- We have also compiled videos for each section of the course that will help you process and digest core information. You will find links to these videos in the Course Management System used by your teacher (Either Blackboard or lumenohm).

- The Exercises that are contained in the Textbook are meant to add to the Essential Practice. You will be working on some of these exercises in class as well as completing some on your own.

- It is essential that you complete both the exercises contained within the textbook as well as the Essential Practice computerized homework in order to cover the full scope of the course.
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# Introduction

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Introduction

Welcome to Calculus with Business Applications! This book with exercises will be an important resource for you as you navigate this course.

- The exercises are set up to be used as a workbook and it is intended that you to fill in sections as if you were taking notes in class.
- The content and examples presented are required learning concepts (except where noted) for the course.
- Your teacher may add other examples and notes to their course, so please see and follow the directions from your particular instructor.

Most of the mathematics courses offered at Dutchess Community College are presented using a mathematical method of teaching called “The Rule of Four”, meaning that as mathematicians we strive to look at complex mathematical concepts in four ways.

1. Numerically, using tables and data
2. Graphically, using a variety of graphs
3. Analytically, using formulas and analysis
4. Verbally, using descriptions in words

This course will utilize the “Rule of Four” while we study Calculus in the Business field.

**Ways to Represent a Function**

- **Symbolic**
  
  \[ f(x) = 2x \]
  
  \[ \text{or } y = 2x \]

- **Graphical**

- **Numeric**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

- **Verbal**

  The cost of the item is twice the original amount.
Unit 1 Section 1: Function Review, Domain and Range, Interval Notation

Unit 1 Section 1: Learning Outcomes:

- Evaluating, solving with equation or graph.
- Write complete sentences to interpret the real-world, contextual meaning of points.
- Properly using function notation. Understand $f(3)$ vs. $f(x) = 3$.
- Revenue = (price per item) \times (Number items sold)
- Cost = (fixed costs) + (cost per item) \times (Number items produced)
- Profit = (revenue) – (cost)
- Use interval notation, inequality notation, sketch intervals

Functions

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask, “If I know one quantity, can I then determine the other?” This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

**Definition of Function**

**Function:** A rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say, “the output is a function of the input.”

**Function Notation**

To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write “height is a function of age”, we could use the descriptive variable $h$ to represent height and we could use the descriptive variable $a$ to represent age.

“height is a function of age” if we name the function $f$ we write

- “$h$ is $f$ of $a$” or more simply $h = f(a)$
- we could instead name the function $h$ and write $h(a)$ which is read “$h$ of $a$”
When you are asked to use function notation, it is essential that you use the full notation with the function name, as well as the indicated input.

Remember we can use any variable to name the function; the notation \( h(a) \) shows us that \( h \) depends on \( a \). The value “\( a \)” must be put into the function “\( h \)” to get a result. Be careful - the parentheses indicate that age is input into the function (Note: do not confuse these parentheses with multiplication!).

**Function Notation**
The notation output = \( f(\text{input}) \) defines a function named \( f \).
This would be read “output is \( f \) of input”

Example 1

A function \( N = f(y) \) gives the number of police officers, \( N \), in a town in year \( y \). What does \( f(2005) = 300 \) tell us?

When we read \( f(2005) = 300 \), we see the input quantity is 2005, which is a value for the input quantity of the function, the year (\( y \)). The output value is 300, the number of police officers (\( N \)), a value for the output quantity. Remember \( N = f(y) \). So this tells us that in the year 2005 there were 300 police officers in the town.

**NOTE:** In the future, you may simply see \( N(2005) \) rather than show \( N = f(2500) \).

**Related Exercises You Should Complete Now**

Work on Exercises 1.1.1 and 1.1.2. Remember that you have both written and video help for these exercises in your course.
Solving and Evaluating Functions:

When we work with functions, there are two typical things we do: evaluate and solve.

- **Evaluating a function** is what we do when we know an input and use the function to determine the corresponding output. Evaluating will always produce one result, since each input of a function corresponds to exactly one output.

- **Solving equations involving a function** is what we do when we know an output and use the function to determine the inputs that would produce that output. Solving a function could produce more than one solution, since different inputs can produce the same output.

Tables as Functions

Functions can be represented in many ways: Words (as we did in the example above), tables of values, graphs, or formulas. Represented as a table, we are presented with a list of input and output values.

**Example 2**

Using the table shown:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

a) **Evaluate** $Q(3)$

Evaluating $Q(3)$ (read: “$Q$ of 3”) means that we need to determine the output value, $Q$, of the function given the input value of $n = 3$. Looking at the table, we see the output corresponding to $n=3$ is $Q=7$, allowing us to conclude $Q(3) = 7$.

b) **Solve** $Q(n) = 6$

Solving $Q(n) = 6$ means we need to determine what input values, $n$, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$.

When we input 2 into the function, we get $Q(2) = 6$.

When we input 4 into the function, we also get $Q(4) = 6$.

Notice that a function CAN have two different inputs mapped to the same output. However, you cannot have an input mapped to two different outputs. Read that carefully!!
Related Exercise You Should Complete Now

This table represents the age of children in years and their corresponding heights. While some tables show all the information we know about a function, this particular table represents just some of the data available for height and ages of children.

<table>
<thead>
<tr>
<th>(input) a, age in years</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(output) h, height inches</td>
<td>40</td>
<td>42</td>
<td>44</td>
<td>47</td>
<td>50</td>
<td>52</td>
<td>54</td>
</tr>
</tbody>
</table>

Could this data represent a function? Support your answer using the definition of function.

Can we represent this in function notation? If yes, show that notation using the variables shown.

Can we determine \( h(15) \)? Can we solve \( h(a) = 47 \)? Why or why not?
Graphs as Functions

Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.

- **Evaluating a function** using a graph requires taking the given input and using the graph to look up the corresponding output.

- **Solving a function equation** using a graph requires taking the given output and looking on the graph to determine the corresponding input.

**Example 3**

Given the graph below,

a) Evaluate $f(2)$.

b) Solve $f(x) = 4$.

---

a) **To evaluate $f(2)$**, we find the input of $x = 2$ on the horizontal axis. Moving up to the graph gives the point (2, 1), giving an output of $y = 1$. So $f(2) = 1$

b) **To solve $f(x) = 4$**, we find the value 4 on the vertical axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: (-1, 4) and (3, 4). These give the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$

This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4.

---

**Related Exercises You Should Complete Now**

Work on Exercises 1.1.3, 1.1.4 and 1.1.5. Remember, you have both written and video help for these exercises within your course.
Recall the Vertical Line Test
The **vertical line test** is a way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line crosses the graph more than once, then the graph has more than one vertical output for the horizontal input associated with that vertical line. Therefore, if a vertical line crosses a graph more than once, the graph does not represent a function, since the definition of function is contradicted.
Functions Involving Formulas

As with tables and graphs, it is common to evaluate and solve functions involving formulas.

- **Evaluating** will require replacing the input variable in the formula with the value provided and calculating a resulting value.

- **Solving** will require replacing the output variable in the formula with the value provided, and solving for the input(s) that would produce that output.

**Example 4**

Given the function \( k(t) = t^3 + 2 \)

a) **Evaluate** \( k(2) \)

b) **Solve** \( k(t) = 1 \)

a) To evaluate \( k(2) \), we plug in the input value 2 into the formula wherever we see the input variable \( t \), then simplify

\[
k(2) = 2^3 + 2
\]

\[
k(2) = 8 + 2
\]

So \( k(2) = 10 \)

b) To solve \( k(t) = 1 \), we set the formula for \( k(t) \) equal to 1, and solve for the input value that will produce that output

\[
k(t) = 1 \quad \text{Substitute the original formula } t^3 + 2 \text{ in for } k(t).
\]

\[
t^3 + 2 = 1 \quad \text{Subtract 2 from each side.}
\]

\[
t^3 = -1 \quad \text{Take the cube root of each side.}
\]

\[
t = -1
\]

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

We want to know if \( k(t) = 1 \) is true when \( t = -1 \).

\[
k(-1) = (-1)^3 + 2
\]

\[
= -1 + 2
\]

\[
= 1 \text{ which was the desired result.}
\]

**Related Exercises You Should Complete Now**

Suppose \( f(x) = 5x - 1 \)

Use appropriate language to explain what the difference is between \( f(4) \) and \( f(x) = 4 \). Then find each.
Domain and Range
One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create. We will discuss this more fully as we continue through the course.

**Domain of a function:** The domain of a function \( f(x) \) is the set of all \( x \)-values which produce real-valued outputs. {INPUT}

**Range of a function:** The range of a function \( y=f(x) \) is the set of all \( y \)-values which the function \( f(x) \) produces. {OUTPUT}

**Practical domain and range:** There are times where the equation form of the function could be defined over a larger set of values, but the context of the problem limits what you can use from a practical point of view. We will discuss this further as we go through the course.

Interval Notation
Oftentimes our domain and range may involve inequalities. For example, the number of items produced, \( n \), may be between 0 and 100, or \( 0 \leq n \leq 100 \). A more compact alternative to inequality notation is *interval notation*, in which intervals of values are referred to by the starting and ending values.

- Curved parentheses are used for strictly less than or strictly greater than, and square brackets are used for less than or equal to or greater than or equal to. In other words, use square brackets if you want to include the boundary value!
- Since infinity is not a number, we can’t include it in the interval, so we always use curved parentheses with \( \infty \) and \(-\infty\).
- The table below will help you see how inequalities correspond to interval notation:

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Interval notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5 &lt; h \leq 10 )</td>
<td>((5, 10])</td>
</tr>
<tr>
<td>( 5 \leq h &lt; 10 )</td>
<td>([5, 10))</td>
</tr>
<tr>
<td>( 5 &lt; h &lt; 10 )</td>
<td>((5, 10))</td>
</tr>
<tr>
<td>( h &lt; 10 )</td>
<td>((-\infty, 10))</td>
</tr>
<tr>
<td>( h \geq 10 )</td>
<td>([10, \infty))</td>
</tr>
<tr>
<td>all real numbers</td>
<td>((-\infty, \infty))</td>
</tr>
</tbody>
</table>
Example 5

Describe the intervals of values shown on the line graph below using set builder and interval notations.

To describe the values, \( x \), that lie in the intervals shown above we would say, “\( x \) is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5.”

As an inequality it is: \( 1 \leq x \leq 3 \) or \( x > 5 \)
In interval notation: \([1,3] \cup (5,\infty)\)

Related Exercises You Should Complete Now

Work on Exercises 1.1.10 through 1.1.16. Remember, you have both written and video help for these exercises within your course.

Example 6

Find the domain of each function:

\( f(x) = 2\sqrt{4-x} \) \hspace{2cm} \( g(x) = \frac{3}{6-3x} \)

a) Since we cannot take the square root of a negative number, we need the inside of the square root to be non-negative. 
\[ 4 - x \geq 0 \] when \( -x \geq -4 \).
We want to know what the values are for \( x \), so we can divide both sides by \(-1\). \( \text{NOTE: When you multiply or divide both sides of an inequality by a negative value, you MUST reverse the inequality sign to preserve a correct statement.} \)
Therefore, our answer will be when \( x \leq 4 \).
The domain of \( f(x) \) is \((\infty, 4]\).

b) We cannot divide by zero, so we need the denominator to be non-zero. 
\[ 6 - 3x = 0 \] when \( x = 2 \), so we must exclude \( 2 \) from the domain.
The domain of \( g(x) \) is \((\infty,2) \cup (2,\infty)\).

Related Exercises You Should Complete Now

Work on Exercise 1.1.17. Remember, you have both written and video help for these exercises within your course.
Important Functions in Business: Cost, Revenue and Profit

The concepts of cost, revenue, and profit are the most important factors in determining success in business. Managing your costs and revenue to maximize your profit is key for every businessperson.

- **Cost** measures the total expenditures made by the business to run the operation: both the “direct” costs involved in creating the goods or service, as well as the “indirect” costs that stem from running a business, such as rent, salaries, and legal or professional fees.

- **Revenue** is the total income for a business and measures all of the money taken into the business through the sales of goods and services.

- **Profit** is the money made by the business and is the key indicator of success. Profit is the total revenue minus the total cost, that is money coming into the business minus the money going out. The mathematical equations of each is shown below.

- The **break-even point** for a company is when the amount of money coming into the company, revenue, is equal to the amount of money going out, cost (Revenue = Cost). At the break-even point the profit of the company is zero.

**Definitions:**

\[
\text{Cost} = \text{fixed costs} + \text{variable cost} \\
= \text{fixed costs} + \left[ \# \text{items} \cdot \text{(cost to produce item)} \right]
\]

\[
\text{Revenue} = (\text{Number of items}) \cdot \text{(Price item is sold at)}
\]

\[
\text{Profit} = \text{Revenue} - \text{Cost}
\]

**Linear Functions**

**RECALL From a Previous Course: Linear Function**

A linear function is a function whose graph produces a straight, non-vertical line. Linear functions have a constant, fixed, rate of change called the slope of the line. \( f(x) = mx + b \) is the equation of a linear function with slope \( m \) and y-intercept \( (0, b) \).

\( f(x) = mx + b \) is called the **slope-intercept form** of a linear function because you can see the slope and the y-intercept in the equation itself. All linear functions can be written in the slope-intercept form, and each line has a unique slope-intercept form.

\( f(x) = m(x - x_1) + y_1 \) is the equation of a linear function with slope \( m \) that passes through \( (x_1, y_1) \).

\( f(x) = m(x - x_1) + y_1 \) is called the **point-slope form** of a linear function because the equation displays a generic point on the line, and the equation also displays the slope of the line. The equation of linear functions can be written in point-slope form in an infinite number of ways because any point on the line can be used in the equation.
The standard form of a linear function is $Ax + By = C$ where A and B and C are real numbers.

Note that all linear functions can be re-written in slope-intercept form.

Example 7

Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, $N$, in his iTunes collection as a function of the number of months, $m$. How many songs will he own in a year?

The initial value for this function is 200, since he currently owns 200 songs, so $N(0) = 200$

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. With this information, we can write the formula:

$$N(m) = 200 + 15m.$$  

$N(m)$ is an increasing linear function.

With this formula we can predict how many songs he will have in 1 year (12 months):

$$N(12) = 200 + 15(12) = 200 + 180 = 380.$$  Marcus will have 380 songs in 12 months.

---

Related Exercises You Should Complete Now

Work on Exercises 1.1.6 and 1.1.7. Remember, you have both written and video help for these exercises within your course.

Example 8

Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so Ilya’s weekly income, $I$, depends on the number of new policies, $n$, he sells during the week. Last week he sold 3 new policies, and earned $760 for the week. The week before, he sold 5 new policies, and earned $920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

The given information gives us two input-output pairs: (3,760) and (5,920). We start by finding the rate of change or slope.

$$m = \frac{920 - 760}{5 - 3} = \frac{160}{2} = 80$$

Keeping track of units can help us interpret this quantity. Income increased by $160 when the number of policies increased by 2, so the rate of change is $80 per policy; Ilya earns a commission of $80 for each policy sold during the week.

We can now write the equation using the point-slope form of the line, using the slope we just found and the point (3,760):

$$I - 760 = 80(n - 3)$$
If we wanted this in function form (slope intercept form), we could rewrite the equation into that form:

\[ I - 760 = 80(n - 3) \]
\[ I - 760 = 80n - 240 \]
\[ I(n) = 520 + 80n \]

This form allows us to see the starting value for the function: 520. This is Ilya’s income when \( n = 0 \), which means no new policies are sold. We can interpret this as Ilya’s base salary for the week, which does not depend upon the number of policies sold.

Our final interpretation: Ilya’s base salary is $520 per week and he earns an additional $80 commission for each policy sold each week.

---

**Related Exercises You Should Complete Now**

If \( f(x) \) is a linear function, \( f(3) = -2 \), and \( f(8) = 1 \), find an equation for the function. Notice that this situation really gives us two ordered pairs: (3, -2) and (8, 1).

First compute the slope of the line. Then use the point–slope form of a line to find the equation.
Exponential Functions

RECALL From a Previous Course: Exponential Functions

The equation of an exponential function can be written in the form

\[ f(x) = a(1 + r)^x \quad \text{or} \quad f(x) = a(b)^x \]

Where \( a \) is positive, and \( r \) is a decimal between 0 and 1.

\( a \) is the initial or starting value of the function when \( x = 0 \).

*The y-intercept is \((0, a)\). \( r \) is the percent growth rate or percent decay rate, written as a decimal. As \( x \) increases by 1 unit, \( y \) increases or decreases by the percentage rate.

*When \( r \) is positive, then the factor \( b \) will be more than 1, and the function \( f(x) \) is increasing.

*When \( r \) is negative, then the factor \( b \) will be less than 1, and the function \( f(x) \) is decreasing.

\( b \) is the factor. As \( x \) increases by 1 unit, \( y \) is multiplied by \( b \).

*When \( b \) is more than 1, then \( r \) is positive, and the function \( f(x) \) is increasing.

*When \( b \) is less than 1, then \( r \) is negative, and the function \( f(x) \) is decreasing.

Example 9

India’s population was 1.14 billion in the year 2008 and is growing by about 1.34% each year. Write an exponential function for India’s population, and use it to predict the population in 2020.

Using 2008 as our starting time \((t = 0)\), our initial population will be 1.14 billion. Since the percent growth rate was 1.34%, our value for \( r \) is 0.0134.

Using the basic formula for exponential growth \( f(x) = a(1 + r)^x \) we can write the formula, \( f(t) = 1.14(1 + 0.0134)^t \)

To estimate the population in 2020, we evaluate the function at \( t = 12 \), since 2020 is 12 years after 2008.

\[ f(12) = 1.14(1 + 0.0134)^{12} \approx 1.337 \text{ billion people in 2020} \]

Related Exercises You Should Complete Now

Work on Exercises 1.1.8 and 1.1.9. Remember, you have both written and video help for these exercises within your course.
Example 10

\( T(q) \) represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured quarterly since January 1, 2010,

Interpret all of the parts of the equation \( T(2) = 86(1.64)^2 = 231.3056 \).

Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2010 this region had 86,000 Android smart phone contracts.

Since \( b = 1 + r = 1.64 \), we know that every quarter the number of smart phone contracts grows by 64%.

\( T(2) = 231.3056 \) means that in the 2\textsuperscript{nd} quarter (or at the end of the second quarter) there were approximately 231,305 Android smart phone contracts.
Exercises for Unit 1 Section 1

Exercise 1.1.1: A function \( N = f(y) \) gives the number of full-time teachers, \( N \), at DCC in any given year, \( y \).

(a) What does \( f(2015) = 283 \) tell us? Use a complete sentence to explain this notation in correct context.

(b) What is the independent variable here and what is its meaning? What is the dependent variable and what is its meaning?

(c) What would the expression \( N = f(2020) \) mean in practical terms?

Exercise 1.1.2: \( C(t) \) billion dollars is the value of cumulative capital investment in the cellular phone industry, where \( t \) is the number of years after 2000.

(a) Identify each of the following:

the input variable:


the output variable:


(b) Write a sentence interpreting the real-world, practical meaning of \( C(9) = 285.1 \)

(c) Write function notation for the statement “In 1990, the cumulative capital investment in the cellular phone industry was $6.3 billion.”
Exercise 1.1.3: The population of a town is \( P(t) = 85.2(0.924)^t \) where \( t \) is the number of years after 1950 and \( P(t) \) is the population of the town in thousands.

(a) Label the graph with the real-world meaning of the axis, and the variables on each axis.

(b) Use the graph to estimate the solution to \( P(t) = 40 \). Then write a complete sentence to interpret the real-world, practical meaning of the answer.

(c) Use the graph to estimate \( P(30) \). Then write a complete sentence to interpret the real-world, practical meaning of the answer.

Exercise 1.1.4: The function \( R(n) = n^3 - 19.5n^2 + 102.5n + 16 \) gives the revenue a company earns where \( n \) is the thousands of employees working that day nationwide, and \( R(n) \) is the revenue in millions of dollars that day. The company is only capable of employing, at most, 13,000 employees on any given day. Below is a graph of \( R(n) \) from \( n = 0 \) to \( n = 13 \). The y-axis spans from -50 up to 250.

(a) Use the graph to estimate the solution to \( R(n) = 200.4 \). Write a complete sentence to interpret the real-world, practical meaning of the answer.

(b) Use the graph to estimate the company’s revenue when employing 8500 employees.
**Exercise 1.1.5:** The function $R(a)$ in the graph below gives the projected revenue, $R(a)$, in dollars, when spending $a$ dollars per day on advertising in a certain market.

(a) Label both axis with the variables and the real-world meaning of each axis.

(b) Estimate the revenue when they spend $30 per day on advertising.

(c) Estimate the advertising amount spent when they earn $70 thousand in revenue.

---

**Exercise 1.1.6:** A company has initial costs of $275,000 and it costs the company $14.80 to produce each sprocket they produce. The company sells sprockets for $26.00 each.

(a) Find a formula for the total cost, $C(n)$, when producing $n$ sprockets.

(b) Find a formula for the total revenue, $R(n)$, when selling $n$ sprockets.

(c) Assuming they sell every sprocket they produce, find a formula for the total profit, $P(n)$. Simplify the equation for the profit.

(d) Identify the break-even point. Show all work, answer with a complete sentence.
Exercise 1.1.7: A company makes a total profit of $145,620 when selling 8400 widgets. They increase profit by $13.25 for every widget sold. Find a formula for P(n), the profit earned when selling n widgets.

Exercise 1.1.8: Demand for widgets at Acme decreases by 4.2% for every dollar they raise the price. They are projected to “sell” 45,000 widgets when they are free. Find a formula for D(p), the number sold when the items are priced at p dollars each.

Exercise 1.1.9: When XYZ company hired a new general manager, they found that revenue increased by 1.8% every month. If revenue was $1.9 million the month the new manager was hired, then find a formula for \( R(m) \), the revenue earned \( m \) months after the new manager started.

Exercise 1.1.10: \( 6 < x \leq 17 \) Draw the interval on the number line and write the interval using interval notation.

Exercise 1.1.11: \( 0 < y < 150 \) Draw the interval on the number line and write the interval using interval notation.
Exercises for Section 1.1

Exercise 1.1.12: 200 ≤ x. Draw the interval on the number line and write the interval using interval notation.

Exercise 1.1.13: 75 ≥ y. Draw the interval on the number line and write the interval using interval notation.

Exercise 1.1.14: Use interval notation and also inequality notation to identify the domain and range of the function.

Exercise 1.1.15: Use interval notation and inequality notation to identify the practical domain and range of the function. Our function represents the total cost of taking a taxi where you have to initially pay $5.00, and then you pay $0.40 for every mile driven. The function could be represented as C(m) = 5 + .40 m. Suppose the taxi will drive you a maximum of 25 miles.
**Exercise 1.1.16:** Use interval notation and also inequality notation to identify the domain and range of the function.

**Exercise 1.1.17** Find the **domain** of each function:

(a) \( f(x) = 4\sqrt{2x} + 6 \)  
(b) \( h(x) = \frac{7}{12-2x} \)  
(c) \( k(x) = \sqrt{6 - 4x} \)
Unit 1 Section 2: Function Behavior

Unit 1 Section 2: Learning Outcomes:
- Identify domain/range in context.
- Identify the intervals on which a function is increasing/decreasing. Use interval notation.
- Identify the intervals on which a function is concave up/down. Use interval notation.
- Give an accurate estimate of the local maximum/minimum points on a function using a graph.
- Give an accurate estimate of the inflection points on a function using a graph.
- Identify end behavior (tail behavior) of a function.
- Write complete sentences to describe real-world meaning of increasing/decreasing, concavity, maximum/minimum, inflection points, and end behavior.

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways.

- We say that a function is **increasing on an interval** if the function values increase as the input values increase within that interval.
- Similarly, a function is **decreasing on an interval** if the function values decrease as the input values increase over that interval.
- Graphically, f is **increasing** (decreasing) if, as we move from left to right along the graph of f, the height of the graph increases (decreases).
- The average rate of change of a function is the slope of the line between two points on the graph of the function.
- If a function is increasing on an interval, then the average rate of change will be positive between any two points in that interval.
- If a function is decreasing on an interval, then the average rate of change will be negative between any two points in that interval.

**Monotonic functions:**

Monotonic functions are functions that move only in one direction, that is they only increase or only decrease. The rate of change for these functions are always positive (increasing functions) or always negative (decreasing functions). We have studied monotonic functions such as linear functions and exponential functions.

Many functions have some intervals of increase and some intervals of decrease.
Figure 3 shows examples of increasing and decreasing intervals on a function.

![Graph showing increasing and decreasing intervals on a function](image)

Figure 3: The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not.

- A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima.

- Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is “local minima.”

- Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is “extremum.”) Often, the term *local* is replaced by the term *relative*. In this text, we will use the term *local*.

**Local Extremes or Relative Extremes**

<table>
<thead>
<tr>
<th>Definitions:</th>
</tr>
</thead>
<tbody>
<tr>
<td>• $f$ has a <strong>local maximum</strong> at $a$ if $f(a) \geq f(x)$ for all $x$ near $a$</td>
</tr>
<tr>
<td>• $f$ has a <strong>local minimum</strong> at $a$ if $f(a) \leq f(x)$ for all $x$ near $a$</td>
</tr>
<tr>
<td>• $f$ has a <strong>local extreme</strong> at $a$ if $f(a)$ is a <strong>local maximum</strong> or <strong>minimum</strong>.</td>
</tr>
<tr>
<td>• The plurals of these are maxima and minima. We often simply say “max” or “min;” it saves a lot of syllables.</td>
</tr>
<tr>
<td>• Some books say “relative” instead of “local.”</td>
</tr>
<tr>
<td>• A point is a local max (or min) if it is higher (lower) than all the <strong>nearby points</strong>. These points come from the shape of the graph.</td>
</tr>
</tbody>
</table>

**Important Notes:**

- Clearly, a function is neither increasing nor decreasing on an interval where it is constant.

- A function is also neither increasing nor decreasing at extrema. Note that we must speak of *local* extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function’s entire domain.

For the function whose graph is shown in **Figure 4**, the local maximum is 16, and it occurs at $x = -2$. The local minimum is $-16$ and it occurs at $x = 2$.

### Local maximum = 16
occurs at $x = -2$

![Graph showing local maximum and minimum](image)

### Example 1

**Finding Local Maxima and Minima from a Graph**

For the function $f$ whose graph is shown in Figure 9, find all local maxima and minima.
**Solution** Observe the graph of \( f \). The graph attains a **local maximum** at \( x = 1 \) because it is the highest point in an open interval around \( x = 1 \). The local maximum is the \( y \)-coordinate at \( x = 1 \), which is 2.

The graph attains a **local minimum** at \( x = -1 \) because it is the lowest point in an open interval around \( x = -1 \). The local minimum is the \( y \)-coordinate at \( x = -1 \), which is -2.

**Use A Graph to locate the Absolute Maximum and Absolute Minimum**

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain.

- The \( y \)-coordinates (output) at the highest and lowest points over the entire domain of a graph are called the **absolute maximum** and **absolute minimum**, respectively.

To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points on the entire domain of the function. See Figure 13.

![Graph showing absolute maximum and minimum](https://via.placeholder.com/150)

**Figure 13**

**Absolute Maxima and Minima**

The **absolute maximum** of \( f \) at \( x = c \) is \( f(c) \) where \( f(c) \geq f(x) \) for all \( x \) in the domain of \( f \).

The **absolute minimum** of \( f \) at \( x = d \) is \( f(d) \) where \( f(d) \leq f(x) \) for all \( x \) in the domain of \( f \).
Absolute Extreme Values

Absolute Extreme Values are either maximum (highest) or minimum (lowest) point on a curve. They are sometimes called global extremes.

Absolute Extreme values can be in the interior of the function or the end points of a function (if endpoints are given).

When the domain of the function involves infinity, or the domain is an open interval, there may be no absolute maximum or minimum value on the curve.
Example 2

Finding Absolute Maxima and Minima from a Graph
For the function $f$ shown in Figure 14, find all absolute maxima and minima.

![Graph of function $f$](image)

**Solution** Observe the graph of $f$.

The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function.

The absolute maximum value is the $y$-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at $x = 3$, because it is the lowest point on the domain of the function’s graph.

The absolute minimum value is the $y$-coordinate at $x = 3$, which is $-10$. 
Example 3
On the curve below we label each of the extremes of this function that is defined on a closed interval (has endpoints).

Absolute maximum
(also local maximum)

Local maximum

Local minimum

Absolute minimum
(also local minimum)

End-Behavior/Tail Behavior of a Function

Oftentimes, we also want to know what happens to a function in the long term. We call this behavior **End-Behavior, or Tail Behavior**, and must recognize that we are looking either far to the left on the graph, or far to the right.

- We ALWAYS describe a graph in terms of the input, so we can say we are considering inputs that are moving far to the right or far to the left.
- **NOTICE:** Typically, we read a graph from left to right, but when we describe end behavior, we can talk about what happens as we move far to the right, or what happens as we move far to the left.

The **right tail end-behavior** of a function describes the behavior of the outputs (y-values) as the inputs (x-values) increase without bound or as we consider the graph behavior as we move indefinitely to the right. We represent the input behavior symbolically as \( x \to \infty \).

The **left tail end-behavior** of a function describes the behavior of the outputs (y-values) as the inputs (x-values) decrease without bound or as we consider the graph behavior as we move indefinitely to the left. We represent the input behavior symbolically as \( x \to -\infty \).

It is possible for the outputs of the function to approach a particular value, or to diverge to \( \infty \) or \(-\infty\) on either or both tails.
Examples of end-behavior/tail behavior:

As \( x \to -\infty, f(x) \to 4 \)
As \( x \to -\infty, f(x) \to -\infty \)
As \( x \to \infty, f(x) \to -4 \)
As \( x \to \infty, f(x) \to 0 \)

Concavity of a Function

Concavity describes the “bend” of a curve. Graphically, a function is concave up if its graph is curved with the opening upward (a in the figure). Similarly, a function is concave down if its graph opens downward (b in the figure).

This figure shows the concavity of a function at several points. Notice that a function can be concave up regardless of whether it is increasing or decreasing.
For example, **An Epidemic:** Suppose an epidemic has started, and you, as a member of congress, must decide whether the current methods are effectively fighting the spread of the disease or whether more drastic measures and more money are needed. In the figure below, \( f(x) \) is the number of people who have the disease at time \( x \), and two different situations are shown. In both (a) and (b), the number of people with the disease, \( f(\text{now}) \), and the rate at which new people are getting sick are the same. The difference in the two situations is the concavity of \( f \), and that difference in concavity might have a big effect on your decision.

In (a), \( f \) is concave down at "now", the slopes are decreasing, and it looks as if it’s tailing off. We can say “\( f \) is increasing more slowly.” It appears that the current methods are starting to bring the epidemic under control.

In (b), \( f \) is concave up, the slopes are increasing, and it looks as if \( f \) will keep increasing faster and faster. It appears that the epidemic is still out of control.

**Concave Up:**
- A graph is said to be concave up if the graph bends upward as we move from left to right.
- The rates of change (slopes) are increasing (the values for the slopes are moving to the right on a number line as we follow the curve from left to right).
- If a function is decreasing and concave up, we say it is decreasing less rapidly. The curve is becoming flatter as we move from left to right.
  - Notice that if a function is decreasing and concave up, the slopes are NEGATIVE, and become less negative as we move from left to right. Therefore, the rate of change (slope) is increasing! Make sure you take time to really think about this!
- If a function is increasing and concave up, we say it is increasing more rapidly. The curve is becoming steeper as we move from left to right.
A graph can be concave up and decreasing ...

A graph can be concave up and increasing.

Concave Down
- A graph is said to be concave down if the graph bends downward as we move from left to right.
- The rates of change (slopes) are decreasing (the values of the slopes are moving to the left on a number line as we move along the curve from left to right).
- If a function is decreasing and concave down, we say it is decreasing more rapidly. The curve is becoming steeper as we move from left to right.
  - Notice that if a function is decreasing and concave down, the slopes are NEGATIVE, and become more negative as we move from left to right. Therefore, the rate of change (slope) is decreasing!
- If a function is increasing and concave down, we say it is increasing less rapidly. The curve is becoming flatter as we move from left to right.
Inflection Points

**Definition:** An inflection point is a point on the graph of a function where the concavity of the function changes, from concave up to down or from concave down to up.

**Example 4**

Which of the labeled points in the graph below are inflection points?

The concavity changes at points b and g.

At points a and h, the graph is concave up on both sides, so the concavity does not change.

At points c and f, the graph is concave down on both sides.

At point e, even though the graph looks strange there, the graph is concave down on both sides – the concavity does not change.

**Inflection Point and Function Behavior:**

A point on the graph where the concavity changes is called a point of inflection. At this point the concavity changes from concave up to concave down OR from concave down to concave up.

- If the function is increasing and changes from concave up to concave down, the function will be increasing most rapidly at the point of inflection.
- If the function increasing and changes from concave down to concave up, the function will be increasing the least rapidly at the point of inflection.
- If the function is decreasing and changes from concave up to concave down, the function will be decreasing the least rapidly at the point of inflection.
- If the function is decreasing and changes from concave down to concave up, the function will be decreasing the most rapidly at the point of inflection.
In the first graph, the inflection point occurs where the graph is decreasing and changes from concave down to concave up. Therefore, we would say the function is decreasing the most rapidly at that point.

In the second graph, the point of inflection occurs where the graph is increasing and changes from concave down to concave up. Therefore, we would say that the function is increasing least rapidly at this point.

**Example 5**

Let \( f(x) = x^3 \), \( g(x) = x^4 \) and \( h(x) = x^{1/3} \). For which of these functions is the point \( (0,0) \) an inflection point?

Graphically, it is clear that the concavity of \( f(x) = x^3 \) and \( h(x) = x^{1/3} \) changes at \( (0,0) \), so \( (0,0) \) is an inflection point for \( f(x) \) and \( h(x) \).

The function \( g(x) = x^4 \) is concave up everywhere so \( (0,0) \) is not an inflection point of \( g(x) \).
Example 6

Sketch the graph of a function with $f(2) = 3$, where $f$ is increasing at $x = 2$, and has an inflection point at $(2,3)$.

Two possible solutions are shown here.

---

**Related Exercises You Should Complete Now**

You should now work on Exercises 1.2.1 through 1.2.4. These exercises use ideas presented so far in this section. Remember, you have both written and video help for these exercises within your course.
Example 7

Assume that the table below captures key information about the continuous function \( f(x) \). Identify where the given function in the table is increasing concave up, increasing concave down, decreasing concave up, decreasing concave down. Use interval notation and ONLY the values given in the table. HINT: Use the Average Rate of Change (ARC) values between consecutive points in the table to help determine concavity.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>ARC</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>-4.5</td>
<td>On ([-4, -3]], -20.5</td>
</tr>
<tr>
<td>-2</td>
<td>-10</td>
<td>On ([-3, -2]], -5.5</td>
</tr>
<tr>
<td>-1</td>
<td>-6.5</td>
<td>On ([-2, -1]], 3.5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>On ([-1, 0]], 6.5</td>
</tr>
<tr>
<td>1</td>
<td>3.5</td>
<td>On ([0, 1]], 3.5</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>On ([1, 2]], -5.5</td>
</tr>
<tr>
<td>3</td>
<td>-22.5</td>
<td>On ([2, 3]], -20.5</td>
</tr>
<tr>
<td>4</td>
<td>-64</td>
<td>On ([3, 4]], -41.5</td>
</tr>
<tr>
<td>5</td>
<td>-132.5</td>
<td>On ([4, 5]], -68.5</td>
</tr>
<tr>
<td>6</td>
<td>-234</td>
<td>On ([5, 6]], -101.5</td>
</tr>
</tbody>
</table>

First notice that the inputs are consistently increasing by 1 unit throughout the table.

We need to think about this situation in 3 stages:

1) Where is the function increasing and where is it decreasing?
2) Where is the function concave up and where is it concave down?
3) Where do the combined categories occur?

**Decreasing:** Since the inputs are always increasing, we just must look for where the output is decreasing. We can see that the function is decreasing on the DOMAIN intervals \((-4, -2)\) and \((1, 6)\). Remember, this is interval notation, so these do not represent ordered pairs. The values are boundaries for the inputs (domain) values.

**Increasing:** The outputs are increasing on the DOMAIN interval \((-2, 1)\). Again, this is not an ordered pair. The notation tells us that when the INPUTS are between –2 and 1, we will see the output values increasing.

If we consider Average Rates of Change (ARC) for this function, we can see that since the inputs are all 1 unit apart, we simply must figure out how far apart the output are and the value we find will be the ARC for that interval \( \frac{\text{change in output}}{\text{change in input}} \). Doing those calculations, we have the ARCs shown in the table above. For example, the ARC between \( x = -4 \) and \( x = -3 \) would be computed as \( \frac{-4.5 - 16}{-3 - (-4)} = \frac{-20.5}{1} = -20.5 \). Since the change in input will always be 1, we can see that the change in output is our ARC in this example. Again, this is because the inputs are all 1 unit apart.
Concave Up: We know that a continuous function is concave up when the ARC increases. Therefore, this function is concave up on the DOMAIN interval (-4, 0).

Concave Down: We know that a continuous function is concave down when the ARC decreases. Therefore, this function is concave down on the DOMAIN interval (0, 6).

To answer the question asked, we need to blend the ideas together by looking where the intervals intersect to find the desired behavior.

Increasing, concave up: on the DOMAIN interval of (-2, 0)
Increasing, concave down: on the DOMAIN interval of (0, 1)
Decreasing, concave up: on the DOMAIN interval of (-4, -2)
Decreasing concave down: on the DOMAIN interval of (1, 6)

Related Exercises You Should Complete Now

You should work on Exercise 1.2.5. Remember, you have both written and video help for this exercise within your course.
Exercises for Unit 1 Section 2

Exercise 1.2.1: Sketch a graph of each function following the given description. NOTE: Many correct graphs can be sketched for these situations.

(a) A continuous function that is decreasing and concave up on the interval \((-\infty, 5]\) and decreasing and concave down on the interval \((5, \infty)\). Label the point of inflection and state how the function is behaving at this point. 

(b) A continuous function that has a local maximum at \(x = 20\), an inflection point at \(x = 25\), and a local minimum at \(x = 30\). Label the point of inflection and state how the function is behaving at this point. 

(c) A continuous function that approaches \(y = 0\) on the right tail, and approaches \(y = 300\) on the left tail. Describe the tail behavior symbolically.

(d) A continuous function that has domain of positive real numbers, \((0, \infty)\), and a range of all real numbers. The function is decreasing everywhere.
**Exercise 1.2.2:** The function \( P(n) = n^3 - 19.5n^2 + 102.5n + 16 \) gives the profit a company earns where \( n \) is the thousands of employees working that day nationwide, and \( P(n) \) is the profit in millions of dollars that day. The company is only capable of employing, at most, 13,000 employees on any given day and the graph below is displaying that.

(a) Label each axis with the real-world meaning of each axis, the variable on each axis, and the scale on each axis.

(b) What is the practical, real-world domain in this problem? Write your answer in interval notation.

(c) Using the graph given, estimate the local maximum point (ordered pair). Interpret the practical meaning using a complete sentence. Remember this is a closed domain.

(d) Using the graph given, estimate when the function was decreasing most rapidly. What is this point called? Interpret the practical significance (real-world significance, using context) of this function at that point.

(e) Using the graph given, identify the interval(s) where the function is increasing using interval notation. Write a complete sentence to interpret the real-world, practical meaning of the answer.

(f) Identify the interval(s) where the function is **increasing and concave down** (occurring at the SAME TIME!). Write a complete sentence to interpret the real-world, practical meaning of the answer.
**Exercise 1.2.3:** The population of a town is \( P(t) = 85.2(0.924)^t \) where \( t \) is the number of years after 1950 and \( P(t) \) is the population of the town in thousands.

(a) Label the graph with the real-world meaning of each axis and the variable for each axis.

(b) Is this function increasing or decreasing? What is the practical, real-world meaning of this?

(c) Is this function concave up or concave down? Explain this behavior in practical, real-world context.

(d) Identify the right-tail behavior of the function. Then write a complete sentence to interpret the real-world, practical meaning of the answer.
**Exercise 1.2.4:** The function $R(a)$ gives the revenue a company earns in dollars, $R(a)$, when spending “$a$” dollars per day on advertising in a certain market.

(a) Describe, using the context and terminology of the problem, the practical interpretation for what happens to $R(a)$ as $a$ increases from $0$ per day up to $50$ per day.

(b) Describe, using the context and terminology of the problem, the practical interpretation for what happens for expenditures beyond $50$ per day.

(c) From the graph, it seems the output of the graph approaches $325,000$ as the inputs increase but never gets any higher. Using the variables in the problem statement, capture this behavior symbolically and then explain using a complete sentence the practical interpretation for this behavior.

(d) Estimate when the revenue is increasing most rapidly and write your answer in a complete contextual sentence.
**Exercise 1.2.5:** Assume that $f(x)$ is a continuous function and that the table below indicates key information about the function. Identify where the given function in the table is increasing concave up, increasing concave down, decreasing concave up, decreasing concave down. Use interval notation. Sketch a graph. HINT: Find the average rate of change between consecutive points on the graph to determine concavity.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>50</td>
</tr>
<tr>
<td>0</td>
<td>70</td>
</tr>
<tr>
<td>2</td>
<td>78</td>
</tr>
<tr>
<td>4</td>
<td>84</td>
</tr>
<tr>
<td>6</td>
<td>86</td>
</tr>
<tr>
<td>8</td>
<td>84</td>
</tr>
<tr>
<td>10</td>
<td>80</td>
</tr>
<tr>
<td>12</td>
<td>72</td>
</tr>
<tr>
<td>14</td>
<td>75</td>
</tr>
<tr>
<td>16</td>
<td>80</td>
</tr>
<tr>
<td>18</td>
<td>90</td>
</tr>
</tbody>
</table>
Unit 1 Section 3: Introduction to Excel

Unit 1 Section 3: Learning Outcomes:
- Perform basic calculations on Excel including $e^x$ and $\ln(x)$.
- Evaluate functions in Excel.
- Create input-output tables in Excel using a given function.
- Create well-labeled scatterplots and smooth-line graphs on Excel.
- Use Solver to solve equations in Excel.
- Using F4 for absolute cell reference.

NOTE: Excel on a Mac is slightly different, but most of the instructions here will work on Macs, and a quick google search can generally help you figure out how to do the same thing on a Mac.

See Videos in Blackboard for extra help on practicing Excel skills.
Excel is a computer software tool that will aid in the calculation, evaluation, analysis, and graphing of mathematical expressions that we study. It will take time and require consistent practice to become familiar with this tool and proficient in its use.

DCC students are given free access to downloading the Office suite, which includes Microsoft Excel. To Install Office 365 Apps, follow the instructions below. Make sure you do this from the computer you want to install the programs on!

1) Access your email from myDCC. Once in your email, click the “grid” icon in the upper left corner.

2) This will open a side menu. At the top of the side menu, click “Office 365.”
3) Clicking this link will open a new window. To the far right of this new window is a link to install office on your computer.

4) Clicking the “Install Office” button will give you two options. For most cases, the top one is what you want.

5) Follow the instructions given on the screen from here.

Starting Excel: Microsoft Excel is a commonly used spreadsheet program. In this section we will learn some basic Excel skills.

1) When we first open Excel, we see a worksheet labeled as “Sheet 1.”

2) We are able to add additional worksheets by clicking the + symbol next to “Sheet 1.”

3) Within each Excel worksheet, Excel labels the columns with letters (A, B, C, …), and Excel labels the rows with numbers (1, 2, 3, …). We refer to the little boxes in an Excel worksheet as Cells.

4) We reference particular Cells within Excel by using the Column-row designation. For example, in the picture above, the grey Cell is referred to as Cell C4.
Creating a Textbox in Excel  
(*Video Link - Creating a Textbox in Excel)  
https://youtu.be/RqlB6x2RVIY

Most people have experience creating textboxes in Microsoft Word. Microsoft Excel also allows users to create textboxes within worksheets. Students can use textboxes to take notes for themselves right next to the work in Excel, and in general textboxes are a nice way for people to create summaries and comments regarding the Excel work that we see in a worksheet.

**To create a textbox in Excel:**

1. Click the Insert tab, and then select “Text Box.”

2. At this point, your cursor will change to look like a Plus sign or down arrow (or ↓).

3. To create the textbox, hold down the left mouse button and drag your cursor down and to the right in order to draw the height and width of the textbox. Release the mouse and the textbox will appear.

4. You can adjust the size of the textbox after creating it. You can also drag the textbox and move it to wherever you like in the worksheet.
Basic Excel Evaluation Skills:

**Calculations in Excel** (*Video Example Here*)  [https://youtu.be/C1eCp7s8Xzg](https://youtu.be/C1eCp7s8Xzg)
Excel can work as any calculator to find the value of any arithmetic expression.

Use Excel to calculate 2+5 and also 7/9, rounding your answers to two decimal places.

**Solution:**
1) Open an Excel Blank Workbook.
2) In Cell A1 type the mathematical expression, 2 + 5, and then hit the enter key.
3) In cell A2 type the mathematical expression, \( \frac{7}{9} \), and then hit the enter key.
4) Notice that Excel DOES NOT calculate the sum of 2+5. Excel will simply refer to the first expression, in cell A1, as text and Excel will refer to the second expression, in cell A2, as a date on the calendar.
5) In order to use Excel as a calculator, we must always begin our expression with an equal sign.
6) Retry these expressions again, using the “=” sign.
7) Notice the first value, =2+5, evaluates just fine as 7. But the second calculation, =7/9, does not. The 7/9 calculation still returns a date. In order to have Excel calculate numerical values, we must format the column properly so that it returns numbers instead of dates.
8) Click your cursor at the top of column A in order to select all cells in column A.

9) Then right click to bring up a menu of options (the Format Cells box shown on the right).

10) Click the “Number” tab, and click “Number” from the list of options.

11) Choose the number of decimal places that you would like displayed in your answers. The default is 2 decimal places.

12) You can also choose here how you would like negative answers to be displayed.

13) Click “OK” to apply your changes.

14) Your answers in column A are now displayed accurately to two decimal places.

Let’s try a new expression. Evaluate the numerical value of the expression: $(14.5 - 11.31)(9.2 - 17^2)$ using cell A3 in your Excel Workbook.

Solution:

1) This expression requires many operations, and just as we would with a calculator, we will use the “*” symbol for multiplication, and the “^” for raising to an exponent. Remember to start your expression in Excel with an “=” sign and use proper mathematical notation. Notice that Excel color codes its parenthesis as you “open” and “close” parenthesis as we use them in the expression.

2) Notice that the expression is also shown in the formula box at the top of the workbook as we type. Be especially careful with entering fractions, to make sure you enclose the entire numerator and entire denominator in parenthesis. The resulting answer is $-892.56$.  

Written by J. Halsey – Updated December 2021 - Dutchess Community College
Calculate \( \sqrt{356} \) in Excel.

**Solution:**
1) Square root functions can be entered in Excel by using the SQRT command. This is one of many “built-in” functions within Excel. You will notice that when you begin to type a command like =SQRT, Excel will display a menu of possible options to choose from.

2) In cell A4, try entering the expression: \( \sqrt{356} \) by typing =sq and you will see the SQRT option displayed.

3) Remember to ALWAYS begin with an “=” sign!

The resulting answer is 18.87.

**Example 1.3.3 (Video Example Here) https://youtu.be/yHVo_mOp4co**

In order to perform operations involving variables in Excel, we must use a process called cell referencing. Add a new sheet to your Excel workbook by clicking on the little + sign at the bottom of the sheet next to “Sheet 1.”

You can name any worksheet by simple clicking on the sheet1 label and typing a new name. Let’s call Sheet1 “Example 1” and let’s call Sheet2 “Example 2.”

**Perform the following operations in this new Excel sheet. Evaluate each expression as directed.**

1) Calculate \( \frac{2-(5.01)^3}{9} \) in cell A2, and label it with the value “A” in cell A1.

2) Store the expression \( \frac{2-(5.01)^2}{9} \) in cell A2. Call this value A, and label it with an “A” in cell A1.
This returns a value of 27.94478 in cell A2.

3) Calculate $20 \sqrt{113}$ in cell B2, and label it with the value “B” in cell A2.

4) Solution:
   Store the expression $20 \sqrt{113}$ in cell B2. Call this value B and label it with a “B” in cell B1.

This returns a value of 212.6029 in cell B2.

5) In cell B3, calculate the value of $2A+5B$ in cell C2 using Excel cell referencing of cells B1 and B2. Label this new value as “C” in cell A3.

Solution:
6) In order to perform this last operation, we must reference the cells that have the numerical values of A and B to find the value of C. We do so by clicking on the cells being referenced in the expression or typing the names of those cells. Notice how Excel again color codes the cells for us for easy reference. Cell B3 now has the expression $2A + 5B$. Excel will automatically pull the values within cells B1 and B2 in order to perform the calculation $2A+5B$.

The resulting value of $2A+5B$ is 1118.904.

Example 1.3.4: Perform the following operations. Evaluate each expression as directed

Store $\frac{3 \ln 5}{7}$ in cell A2. Call this value A and label it with an A in cell A1

Store $20 e^{1.2}$ in cell B2. Call this value B and label it with a B in cell B1.

Store $2A + 4B$ in cell C2. Call this value C and label it with an C in cell C1.

Solution:
To input the natural log function, you first type LN( and then type the value you are taking the Ln of in ( ).

To create $e^{1.2}$, you type EXP ( and then type the exponent in the ( )

Written by J. Halsey – Updated December 2021 - Dutchess Community College
Related Exercises You Should Complete Now

Work on Exercise 1.3.1. Remember, there are written solutions and videos in your course.

Example 1.3.5: Inp(*Video Example Here) https://youtu.be/VVwLK7A2o8s

Use Excel to create an input-output table from \( x = -10 \) to \( x = 10 \) in increments of one for the function \( f(x) = 7x^2 + 5x + 10 \).

Solution:

We can use our knowledge of cell references to produce this input/output tables in Excel. We can start by adding a new sheet in our Excel workbook, naming it Example 3.

Next, in cell A1, type “x”, and in cell B1 type “\( f(x) \)”.

1) In column A we could fill in the x-values by simply entering the values \(-10, -9, -8, \) etc. However, this will not be feasible in the future when we have many more x-values. Luckily, Excel has a feature that can help us with this task: the Fill feature. To use the fill feature, first type the first number in our domain (the first x-value), in this case \(-10\) in cell A2 and hit enter.

2) Then click back into cell A2 and click on Excel’s Fill feature which is found in the upper right corner of the Home menu in the editing tab, click on the little arrow to the right of “Fill” to see the dropdown menu. Then click on “Series” within the Fill menu.
3) Select the “Columns” button in order to direct Excel to automatically fill in the column in which your cursor currently sits.

4) Next type in your “Step value”, in this case our increment is 1.

5) Lastly fill in your “Stop value”, in this case 10. Then click “OK”.

6) You will notice that your column A is now automatically filled with your correct values starting at -10, incrementing in units of 1, and stopping at 10.

7) Next, type the expression for \( f(x) \) in cell B2, using a cell reference for cell A2 to put the x-value into the calculation as shown below.

8) To evaluate the remaining cells in column B there are two techniques that work equally well. Putting your cursor in the bottom right of the highlighted cell B2 box, you will notice that the cursor changes from \( \rightarrow \) to \( \leftrightarrow \) to \( \uparrow \). Once you see the bold black “plus” sign you can hold down the left mouse button and drag the formula down the column to evaluate the function at each of the other input values. OR Once you see the bold “plus” sign you can double click the mouse and the formula will evaluate the function at each of the other input values.
9) If we wanted to produce the table with different input values, we could simply add that input value to the list in column A, or refill the table with different input values.

10) For example, if we would like to see the value of this function from -2 to 5 using increments of 0.5, then we can reproduce the table by editing the input values using the fill feature.

11) We could also manually type in any input values we wanted to produce the desired output. Say we wanted the following outputs of this function: $f(12.2), f(22), f(19), f(0.23), f(11.3)$

Installing Solver in Excel

- **Video Link: Installing Solver Add-In on a MAC**
  [https://youtu.be/AmfKxpPZR2w](https://youtu.be/AmfKxpPZR2w)

- **Video Link: Installing Solver Add-In in Windows**
  [https://youtu.be/RX9NJmKsOrQ](https://youtu.be/RX9NJmKsOrQ)

Using Solver

**Example 1.3.7:** ([Video Example Here](https://youtu.be/rNKPufAXdCE))

Use Excel’s Solver tool to solve the equation $f(x) = 4 - 2x = 0$.

**Solution:**

Excel has a tool called Solver which will solve a given function for a desired input value. In other words, Solver will give you the input value to produce a given output value. We must be careful when using the Solver command when there is more than one input value that will give us the same output value.
1) We want to have Excel solve the following equation. \( f(x) = 4 - 2x = 0 \). We first produce an input/output table for this function using any desired input values. Random input values were chosen here.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>( f(x) = 4 - 2x )</td>
</tr>
<tr>
<td>-1</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>7</td>
<td>-10</td>
</tr>
</tbody>
</table>

2) We can see from the table that perhaps the function will have an OUTPUT of zero somewhere between \( x = 0 \) and \( x = 3 \) since the \( y \)-values change from positive to negative somewhere in between \( x=0 \) and \( x=3 \). Notice that the answer we seek here is the \( x \)-value that will produce an output of zero, and not finding \( f(0) \).

3) Clicking on the DATA tab at the very top of the Excel workbook, you will see Solver (once it is installed) all the way on the right, under the analysis tab.

4) Next, click back in Column B of your table in an output value cell that has an output value that is close to the value for which we are solving. In this case let’s click on the output of -2 since that is close to \( y=0 \) that we are trying to solve for.

5) Next click on Solver to produce the following.
6) Putting your cursor in the “Set Objective” area, click on the output cell that has the output of -2 (cell B4). This will tell Excel that we want it to use the formula in cell B4 to solve.

7) Next click on the “Value of” button and type the desired output value in the box next to it, in this case we want a value of zero. This will tell Excel that we want it to solve for when the formula in cell B4 is equal to a value of 0.

8) Next put your cursor in the “By Changing Variable Cells” and click on the input cell that we want to change to solve this equation, in this case cell A4. This will tell Excel which x-value cell to change to solve the equation.

9) Next click the Solve button at the bottom. Excel will change the x-value in cell A4 in order to make cell B4 equal 0 (or very close to 0).

10) The following message will appear and we can simply click the “OK” button to get the solution.

11) Notice that the table values changed: B4 will have a value of zero or very close to zero, and A4 will have the corresponding x-value that produces a y-value of 0.

Using Solver to Find More Than One Solution

Example 1.3.8: (*Video Example Here) https://youtu.be/iU_sT4vyfGE

Use Excel to find all solutions to the following equation. \( g(x) = -x^3 + 2x = 0 \).

Solution:
1) Let’s first produce an input/output table.
2) Notice that the table produces one correct answer: when \( x = 0 \), \( g(x) = 0 \).

3) But there are other answers to this equation. Somewhere between \( x = -2 \) and \( x = -1 \) the function probably equals zero, and again somewhere between \( x = 1 \) and \( x = 2 \) the output looks to have been zero. We know this because there is a sign of the output value in between these \( x \)-values.

4) Here Excel’s Solver will need a little help to produce the correct input value. We need to give Excel some direction about which \( x \)-value we are seeking when we ask it to solve for the \( x \)-value that makes \( y=0 \). We can do that by selecting a cell for the “By Changing Variable Cells” that is closest to the \( x \)-value that we are seeking.

5) We begin as we did before, using Solver. Let’s find the zero that occurs between \( x = -2 \) and \( x = -1 \) first.

6) Set the objective as cell B6 by placing the cursor in the “Set Objective” box and clicking on cell B6.

7) Place the cursor in the “By Changing Variable Cells” and click on cell A6. This is the closes cell to the \( x \)-value we seek, and it will serve as a “seed” starting point for Excel to search for the \( x \)-value. Since it is closes to the \( x \)-value we seek, it will identify that solution.

8) *NOTE:* You must un-check the “Make Unconstrained Variables Non-Negative” when your \( x \)-value solution may be negative.

9) Click “Solve” and Excel will change cell A6 to the \( x \)-value closest to the value in A6 that solves the equation. In this case, that \( x \)-value is approximately -1.41421.
10) Notice the output is not quite zero but very close, giving the answer as scientific notation. This is the number 0.000000934813. So, essentially, it is y=0.

11) We could format column A and/or column B to display the answers in standard form with 6 decimal places as we would like by again right clicking on the column, and formatting the output to a number to 6 decimal places.

12) Repeating the process of using Solver, we could find the third value where the output of the function is equal to zero between the input values of x = 1 and x = 2.
13) Therefore, the three input values that satisfy the equation \( g(x) = -x^3 + 2x = 0 \) are \( x = 0 \), and \( x = -1.41421 \), and \( x = 1.4142 \).

**Related Exercises You Should Complete Now**

**Work on Exercises 1.3.2, 1.3.3, 1.3.5.** Remember there are written solutions as well as solution videos in your course.

**Creating a Graph From A Table**

**Example 1.3.10: ([Video Example Here](https://youtu.be/H7jvAFlyKS8))**

Using Excel, sketch a Scatterplot and Smooth-line graph of the following data table.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>x</td>
</tr>
<tr>
<td>2</td>
<td>-5</td>
</tr>
<tr>
<td>3</td>
<td>-4</td>
</tr>
<tr>
<td>4</td>
<td>-3</td>
</tr>
<tr>
<td>5</td>
<td>-2</td>
</tr>
<tr>
<td>6</td>
<td>-1.41421</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1.4142</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
</tr>
</tbody>
</table>
Solution:
1) Copy the table into Excel as it is shown. Next, highlight all the data from cell A1 through cell B11.

2) Click on the INSERT tab. In the Charts group, click on the arrow next to the picture of a scatter plot to produce a drop-down menu.

3) Click on the upper middle graph option to produce a scatterplot with data points.

4) Or, select the upper right graph option to produce a smooth line graph.

Creating a Graph From A Function

Example 1.3.11: G (*Video Example Here) [https://youtu.be/DAHAVncsxQw](https://youtu.be/DAHAVncsxQw)

Using Excel, produce an input/output table for the function \( f(x) = 8x^2 - 7x + 3 \). Then sketch a Smooth-line graph of the function. Use input values from -10 to 10.
**Solution:** We first create the input-output table. Then highlight it and click Insert, and click on smooth-line graph.

![Input-output table and smooth-line graph]

**Absolute Cell Reference**

**ABSOLUTE cell reference:**
When you have an absolute cell reference, the value does not change because you have **FIXED** it absolutely.

Use the F4 function (On a MAC, you would press fn + F4) once you have clicked on a cell that you want to use in your formula to create the dollar signs around the cell reference to hold the value constant.

Example: $B$7 will reference cell B7 and fix that cell reference even when we drag a formula down.

You could also simply type the $ and the Column Letter followed by $ and the Row Number.

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**Related Exercises You Should Complete Now**

**Work on Exercises 1.3.4, 1.3.6, 1.3.7.** Remember you have access to the answers in your course.
What Makes a Good Graph?

Instructions for making changes to a graph on a Macbook are shown in the Instructional Videos for the course: How to Edit and Add to a Graph in Excel (https://youtu.be/6nccLER03uM). You would follow similar patterns on a PC.

You will be creating graphs on a regular basis for this course. You must make sure that each graph you create for graded material includes the following adjustments:

- **The graph MUST show all significant features of the function**
  - You want to choose inputs and outputs that will allow you to see important parts of the function.
  - Much of how you determine what types of values to use for input and output depends on whether you are looking at a function in general, or if you are looking at a function in a contextual setting.
  - You also need to consider whether you are interested in the “middle” behavior of the graph, or the “end” behavior.
  - Getting an appropriate view for your graph often takes several attempts.
  - Be willing to spend the time thinking about the graph you see, and whether you need to change your inputs.
  - Notice in the graph below, you really can’t easily tell what is happening in the middle of the graph.

**Graph 1 – Initial View of P(t)**

<table>
<thead>
<tr>
<th>t</th>
<th>P(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>44900</td>
</tr>
<tr>
<td>-8</td>
<td>12980</td>
</tr>
<tr>
<td>-6</td>
<td>-460</td>
</tr>
<tr>
<td>-4</td>
<td>-3340</td>
</tr>
<tr>
<td>-2</td>
<td>-1660</td>
</tr>
<tr>
<td>0</td>
<td>500</td>
</tr>
<tr>
<td>2</td>
<td>980</td>
</tr>
<tr>
<td>4</td>
<td>-460</td>
</tr>
<tr>
<td>6</td>
<td>-2140</td>
</tr>
<tr>
<td>8</td>
<td>-460</td>
</tr>
<tr>
<td>12</td>
<td>36980</td>
</tr>
<tr>
<td>14</td>
<td>89540</td>
</tr>
<tr>
<td>16</td>
<td>179060</td>
</tr>
</tbody>
</table>

- The units on the axes must show an appropriate scale that makes it easy to read values from the graph.
For most graphs, you will need to spend some time changing the minimum and maximum values that show on your axes, as well as the scaling.

Notice that in Graph 1 above that your input values go out past the data in the table, though the graph only appears for the data shown.

By adjusting the vertical scaling for the graph shown above, you can get a much clearer picture.

Notice that by adjusting the horizontal minimum and maximum as shown in Graph 2, you can adjust your window so that only the data in the table shows as input.

Also, by adjusting the scaling on the axis, you can get a much clearer idea of where key behavior occurs.

**Graph 2 - Adjusted**

- **The graph MUST have an appropriate title.**
  - Typically, the graph name will be the function name at the top of the output column. You can see that in the first graph shown.
  - This may be appropriate if you are working with just a general function.
  - However, if you are working in context, your graph title should clearly indicate the scenario that you are considering.
  - Notice that has been changed in Graph 2.

- **The Axes MUST have appropriate labels.**
  - If you are working in a contextual situation, it must be very clear what your variables represent in that context.
  - You MUST include the units where appropriate.
  - Notice that the contextual meaning of the input and output has been added to Graph 2.
Exercises for Section 1.3

**Exercise 1.3.1:** Use Excel for the questions below, and report answers rounded to three decimal places.

Save your Excel work so you can reference it later! Make notes to yourself in Excel! Calculate the following expressions by typing the entire expression in Excel:

(a) \((37.5 - 8.76)(4.92 - e^3)\)
(b) \(750e^2 - 5(1.02)^7\)
(c) \(\frac{750}{350 + 4e^{-2}}\)
(d) \(7(1.03)^3 + \ln(9)\)
(e) \((6)^{1.7} + \ln(8)\)
(f) \((e)^{-0.03} + 8^{(-0.04+95-7.8)}\)

**Exercise 1.3.2:** Use Excel to answer the following questions by first inputting the value of the variable and then evaluating. using a cell reference or solving using the solver feature.

1) Evaluate \(g(t) = 6 \ln(40t - 3)\) at each of the following input values, by creating an input-output table for \(t\) and \(g(t)\). \(t = 0.076, 0.09, \frac{5}{6}, 3, 12\). Hint: Remember to format your cell to a number!

2) Calculate the input value(s) of \(g(t) = 6 \ln(40t - 3)\) corresponding to \(g(t) = 35\). Round the answer to 3 decimal places. Hint: Remember to use Solver here! The OUTPUT = 35 not the input!

**Exercise 1.3.3:** For \(f(x) = x^3 + x^2 - 12x\), calculate all solutions to \(f(x) = 10\). Note: Use Solver here! ALSO Note: there are three solutions. Write a brief explanation to yourself in a textbox to remind yourself how you did this. Round the answer to 3 decimal places

Written by M. Johnson and S. Taylor - Dutchess Community College
**Exercise 1.3.4:** Use Excel to complete each of the following:

1) Create a SCATTERPLOT graph of the function below.

2) Then use Excel to create a SMOOTHLINE graph with no markers.

   **NOTE:** You will have 2 separate graphs for the function!!

3) Graph \( f(x) = 16 - 5x - x^2 \) using a window from \( x = -30 \) to \( x = 30 \). Make a note for yourself in a textbox to remind yourself how you did this.

**Exercise 1.3.5:** Suppose \( P(t) = \frac{425000}{1+50000e^{-0.09t}} \) where \( t \) represents the number of months after a new lake is stocked with fish and \( P(t) \) represents the approximate number of fish in the lake.

1) Use Excel to create a Smooth-line (with no markers) graph of the function for the first 200 months. Make your table increment by 5 months at a time. Label the horizontal and vertical axis with the real-world, practical meaning of each.

2) When will there be 300,000 fish in the pond? Use Excel to solve. Write a complete sentence to answer the question.

3) Identify the right-tail behavior of the function. Write the behavior in REAL-WORLD WORDS WITH PRACTICAL MEANING. (using the context of the problem).

4) Where is the function increasing and concave down?
   - **(a)** Give the interval using interval notation
   - **(b)** Give the interval using inequality notation.
   - **(c)** Write a complete sentence to interpret the real-world, practical meaning of this increasing/concave down behavior.

**Exercise 1.3.6:** Use Excel to create an input-output table for the function \( f(x) = mx + b \ldots \)

1) First label the constants/parameters by putting the letter m in cell A1, the letter b in...
cell B1. Then you will put the value of m in cell A2, and the value of b in cell B2 in the next problem.

2) Create an input-output table for f(x) when \( m = 2 \) and \( b = -5 \) by typing the given values in cells A2 and B2. **Make these values with Absolute Cell References, since we do NOT want these values to update as we copy the formula.**

3) You will then create your input-output table by creating the column of x values (use from -5 to 5 with steps size 1), and then creating the column of y values. **Note:** When you are creating the function values, you will select cell A2 for m and cell B2 for b in your formula.

4) Create the graph on the window from \( x = -5 \) to \( x = 5 \).

5) Create an input-output table for f(x) when \( m = -3 \) and \( b = 15 \) by changing the values in cells A2 and B2. **Note:** When you change these values in cells A2 and B2, the rest of your worksheet should automatically update to reflect this new function.

6) Create the graph on the window from \( x = -5 \) to \( x = 5 \).

**Exercise 1.3.7:** Use Excel to create an input-output table for the function:

\[
 f(x) = ax^2 + bx + c
\]

1) First label the constants/parameters by putting the letter a in cell A1, the letter b in cell B1, and the letter c in cell C1. Then you will put the value of a in cell A2, and the value of b in cell B2 and value of c in cell C2 in the next problem.

2) Create an input-output table for f(x) when \( a = -2 \) and \( b = 3 \) and \( c = 5 \) by first typing the values in cells A2, B2, and C2. Remember to use Absolute Cell references in the formula for your output column when you refer to these cells.

3) Create the graph on the window from \( x = -5 \) to \( x = 5 \).

4) Create an input-output table for f(x) when \( a = 3 \) and \( b = -4 \) and \( c = -7 \) by changing the values in cells A2, B2, and C2.

5) Create the graph on the window from \( x = -5 \) to \( x = 5 \).
Unit 1 Section 4 : Using Models and Creating Formulas of Models with Excel

Regression

Learning Outcomes

• Given an equation or table of values or a graph or plotted points, understand what TYPE of function it is or what TYPE of function it best resembles. Find the formula of best-fit if needed.

• Given data points in a table or a graph, identify the type of function that would best model the data and justify the model type. Choose the best type of function based on: Increasing/decreasing and concavity, End-behavior, Intercepts, Domain and range

• Understand that sometimes a model will only "fit" the data in some ways, and be a "bad fit" in other ways. Be able to explain that there is "model break-down" but you have still selected that model for a reason.

• Use Excel to find the equation of best-fit. Use the equation of best fit as the model to answer questions.
  o Find the most appropriate best-fit model based on the behavior of the given data.
  o Store the parameters of the equation in Excel to 10 decimal places, but report the parameters of the equation to 3 decimal places.
  o Use the equation of the model to answer questions (such as those answered in the previous sections).

Basics of Finding a Best-Fit Regression Model for the data:

1) Always align your data first (example: “years since 2000” so $t = 0$ means the year 2000).

2) When selecting an appropriate function type to fit your data, you will ONLY use one of the following functions throughout the semester:
   • linear, exponential, quadratic, logarithmic, logistic, cubic.
   • Do not use any of the other model types this semester.

(a) If the data has a clear point of inflection, then you must use a function that has an inflection point (logistic or cubic).

(b) If the data has a turning point, then you must use a function that has a turning point (quadratic or cubic).

(c) Sometimes you will select a type of function that fits the data in many ways, but is not a good fit in other ways. For example, your model may fit the data in the short-term, but in the long term you would expect that the data would not follow the pattern of the chosen function. When this happens, you will make your function selection, but then it is important to explain your expectations about model break-down.

3) When you have found the equation of the best-fit model, you should always report the following:
   • The equation of the model using function notation.
   • The real-world, contextual meaning of both variables in your model.
• The domain for which your model works in the real-world scenario.
• Any further comments/discussion about the goodness/badness of fit for your selected model. For example, if the model only makes sense over a short period of time, it should be clearly noted that there is model break-down after a certain time.

**Linear Functions:**
• The graph is a straight line.
• The graph has a constant rate of change. Every time x increases by one unit you ADD or SUBTRACT the same amount to the y-value.
  - The slope is the constant rate of change.
  - \( Slope = \frac{\text{rise}}{\text{run}} \) OR \( \frac{\Delta y}{\Delta x} \)
  - Positive slope ↔ Function increases from left to right.
  - Negative slope ↔ Function decreases from left to right.
  - Slope is the number 0 ↔ Function is flat/horizontal
  - Slope is undefined ↔ Function is vertical

**Equation of a line:**
An equation of a linear function is a function of the form:

\[
f(x) = mx + b \quad \text{where} \quad m \text{ is the slope and} \quad (0, b) \text{ is the y-intercept}
\]
or

\[
f(x) = m(x - x_1) + y_1 \quad \text{where} \quad m \text{ is the slope and} \quad (x_1, y_1) \text{ is any point on the line}
\]

**Exponential Functions:**

\[
f(x) = a(b)^x \quad \text{or} \quad f(x) = ae^{rt}
\]

- \((0, a)\) is the y-intercept,
- \(b\) is the growth/decay factor
- \(b > 1\): \(f(x)\) is increasing.
- \(0 < b < 1\): \(f(x)\) is decreasing.
- \(b = 1 + r\) where \(r\) is the decimal form of the growth/decay rate.
- \(r\) is positive for growth and negative for decay.
- Constant \textbf{PERCENT} growth/decay rate: \((\text{factor-1}) \times 100\) or \((b - 1) \times 100\).
- When we discuss rate, we always refer to it as a percent, though in the formula for \(b\), it MUST be inserted as a decimal

- \((0, a)\) is the y-intercept
- \(r\) is the continuously compounding rate in decimal form.
- \(\text{positive} \; r\) means \(f(x)\) increasing
- \(\text{negative} \; r\) means \(f(x)\) decreasing
- \(\text{Constant continuous compounding} \quad \text{PERCENT} \; \text{rate is} \; r \times 100\)
EXPONENTIAL GROWTH
(Increasing Function)

EXPONENTIAL DECAY
(Decreasing Function)

Quadratic Functions (2\textsuperscript{nd} Degree Polynomial Functions)

Equation can be written in:

- **Standard form**: \( f(x) = ax^2 + bx + c \), where \( a \neq 0 \)
- **Vertex form**: \( f(x) = a(x - h)^2 + k \), where \((h, k)\) is the vertex (turning point)
- The graph is a parabola.
- Parabola opens upward if the leading coefficient is positive, \( a > 0 \)
- Parabola opens downward if the leading coefficient is negative, \( a < 0 \)
- The graph has a vertex. The vertex is the maximum point if it is open down, and the vertex is the minimum point if the parabola is open up.

Quadratic Functions and Possible Number of Roots:

The solution(s) to \( 0 = ax^2 + bx + c \) is \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

<table>
<thead>
<tr>
<th>Two Real Roots</th>
<th>One Real Root</th>
<th>No Real Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Graph" /></td>
<td><img src="image2.png" alt="Graph" /></td>
<td><img src="image3.png" alt="Graph" /></td>
</tr>
<tr>
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<td><img src="image6.png" alt="Graph" /></td>
</tr>
<tr>
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<td><img src="image8.png" alt="Graph" /></td>
<td><img src="image9.png" alt="Graph" /></td>
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<tr>
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<td><img src="image14.png" alt="Graph" /></td>
<td><img src="image15.png" alt="Graph" /></td>
</tr>
<tr>
<td><img src="image16.png" alt="Graph" /></td>
<td><img src="image17.png" alt="Graph" /></td>
<td><img src="image18.png" alt="Graph" /></td>
</tr>
</tbody>
</table>

Created by Maryanne Johnson, Dutchess Community College, Summer 2020
Logarithmic Functions:

\[ f(x) = a \cdot \ln(x) + b \]

- If \( a > 0 \), \( f(x) \) is Increasing without bound.
- \( f(x) \) is concave down
- As \( x \to 0 \), the y-value \( \to -\infty \).
- Domain: the interval \((0, \infty)\)
- Range: the interval \((-\infty, \infty)\)

- If \( a < 0 \), \( f(x) \) is Decreasing without bound.
- \( f(x) \) is concave up
- As \( x \to 0 \), the y-value \( \to \infty \).
- Domain: the interval \((0, \infty)\)
- Range: the interval \((-\infty, \infty)\)

Logistic Function:

\[ f(x) = \frac{L}{1 + Ae^{-Bx}} \]

- Logistic functions have one inflection point.
- Logistic functions approach \( y = 0 \) on one tail, and approach \( y = L \) on the other tail.
- Domain: the interval \((-\infty, \infty)\)
- Range: the interval \((0, L)\)
Cubic Functions:

\[ f(x) = ax^3 + bx^2 + cx + d \]

- Cubic functions have one inflection point.
- Cubic functions SOMETIMES have a local maximum and local minimum.
- Domain: the interval \((-\infty, \infty)\)
- Range: the interval \((-\infty, \infty)\)

**Related Exercises You Should Complete Now**

Work on Exercises 1.4.1 through 1.4.6. Remember that you have both written and video help for these exercises in your course.
# Quick Summary - Library of Functions

<table>
<thead>
<tr>
<th>Function Type</th>
<th>Equation</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Function</td>
<td>( f(x) = k )</td>
<td>((-\infty, \infty))</td>
<td>( k )</td>
</tr>
<tr>
<td>Linear Function</td>
<td>( f(x) = m(x - x_1) + y_1 )</td>
<td>((-\infty, \infty))</td>
<td>((-\infty, \infty))</td>
</tr>
<tr>
<td></td>
<td>( f(x) = mx + b )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exponential Function</td>
<td>( f(x) = a(b)^x )</td>
<td>((-\infty, \infty))</td>
<td>((0, \infty))</td>
</tr>
<tr>
<td></td>
<td>( f(x) = a e^{kt} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quadratic Function</td>
<td>( f(x) = a(x-h)^2 + k )</td>
<td>((-\infty, \infty))</td>
<td>([k, \infty) OR (-\infty, k])</td>
</tr>
<tr>
<td></td>
<td>( f(x) = ax^2 + bx + c )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Cubic Function

\[ f(x) = ax^3 + bx^2 + cx + d \]

- **Domain:** \((-\infty, \infty)\)
- **Range:** \((-\infty, \infty)\)
- Can have two turning points or no turning points.

### Logarithmic Function

\[ f(x) = a \cdot \ln(x) + b \]

- **Domain:** \((0, \infty)\)
- **Range:** \((-\infty, \infty)\)

### Logistic Function

\[ f(x) = \frac{L}{1 + Ae^{-Bx}} \]

- **Domain:** \((-\infty, \infty)\)
- **Range:** \((0, L)\)
Exercises for 1.4 Section 4

Exercise 1.4.1: Review Ideas From Previous Sections

The function \( D(x) = \frac{100}{1 + 0.167 e^{0.51x}} \) gives the percent of residential internet access that is dial-up access only. In this function, \( x \) represents the number of years since 2000.

(a) Identify the right-tail behavior. Write a complete sentence to interpret the real-world, contextual meaning of the right-tail behavior.

(b) Estimate the inflection point of the function \( D(x) \) using the graph shown above. Describe in context what is happening at this point.

(c) Identify the interval where the function is decreasing and concave down (both at the same time). Use interval notation. Write a complete sentence to interpret the real-world, contextual meaning of this behavior in practical terms.

(d) Identify the interval where the function is decreasing and concave up. Use interval notation. Write a complete sentence to interpret the real-world, contextual meaning of this.
Exercise 1.4.2: DO THIS PROBLEM IN EXCEL. Use a Text box to type in responses to the questions. Make sure you label the responses according to the question number.

At a peach packaging plant, peaches are brought in, and then they are processed and sent for canning. The table below shows the amount of peaches in the processing area after they are brought in.

<table>
<thead>
<tr>
<th>Time (hours)</th>
<th>Peaches (in thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>0.25</td>
<td>38</td>
</tr>
<tr>
<td>0.50</td>
<td>32</td>
</tr>
<tr>
<td>0.75</td>
<td>24</td>
</tr>
<tr>
<td>1.00</td>
<td>17</td>
</tr>
</tbody>
</table>

(a) Plot the data in Excel.

(b) Identify which type of function would serve as the best-fit model. Write a clear explanation to justify the type of function you selected, and then find the equation of the model and define your variables.

(c) Graph the model from time 0 to time 1.5 hours. Use a Smooth-line curve.

(d) Use the model to identify the number of peaches in the processing area after 0.6 hours. Write a clear contextual sentence that addresses what you found.

(e) Use the model to identify the amount of time until there are 35 thousand peaches in the processing area. Write a clear contextual sentence that addresses what you found.

(f) Write a complete sentence to interpret the contextual meaning of the slope.

Exercise 1.4.3: DO THIS PROBLEM IN EXCEL. Use a Text box to type in responses to the questions. Make sure you label the responses according to the question number.

According to the Forrester Research Interactive Advertising Forecast (April 2009), spending on online marketing is projected to increase. The Forrester Research projections are given in the table.

<table>
<thead>
<tr>
<th>Year</th>
<th>Spending (in billions of dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2008</td>
<td>23.073</td>
</tr>
<tr>
<td>2009</td>
<td>25.577</td>
</tr>
<tr>
<td>2010</td>
<td>29.012</td>
</tr>
<tr>
<td>2011</td>
<td>34.077</td>
</tr>
<tr>
<td>2012</td>
<td>40.306</td>
</tr>
<tr>
<td>2013</td>
<td>47.378</td>
</tr>
<tr>
<td>2014</td>
<td>54.956</td>
</tr>
</tbody>
</table>

(a) Plot the data in Excel aligning the data using the year 2000 as $t = 0$.

(b) Identify which type of function would serve as the best-fit model. Write a clear explanation to justify the type of function you selected, and then find the equation.
of the model and define your variables.

(c) Graph the *model* for the years 2008 through 2020, again using the year 2000 as \( t = 0 \). Use a smooth-line curve.

(d) Use the *model* to identify the expected spending in 2015. Write a clear contextual sentence addressing what you found.

(e) Use the *model* to identify when spending will reach $75,430,000,000. Write a clear contextual sentence addressing what you found.

(f) Write a complete sentence to interpret exactly how the spending is increasing each year according to the model using contextual meaning of this behavior in practical terms.

**Exercise 1.4.4:** The table gives the price, in dollars, of a round-trip flight from Denver to Chicago on a certain airline and the corresponding monthly profit for that airline on that route. DO THIS PROBLEM IN EXCEL. Use a Text box to type in responses to the questions. Make sure you label the responses according to the question number.

<table>
<thead>
<tr>
<th>Ticket Price (dollars)</th>
<th>Profit (in thousands of dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>3080</td>
</tr>
<tr>
<td>250</td>
<td>3520</td>
</tr>
<tr>
<td>300</td>
<td>3760</td>
</tr>
<tr>
<td>350</td>
<td>3820</td>
</tr>
<tr>
<td>400</td>
<td>3700</td>
</tr>
<tr>
<td>450</td>
<td>3380</td>
</tr>
</tbody>
</table>

(a) Plot the data in Excel. Identify which type of function would serve as the best-fit model. Write a clear explanation to justify the type of function you selected, and then find the equation of the model and define your variables.

(b) Graph the *model* for ticket prices from $150 up to $600. Use a smooth-line curve.

(c) Use the *model* to identify the expected profit if they sell tickets for $375. Write a clear contextual sentence addressing what you found.

(d) Use the *model* to identify the ticket price(s) that will result in a profit of $3.4 million. Write a clear contextual sentence addressing what you found.

(e) Use the graph of the *model* to estimate the maximum profit. Write a complete sentence to identify the ticket price they should charge and the profit they would earn to achieve the maximum profit.

**Exercise 1.4.5:** The table below shows the number of tickets that people will buy when the ticket price is \( x \) dollars. DO THIS PROBLEM IN EXCEL. Use a Text box to type in responses to the questions. Make sure you label the responses according to the question number.
(a) Plot the data in Excel. Identify which type of function would serve as the best-fit model. Write a clear explanation to justify the type of function you selected, and then find the equation of the model and define your variables. BE CAREFUL HERE TO THINK ABOUT THE END BEHAVIOR ON THE RIGHT TAIL THAT MAKES SENSE IN REALITY!

(b) Graph the model for ticket prices from $1 up to $75. Use a smooth-line curve.

(c) What domain makes sense in reality for this model? Explain your answer in context.

(d) Use the model to identify the ticket price(s) that will result in virtually no tickets sold. Write a clear contextual sentence discussing what you found.

(e) Describe the behavior of the model (increasing/decreasing/concavity). Explain the real-world meaning of that behavior.

<table>
<thead>
<tr>
<th>Ticket Price (dollars)</th>
<th>Demand (tickets)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1000</td>
</tr>
<tr>
<td>20</td>
<td>600</td>
</tr>
<tr>
<td>30</td>
<td>400</td>
</tr>
<tr>
<td>40</td>
<td>224</td>
</tr>
<tr>
<td>50</td>
<td>100</td>
</tr>
</tbody>
</table>
**Exercise 1.4.6:** The data below gives the number of nonprofit national and binational chambers of commerce in the between 1998 and 2007. DO THIS PROBLEM IN EXCEL. Use a Text box to type in responses to the questions. Make sure you label the responses according to the question number.

<table>
<thead>
<tr>
<th>Year</th>
<th>Chambers of Commerce</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998</td>
<td>129</td>
</tr>
<tr>
<td>2000</td>
<td>143</td>
</tr>
<tr>
<td>2001</td>
<td>142</td>
</tr>
<tr>
<td>2002</td>
<td>141</td>
</tr>
<tr>
<td>2003</td>
<td>139</td>
</tr>
<tr>
<td>2004</td>
<td>136</td>
</tr>
<tr>
<td>2005</td>
<td>135</td>
</tr>
<tr>
<td>2006</td>
<td>137</td>
</tr>
<tr>
<td>2007</td>
<td>169</td>
</tr>
</tbody>
</table>

(a) Plot the data in Excel aligning the data using the year 2000 as $t = 0$.

(b) Identify which type of function would serve as the best-fit model. Write a clear explanation to justify the type of function you selected, and then find the equation of the model and define your variables.

(c) Graph the model for the years 1998 through 2010, again aligning the data using the year 2000 as $t = 0$. Use a smooth-line curve.

(d) Identify the right-tail end-behavior for this model. Explain the real world meaning of this if it were the case that the real-world data continued to follow the trend of the model.

(e) Using the graph, estimate the inflection point (state as an ordered pair please). Write a clear contextual sentence that addresses what the significance of this point is.

(f) Identify the interval on which the function is increasing and concave up (both at the same time). Interpret the real-world meaning of this behavior in a clear contextual sentence.

(g) Identify the interval on which the function is increasing and concave down (both at the same time). Interpret the real-world meaning of this behavior in a clear contextual sentence.
Unit 2 Section 1: Average Rate of Change and instantaneous Rate of Change

Unit 2 Section 1: Learning Outcomes
Average Rate of Change
Why Think About Average Rate of Change?
Finding the Average Rate of Change of a Function
Computing Average Rate of Change from a Graph
Terminology
What does the sign and magnitude (size) of the ARC tell us about a function?
Rate of Change at a Point
Instantaneous Rate of Change and Tangent Lines
What is a Tangent Line?
A Graphical View of Instantaneous Rate of Change
Using Excel to Fully Explore Instantaneous Rate of Change at a Point

Exercises for Unit 2 Section 1

Unit 2 Section 2: The Derivative - Definition of the Derivative at a Point

Unit 2 Section 2: Learning Outcomes
The Derivative at a Point:
What does the sign of the derivative at a point mean?
Interpreting a Derivative Value:
Rates in Real Life
Business and Economics Terms
What if the Derivative Doesn’t Exist?
Where can a slope not exist?
Where can a tangent line not exist?

Exercises for Unit 2 Section 2

Unit 2 Section 3: Derivative As A Function

Unit 2 Section 2: Learning Outcomes:
Excel Outcomes:
Recall From Earlier Sections
The Derivative as a Function
Concavity and First Derivative
First Derivative Information About Shape (Part 1)
Interplay Between a Graph of a Function and the Derivative
Sketching a Graph of the Derivative From Information About a Function
Differentiability:
Unit 2 Section 1: Average Rate of Change and instantaneous Rate of Change

Unit 2 Section 1: Learning Outcomes
Average Rate of Change
- Explain what is meant by average rate of change (ARC) and how it is calculated.
  - What are the units of the ARC value?
  - Slope of the secant line between the points \((a, f(a))\) and \((b, f(b))\)
  - \(ARC = \frac{f(b)-f(a)}{b-a}\)
  - Units of average rate of change are (units of y-value) PER (units of x-value)
    Examples: dollars per item, employees per year, dollars of profit per dollar of advertising
- How can average rate of change be seen on the graph of a function (i.e., geometrically with secant line)?
  - Slope of the secant line between the points \((a, f(a))\) and \((b, f(b))\)
- What the sign and magnitude of the average rate of change tell us about the function:
  - Positive ARC means the function increases on average on that interval
  - Negative ARC means the function decreases on average on that interval
  - ARC is 0 means the function is constant (horizontal) on average on that interval
  - Steep graphs have ARC that has a larger absolute value

Instantaneous Rate of Change
- Explain what is meant by instantaneous rate of change at a point (IRC).
  - **slope of the tangent line at that point**
  - **limit of the slopes of the secant lines at that point**
  - **limit of the average rates of change at that point**
- Estimate an IRC value from a graph.
  - **the position of the tangent line is a limiting position of secant lines.**
- How are instantaneous rates of change different from average rates of change? How are they related?
  - **instantaneous rate of change is a limit of average rates of change.**
- How are tangent lines different from secant lines? How are they related?
  - **the slope of a tangent line is a limit of slopes of secant lines.**
- Interpret instantaneous rate of change in context.
  - **since it is a rate, it will have units of (output divided by input)**
- Be able to sketch tangent lines on the graph and explain how the IRC at that point is related to that tangent line (it is the slope of that tangent line).
• Understand how the IRC is positive if function is increasing, negative if function is "flat" at that instant.

• Understand how the IRC is increasing if function is concave up, decreasing if function is concave down.

Excel Objectives
• Calculate ARC in Excel, estimate IRC in Excel

Why Think About Average Rate of Change?
Gasoline costs have experienced some wild fluctuations over the last several decades. Table 1[5] lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

<table>
<thead>
<tr>
<th>y</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
</tr>
</thead>
<tbody>
<tr>
<td>C(y)</td>
<td>2.31</td>
<td>2.62</td>
<td>2.84</td>
<td>3.30</td>
<td>2.41</td>
<td>2.84</td>
<td>3.58</td>
<td>3.68</td>
</tr>
</tbody>
</table>

Table 1

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from $2.31 to $3.68, an increase of $1.37. While this is interesting, it might be more useful to look at how much the price changed per year. In this section, we will investigate changes such as these.

Finding the Average Rate of Change of a Function
The price change per year is a rate of change because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the average rate of change over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

\[
\text{Average rate of change} = \frac{\text{Change in output}}{\text{Change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as “delta-y over delta-x” or “the change in y divided by the change in x.” Occasionally we write △f instead of △y, which still represents the change in the function’s output value resulting from a change to its input value. It does not mean we are changing the function into some other function.
In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

\[
\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7 \text{ years}} \approx 0.196 \text{ dollars per year}
\]

On average, the price of gas increased by about 19.6 cents each year.

**Computing Average Rate of Change from a Graph**

**Example 1**

Given the function \( g(t) \) shown in Figure 1, find the average rate of change on the interval \([-1, 2]\).

![Figure 1](image1.png)

**Solution:** At \( t = -1 \), Figure 2 shows \( g(-1) = 4 \). At \( t = 2 \), the graph shows \( g(2) = 1 \).

![Figure 2](image2.png)

The horizontal change \( \Delta t = 3 \) is shown by the red arrow, and the vertical change \( \Delta g(t) = -3 \) is shown by the turquoise arrow. The output changes by \(-3\) while the input changes by \(3\), giving an average rate of change of

\[
\frac{1 - 4}{2 - (-1)} = \frac{-3}{3} = -1
\]
Looking at the graph, we clearly see that the function decreases on the interval from (-3, 1) and then increases on the interval (1, 2). However, over the interval [-3, 2] the graph decreases more on average than it increases. Therefore, our average rate of change is negative.

We can calculate the Average Rate of Change (ARC) between any two points on a function by calculating the slope of the line between the given points.

\[
slope = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}
\]

\[
ARC = \frac{\text{change in output}}{\text{change in input}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

**Terminology**

- The line formed between the given points is called a **secant line** to the curve.
- The slope of the secant line between any two given points \((a, f(a))\) and \((b, f(b))\) is the average rate of change between these two given points. **Slope of the secant line** = \(\frac{f(b) - f(a)}{b - a}\)
- The units of this ARC are = \(\frac{\text{output units}}{\text{input units}}\)

**What does the sign and magnitude (size) of the ARC tell us about a function?**

- + Positive ARC \(\rightarrow\) the function increases on average on the given interval.
- - Negative ARC \(\rightarrow\) the function decreases on average on the given interval.
- Zero ARC \(\rightarrow\) the function remains constant on the given interval. (Secant line is a Horizontal line)
- The **Magnitude** of the ARC tells us the relative steepness of the curve on the given interval.
- Steeper graphs have ARC that has a larger absolute value.
Rate of Change at a Point

We know how to find the rate of change over an interval, and can think of this rate of change as the slope of a secant line between two points, but what if we want to know how a function is changing at a point?

We need two points in order to determine the slope of a secant line. How can we find a slope of a curve, at just one point?

The answer, as suggested in Figure 2 is to find the slope of the tangent line to the curve at that point. Most of us have an intuitive idea of what a tangent line is. Unfortunately, “tangent line” is hard to define precisely.

**Definition:** A **secant line** is a line between two points on a curve.

**Can’t-quite-do-it-yet Definition:** A **tangent line** is a line at one point on a curve …. that does its best to be the curve at that point?

It turns out that the easiest way to define the tangent line is to define its slope.

**Instantaneous Rate of Change and Tangent Lines**

Suppose we drop a tomato from the top of a 100 foot building and time its fall.

Some questions are easy to answer directly from the table:

(a) How long did it take for the tomato to drop 100 feet? (2.5 seconds)
(b) How far did the tomato fall during the first second? (100 – 84 = 16 feet)
(c) How far did the tomato fall during the last second? (64 – 0 = 64 feet)
(d) How far did the tomato fall between \( t = .5 \) and \( t = 1 \)? \((96 - 84 = 12 \text{ feet})\)

Some other questions require a little calculation:

(e) What was the average rate of change (ARC) of the tomato during its fall?

\[
\text{Average rate of change} = \frac{\text{distance fallen}}{\text{total time}} = \frac{\Delta \text{position}}{\Delta \text{time}} = \frac{-100 \text{ ft}}{2.5 \text{ s}} = -40 \text{ ft/s}.
\]

(f) What was the average rate of change between \( t=1 \) and \( t=2 \) seconds?

\[
\text{ARC} = \frac{\Delta \text{position}}{\Delta \text{time}} = \frac{36 \text{ ft} - 84 \text{ ft}}{2 \text{ s} - 1 \text{ s}} = \frac{-48 \text{ ft}}{1 \text{ s}} = -48 \text{ ft/s}.
\]

Some questions are more difficult.

(g) How fast was the tomato falling 1 second after it was dropped?

This question is significantly different from the previous two questions about average rate of change. Here we want the instantaneous rate of change (IROC), the rate of change at an instant in time. Unfortunately, the tomato is not equipped with a speedometer so we will have to give an approximate answer.

One crude approximation of the instantaneous rate of change after 1 second is simply the average rate of change (called the average velocity) during the entire fall, \(-40 \text{ ft/s}\). But the tomato fell slowly at the beginning and rapidly near the end so the "\(-40 \text{ ft/s}\)" estimate may or may not be a good answer.

We can get a better approximation of the instantaneous rate of change at \( t=1 \) by calculating the average rates of change over a short time interval near \( t = 1 \).

- The average rate of change between \( t = 0.5 \) and \( t = 1 \) is \(\frac{-12 \text{ feet}}{0.5 \text{ s}} = -24 \text{ ft/s} \),
- The average rate of change between \( t = 1 \) and \( t = 1.5 \) is \(\frac{-20 \text{ feet}}{.5 \text{ s}} = -40 \text{ ft/s} \) so we can be reasonably sure that the instantaneous rate of change is between \(-24 \text{ ft/s}\) and \(-40 \text{ ft/s}\).

**Major Idea!! VERY IMPORTANT!!**

In general, the shorter the interval over which we calculate the average rate of change, the better the average rate of change will approximate the instantaneous rate of change.

The average rate of change over an interval is \(\frac{\Delta \text{Output}}{\Delta \text{Input}}\), which is the slope of the **secant line** through two points on the graph of Output (Vertical Axis) versus Input (Horizontal Axis).

The instantaneous rate of change at a particular time and height is the slope of the **tangent line** to the graph at a given point.
What is a Tangent Line?

- We have seen that we can easily find the average rate of change between two points by using slope. So, the idea of the secant line drawn to a curve is quite easy to understand.

- The idea of a tangent line to a curve is more complicated. In geometry, the tangent line (or simply tangent) to a curve at a given point is the straight line that "just touches" the curve at that point. Mathematicians define it as the line through a pair of infinitely close points on the curve. The word "tangent" comes from the Latin tangere, "to touch".

A Graphical View of Instantaneous Rate of Change

So how do we determine the slope of a curve at only one point (instantaneous rate of change)?

Notice: As the second point gets closer and closer to the point where \( x = a \), the secant line seems to be approaching the tangent line. This means that as the distance between the two inputs (\( h \)) gets closer and closer to 0, the slope of the secant line will get closer and closer to the slope of the tangent line.
Average rate of change  = \frac{\Delta \text{Output}}{\Delta \text{Input}} = \text{slope of the secant line through 2 points.}

Instantaneous rate of change  = \text{slope of the line tangent to the graph.}

Example 2:

Now let's look at the problem of finding the slope of the line L (Figure 6) which is tangent to \( f(x) = x^2 \) at the point (2,4).

We will use the idea that secant lines over small intervals approximate the tangent line.

We can see that the line through (2,4) and (3,9) on the graph of \( f(x) \) is an approximation of the slope of the tangent line and we can calculate that slope exactly: \( m = \frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5 \). But \( m = 5 \) is only an estimate of the slope of the tangent line and not a very good estimate. It's too big. Notice that the secant line is steeper than the tangent line.

We can get a better estimate by picking a second point on the graph of \( f(x) \) which is closer to (2,4). Notice that the point (2,4) is fixed and it must be one of the points we use.
From Figure 7, we can see that the slope of the line through the points (2,4) and (2.5,6.25) is a better approximation of the slope of the tangent line at (2,4): \( m = \frac{\Delta y}{\Delta x} = \frac{6.25 - 4}{2.5 - 2} = 2.25 / .5 = 4.5 \), a better estimate, but still an approximation.

We can continue picking points closer and closer to (2,4) on the graph of \( f(x) \), and then calculating the slopes of the lines through each of these points and the point (2,4).

**Notice**: We pick points on either side of the point of interest

<table>
<thead>
<tr>
<th>Points to the left of (2, 4)</th>
<th>Points to the right of (2, 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( f(x) = x^2 )</td>
</tr>
<tr>
<td>-----------------------------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td>1.5</td>
<td>2.25</td>
</tr>
<tr>
<td>1.9</td>
<td>3.61</td>
</tr>
<tr>
<td>1.99</td>
<td>3.9601</td>
</tr>
</tbody>
</table>

The only thing special about the \( x \)-values we picked is that they are numbers which are close, and very close, to \( x = 2 \), and that we use values on both sides of 2. Someone else might have picked other nearby values for \( x \).

As the points we pick get closer and closer to the point (2,4) on the graph of \( f(x) = x^2 \), the slopes of the lines through the points and (2,4) become better approximations of the slope of the tangent line, and these slopes are getting closer and closer to 4. Notice that one of our tables shows slopes that are less than 4, and our other table shows slopes that are greater than 4.

**Related Exercises You Should Complete Now**

**Work on Exercises 2.1.1 through 2.1.5.** Remember, you have written answers and videos for these exercises available in your course.

**Using Excel to Fully Explore Instantaneous Rate of Change at a Point**

We can use Excel to see this clearly what happens to ARC for even smaller intervals or “gaps” between input values. Let’s use the function in Example 2, and the same point. Our goal will be to “get close” to 2 from each side, and then compute the ARC between the point (2,4) and each new point. Follow the steps shown below:

1. Set up your spreadsheet so that it is clear what your gap between inputs is, and so that you have two tables with inputs and outputs. Each table should start with the point (2, 4). One table should show inputs approaching 2 from the left (values less than but getting closer to 2), and the other table should show inputs approaching 2 from the right (values greater than but getting closer to 2).

   a. To help create values that get closer and closer to 2 from both sides, let’s follow a consistent process. We will start by creating a column that represents the “gap” we want between inputs. We will start with a gap of 0, and then use gaps
which are decreasing powers of 10. That means that our gap column will include the values 0, .1, .01, .001, .0001, .00001. Hopefully you agree that these “gaps” will bring your inputs VERY close to 2!

b. One table should have input values starting at 2 and then inputs which get closer and closer to 2 from the left. That means we will be subtracting .1, .01, .001, .0001, .00001 and finally .000001 to 2 for our input column.

c. The second table should have input values starting at 2, and then inputs which add the gaps above to create each successive input value.

2. Now compute the output values for each set of inputs.

3. Finally, compute the ARC between each point and the starter point (2, 4). **NOTE:** This is a good place to use an absolute cell reference! See the first ARC formula below. Notice the $ signs around cell C3 and B3 mean that as we copy the formula down, these values will not change. However, C4 and B4 will be updated as we copy down.

Your final tables should look like the one shown below.

<table>
<thead>
<tr>
<th>Gap</th>
<th>x</th>
<th>f(x) = x^2</th>
<th>ARC between point and (2, 4)</th>
<th>Gap</th>
<th>x</th>
<th>f(x) = x^2</th>
<th>ARC between point and (2, 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>3.61 (C4-$C$3)/(B4-$B$3)</td>
<td>0.1</td>
<td>1.9</td>
<td>3.61</td>
<td>3.9</td>
</tr>
<tr>
<td>0.01</td>
<td>1.99</td>
<td>3.9601</td>
<td>3.99</td>
<td>0.001</td>
<td>1.999</td>
<td>3.9996001</td>
<td>3.999</td>
</tr>
<tr>
<td>0.0001</td>
<td>1.9999</td>
<td>3.99960001</td>
<td>3.9999</td>
<td>0.0001</td>
<td>1.99999</td>
<td>3.9999600001</td>
<td>3.99999</td>
</tr>
<tr>
<td>0.00001</td>
<td>1.999999</td>
<td>3.999996000001</td>
<td>3.999999</td>
<td>0.00001</td>
<td>1.9999999</td>
<td>3.999999600000001</td>
<td>3.9999999</td>
</tr>
</tbody>
</table>

Think about what these tables are showing you. It is important that you learn to “speak” this language clearly!
- As we let the inputs approach 2 from the left, our values for the ARC (or the slopes of the secant lines) are getting closer and closer to 4 from below.
- As we let the inputs approach 2 from the right, our values for the ARC (or slopes of the secant lines) are getting closer and closer to 4 from above.
- We say that the slopes of the secant lines (ARC) approach a limiting value. We call this limiting value the slope of the tangent line.
- Therefore, we can confidently say that the Instantaneous Rate of Change (IROC) or slope of the tangent line is 4.

**Example 3 – Shorter Way to Approximate IROC**

Use Excel to compute an approximation for the Instantaneous Rate of Change (IROC) for the function \( g(x) = x^3 - 3x^2 \) at \( x = -2 \).

We could create the entire table as we did above, however, once you have clearly seen this idea play out, we can also simply find the ARC for a point very close to \((-2, g(-2))\) or \((-2, -20)\) but on the left, and then do the same for a point on the right.

**We will establish the practice of using a “gap” value with either 4 or 5 leading 0’s after the decimal (0.00001 or 0.000001) as being “close enough”.** We can see from our full table above that both values get us excellent approximations for the slope of the tangent line. We can create the table shown below using a gap of 0.000001:

<table>
<thead>
<tr>
<th></th>
<th>Approaching -2 from the left</th>
<th>Approaching -2 from the right</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>gap</td>
<td>( x )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>0.000001</td>
<td>-2.000001</td>
</tr>
</tbody>
</table>

This tells us that it seems that the limiting value for the ARC will be 24. We therefore conclude that the slope of the tangent line at \( x = -2 \) is 24. Looking at the graph near the point, we see this seems reasonable. If we increase the input by 1 from the point, the output along the tangent line seems to have increased by 24 units (which is how we would interpret the slope.)

**Related Exercises You Should Complete Now**

**Work on Exercise 2.1.6.** Remember you have the written answer key and videos in your course.
Example 4

Below is the graph of \( y = g(x) \). At what values of \( x \) does the graph of \( y = g(x) \) below have horizontal tangent lines?

The tangent lines to the graph of \( g(x) \) are horizontal (slope = 0) when \( x \approx -1, 1, 2.5, \) and 5. This tells us that the instantaneous rate of change at these inputs is 0. We can see that at each of these points, the curve is changing from decreasing to increasing, or vice versa.
Exercises for Unit 2 Section 1

Exercise 2.1.1  Do this!
The graph below is the graph of \( y = f(x) \). We want to find the slope of the tangent line at the point \( (1, 2) \).

1) First, draw the secant line between \( (1, 2) \) and \( (2, -1) \) and compute its slope.
2) Now draw the secant line between \( (1, 2) \) and \( (1.5, 1) \) and compute its slope.
3) Compare the two lines you have drawn. Which would be a better approximation of the tangent line to the curve at \( (1, 2) \)?
4) Now draw the secant line between \( (1, 2) \) and \( (1.3, 1.5) \) and compute its slope. Is this line an even better approximation of the tangent line?
5) Now draw your best guess for the tangent line and measure its slope. Do you see a pattern in the slopes?

You should have noticed that as the interval got smaller and smaller, the secant line got closer to the tangent line and its slope got closer to the slope of the tangent line. That’s good news – we know how to find the slope of a secant line.

Exercises 2.1.2: The number of AIDS cases diagnosed from 2000 through 2020 can be modeled as
\[
f(x) = 323,000(1.06)^x \text{ where } x \text{ is the number of years since 2000 and } f(x) \text{ is the number of AIDS cases.}
\]

(a) Use the model equation to calculate the average rate of change in AIDS cases from 2001 to 2015. Interpret this value in context using a complete sentence.
(b) Show how your answer to part a) can be interpreted on a graph of the function (sketch and show by hand or in Excel).

**Exercises 2.1.3:** The amount of surcharge for non-account holders at an ATM can be modeled as

\[ s(t) = 0.72(1.081)^t \]

where \( t \) is the number of years since 1995.

(a) Identify the average rate of change from \( t = 0 \) to \( t = 3 \). Write a complete sentence to interpret the meaning.

(b) Identify the average rate of change from \( t = 9 \) to \( t = 12 \). Write a complete sentence to interpret the meaning.

(c) **Describe how the average rate of change is changing.** Using a complete sentence, explain how that can be interpreted in real-world terms.
**Exercise 2.1.4:** Suppose \( f(x) = -x^2 + 6x + 3 \).

(a) Calculate the average rate of change of \( f(x) \) from \( x = 1 \) to \( x = 1.5 \).
- Show the function notation here.
- Calculate this ARC using Excel.
- Then sketch and label the graphical interpretation of your calculation on the graph below.

(b) Now use Excel to calculate the average rate of change of \( f(x) \) from \( x = 1 \) to \( x = 1.00001 \). Place your answer here.

(c) Use Excel to calculate the average rate of change of \( f(x) \) from \( x = 0.99999 \) to \( x = 1 \). Place your answer here.

(d) Using your answers from parts b and c above, the instantaneous rate of change of \( f(x) \) at \( x = 1 \) seems to be what value?

(e) Sketch a graphical representation of your answer to part (d) on the graph of \( f(x) \) below. Label the value from part (d) and the point @ \( x = 1 \).
**Exercise 2.1.5:** Suppose \( f(x) = -x^2 + 6x + 3 \).

(a) Calculate the average rate of change of \( f(x) \) from \( x = 6.2 \) to \( x = 7 \).
   - Show the function notation here.
   - Calculate this ARC using Excel.
   - Then sketch and label the graphical interpretation of your calculation on the graph below.

(b) Now use Excel to calculate the average rate of change of \( f(x) \) from \( x = 6.2 \) to \( x = 6.20001 \). Place your answer here.

(c) Use Excel to calculate the average rate of change of \( f(x) \) from \( x = 6.1999999 \) to \( x = 6.2 \). Place your answer here.

(d) Using your answers from parts b and c above, the instantaneous rate of change of \( f(x) \) at \( x = 6.2 \) seems to be what value?

(e) Sketch a graphical representation of your answer to part (d) on the graph of \( f(x) \) below. Label the value from part (d) and the point \( @ x = 6.2 \).
Exercise 2.1.6: The population of Park City, Utah can be modeled as

\[ s(t) = 0.0491t^3 - 6.0931t^2 + 155.87t + 3784 \]

where \( t \) is the number of years since 1870, and \( s(t) \) is the population of Park City.

(a) Use Excel to find a good estimate of the instantaneous rate of change of \( s(t) \) for \( t = 59 \). Place your answer here and then interpret the value you found from part (a) in real-world, contextual terms, using a complete sentence.

(b) First label the axes on the graph below in context. Then show a graphical representation of your answer to part (a) on a graph of the function \( s(t) \) below.

(c) Use Excel to find a good estimate of the instantaneous rate of change of \( s(t) \) for \( t = 88 \). Place your answer here, and then interpret the value in real-world, contextual terms, using a complete sentence. Show a graphical representation of your answer to part (c) on a graph of the function \( s(t) \) below.
Unit 2 Section 2: The Derivative - Definition of the Derivative at a Point

Unit 2 Section 2: Learning Outcomes

- Recognize and use the various synonyms for the derivative
  - Instantaneous rate of change
  - Slope of tangent line
  - Limit of average rates of change
  - Limit of the slope of secant lines

- Recognize and use derivative notation to represent the derivative of the function.
  For example, the derivative of a function f(t) at t = 3:
  - f’(3)
  - \( \frac{df}{dt} \) t = 3
  - “f prime at 3”
  - The derivative of f(t) at t = 3
  - The instantaneous rate of change of f(t) at t = 3
  - The slope of the tangent line at t = 3.

- Interpret the real-world, contextual meaning of the derivative (including appropriate units!) including understanding Marginal Revenue, Marginal Cost, and Marginal Profit.

Excel Outcome

- Find an estimated derivative value numerically (in Excel) using an average rate of change on a very small interval

Recall Previous Material: (Go back and review material in 2.1!)

The Average Rate of Change (ARC) of a function over a domain interval = Slope of a Secant line connecting the two associated points on the graph of the function.

The Instantaneous Rate of Change (Derivative) of a function at a particular domain value = Slope of a Tangent line to the graph of the function at the point of tangency.

Slope of Tangent Line \( \approx \) Slope of a Secant Line where the domain interval including the specific point of interest and a point very close by is very small, that is when the two points are very close.

The tangent line problem and the instantaneous rate of change problem we also considered in section 2.1 are the same problem. In each problem we wanted to know how rapidly something was changing at an instant in time, and the answer turned out to be finding the slope of a tangent line, which we approximated with the slope of a secant line. This idea is the key to defining the slope of a curve.
The Derivative at a Point:

**Definitions for Derivative**
- The **derivative** of a function $f$ at a point $(x, f(x))$ is the instantaneous rate of change.
- The **derivative** is the slope of the tangent line to the graph of $f(x)$ at the point $(x, f(x))$.
- The **derivative** is the slope of the curve $f(x)$ at the point $(x, f(x))$.
- A function is called **differentiable** at $(x, f(x))$ if its derivative exists at $(x, f(x))$.

**Notation for the Derivative:**
- The **derivative of $y = f(x)$ with respect to $x$** is written as $f'(x)$ (read aloud as “f prime of x”), or $y'$ (“y prime”)
  - or $\frac{dy}{dx}$ (read aloud as “dee why dee ex”), or $\frac{df}{dx}$
- The notation that resembles a fraction is called **Leibniz notation**. It displays not only the name of the function ($f$ or $y$), but also the name of the variable (in this case, $x$). It looks like a fraction because the derivative is a slope. In fact, this is simply $\frac{\Delta y}{\Delta x}$ written in Roman letters instead of Greek letters.

$$\frac{dR}{dp}\bigg|_{p=6.7}$$  The line next to the derivative notation means that you want to evaluate the derivative of the function $R(p)$ at the point 6.7

**Verb forms:**
- We **find the derivative** of a function, or **take the derivative** of a function, or **differentiate** a function.
- We use an adaptation of the $\frac{dy}{dx}$ notation to mean “find the derivative of $f(x)$:”
  $$\frac{d}{dx}(f(x)) = \frac{df}{dx}$$

**Practical Definition:**
- The derivative can typically be approximated by looking at an average rate of change, or the slope of a secant line, over a very tiny interval. The tinier the interval, the closer this is to the true instantaneous rate of change, slope of the tangent line, or slope of the curve.

**Looking Ahead:**
- We will have methods for computing exact values of derivatives from formulas soon. However, if the function is given to you as a table or graph, you will still need to work with a numerical approximation.

**Interpreting a Derivative Value at a Point:**
- It is impossible to interpret an **instantaneous** rate of change! Therefore, our convention is to interpret the value of the derivative to be **how the outputs would change over a 1 unit increase for the input.**
What does the sign of the derivative at a point mean?

Remember that when we found that an average rate of change was positive over an interval, it meant that, on average, the function was increasing over that interval.

We now know that the derivative can be very well approximated by considering the average rate of change between the point of interest and a point that is very close to the point of interest, or over some very small interval.

**IMPORTANT NOTE:** In Excel, we will typically use $a$ and $a + .000001$ as the inputs for computing our average rate of change, and recognize that the value we produce will be a very good approximation for the instantaneous rate of change or derivative at $x = a$.

Go back and review the work we did in the previous section. While we could work from both above and below the point of interest, and then average the two values we get for ARC, we see that the value we compute just using the $a + .000001$ seems to give us a good estimate on its own.

Also, since we interpret a derivative in terms of the input increasing, we will tend to choose our second point by using an input just to the right of the input of the point of interest – so by adding $.000001$. Since we are moving to the right with our inputs, we can see that a positive rate of change will mean that the function is increasing, and a negative rate of change will mean that that function is decreasing.

This matches what we said about our average rates of change!

- **If $f'(a)$ is positive, we know that $f(x)$ is increasing at $x = a$** for at least some small interval to the right since the tangent line will have a positive slope, so the curve will be moving up from left to right.

- **Likewise, if we know that $f(x)$ is increasing over an interval, we can say that the derivative values will be positive on that interval.** Remember, we always use intervals that do not include the endpoints.

- **If $f'(a)$ is negative, we know that $f$ is decreasing at $x = a$** for at least some small interval to the right since the tangent line will have a negative slope, so the curve will be moving down from left to right.

- **Likewise, if we know that $f(x)$ is decreasing over an interval, we can say that the derivative values will be negative on that interval.**

- **If $f'(a)$ is 0, we know that the tangent line will be horizontal.**

**Interpreting a Derivative Value:**

When we interpret a derivative value at a point, we are trying to grapple with interpreting an instantaneous rate of change. That is very difficult to wrap our minds around! What we do then is assume that the derivative tells us how the function values will change over the next unit increase in our inputs.
• If we see that \( f'(3) = 6 \), we will say that if we are at an input of 3, we expect the function outputs to increase by 6 for a 1 unit increase in the input.

• If we see that \( f'(5) = -2 \), we will interpret that to mean that if we are at an input of 5, we expect the function outputs to decrease by 2 for a 1 unit increase in the input.

Rates in Real Life
So far, we have emphasized the derivative as the slope of the line tangent to a graph. That interpretation is very visual and useful when examining the graph of a function, and we will continue to use it. Derivatives, however, are used in a wide variety of fields and applications, and some of these fields use other interpretations. The following are a few interpretations of the derivative that are commonly used.

General
• Rate of Change: \( f'(x) \) is the rate of change of the function at \( x \). If the units for \( x \) are years and the units for \( f(x) \) are people, then the units for \( \frac{df}{dx} \) are \( \frac{\text{people}}{\text{year}} \), a rate of change in population. If we found that \( f'(3) = 10 \), we would interpret that to mean that if we are at 3 years, we expect the number of people to increase by 10 over the next year.

Graphical
• Slope: \( f'(x) \) is the slope of the line tangent to the graph of \( f(x) \) at the point \((x, f(x))\).

Business
• Marginal Cost, Marginal Revenue, and Marginal Profit: We'll explore these terms in more depth later in the section.

• Basically, the marginal cost is approximately the additional cost of making one more object once we have already made \( x \) objects.

• If the units for \( x \) are bicycles and the units for \( f(x) \) are dollars, then the units for \( f'(x) = \frac{df}{dx} \) are \( \frac{\text{dollars}}{\text{bicycle}} \), the cost per bicycle.

In business contexts, the word "marginal" usually means the derivative or rate of change of some quantity.

One of the strengths of calculus is that it provides a unity and economy of ideas among diverse applications. The vocabulary and problems may be different, but the ideas and even the notations of calculus are still useful.

Business and Economics Terms
Suppose you are producing and selling some item. The profit you make is the amount of money you take in minus what you have to pay to produce the items. Both of these quantities depend on how many you make and sell. (So we have functions here.) Here is a list of
definitions for some of the terminology, together with their meaning in algebraic terms and in graphical terms.

Your cost is the money you have to spend to produce your items.

The Total Cost (C) for q items is the total cost of producing them. It’s the sum of the fixed cost and the total variable cost for producing q items.

The Marginal Cost (MC) at q items is the cost of producing the next item. Really, it’s

\[ MC(q) = C(q + 1) - C(q). \]

In many cases, though, it’s easier to approximate this difference using calculus (see Example below). And some sources define the marginal cost directly as the derivative,

\[ MC(q) = C'(q). \]

In this course, we will use both of these definitions. This is largely because when we interpret an instantaneous rate of change, it is difficult to describe it in practical terms.

Therefore, we interpret the derivative to be how we would expect the output to change if we were to change the input by one unit.

The units on marginal cost is cost per item.

**Why is it OK that are there two definitions for Marginal Cost (and Marginal Revenue, and Marginal Profit)?**

We have been using slopes of secant lines over tiny intervals to approximate derivatives. In this example, we’ll turn that around – we’ll use the derivative to approximate the slope of the secant line.

Notice that the “cost of the next item” definition is actually the slope of a secant line, over an interval of 1 unit:

\[ MC(q) = C(q + 1) - C(q) = \frac{C(q+1)-C(q)}{1} \]

If our input quantities are large, we can think of this as an approximation of the derivative of the cost function at q:

\[ MC(q) \approx C'(q) \]

In practice, these two numbers are so close that there’s no practical reason to make a distinction. For our purposes, the marginal cost is the derivative which is the cost of the next item.
Demand is the functional relationship between the price $p$ and the quantity $q$ that can be sold (that is demanded). Depending on your situation, you might think of $p$ as a function of $q$, or of $q$ as a function of $p$.

Your revenue is the amount of money you take in from selling your products. Revenue is price $\times$ quantity sold.

The Total Revenue ($R$) for $q$ items is the total amount of money you take in for selling $q$ items.

The Marginal Revenue (MR) at $q$ items is the cost of producing the next item,

$$MR(q) = R(q + 1) - R(q).$$

Just as with marginal cost, we will use both this definition and the derivative definition

$$MR(q) = R'(q).$$

The Profit ($P$) for $q$ items is $R(q) - C(q)$, the difference between total revenue and total costs.

The Marginal Profit at $q$ items is $P(q + 1) - P(q)$, or $P'(q)$.

Summary of Marginal Cost, Revenue and Profit as functions of $x$

Marginal Cost: $C'(x)$ gives the marginal cost of producing the $x + 1^{st}$ unit (when $x$ is units produced). $C'(x)$ is the additional cost added when producing the $x + 1^{st}$ unit.

Marginal Revenue: $R'(x)$ gives the marginal revenue of selling the $x + 1^{st}$ unit (when $x$ is units sold). $R'(x)$ is the additional revenue added when selling the $x + 1^{st}$ unit.

Marginal Profit: $P'(x)$ gives the marginal profit of producing and selling the $x + 1^{st}$ unit (when $x$ is units produced/sold). $P'(x)$ is the additional profit added when producing and selling the $x + 1^{st}$ unit.

Important Notes:

Positive Marginal Profit does NOT mean that the profit is positive!!

- The profit could be negative and increasing!
- That would mean the company profit is “in the red” but going up!
- So, the profit (value of the function) would be negative while the marginal profit (derivative) would be positive.
Negative Marginal Profit does NOT mean that the profit is negative!!!

- The profit could be positive and decreasing!
- That would mean the company profit is positive but going down!
- So, the profit (value of the function) would be positive while the marginal profit (derivative) would be negative.

<table>
<thead>
<tr>
<th>The function $f(x)$</th>
<th>The derivative $f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The revenue function is increasing.</td>
<td>The marginal revenue function is positive (the derivative of revenue is positive)</td>
</tr>
<tr>
<td>The revenue function is decreasing.</td>
<td>The marginal revenue function is negative (the derivative of revenue is negative)</td>
</tr>
<tr>
<td>The cost function is increasing.</td>
<td>The marginal cost function is positive (the derivative of cost is positive)</td>
</tr>
<tr>
<td>The cost function is decreasing.</td>
<td>The marginal cost function is negative (the derivative of cost is negative)</td>
</tr>
<tr>
<td>The profit function is increasing.</td>
<td>The marginal profit function is positive (the derivative of profit is positive)</td>
</tr>
<tr>
<td>The profit function is decreasing.</td>
<td>The marginal profit function is negative (the derivative of profit is negative)</td>
</tr>
</tbody>
</table>

**Related Exercises You Should Complete Now**

Work on Exercises 2.2.1 through 2.2.3 and 2.2.6, 2.2.7 and 2.2.8. Remember, you have written answers and videos available in your course.
Recall what the derivative tells us about the function:

- If \( f(x) \) is increasing over an interval, then \( f'(x) > 0 \) on that interval.
- If \( f(x) \) is decreasing over an interval, then \( f'(x) < 0 \) on that interval.
- If \( f(x) \) has a local maximum or minimum on an interval, then \( f'(x) = 0 \) at that point.

Related Exercises You Should Complete Now

Work on Exercises 2.2.4 and 2.2.5. Remember, you have written answers and videos available in your course.

What if the Derivative Doesn’t Exist?

A function is called differentiable at a point if its derivative exists at that point.

We’ve been acting as if derivatives exist everywhere for every function. This is true for most of the functions that you will run into in this class. But there are some common places where the derivative doesn’t exist.

Remember that the derivative is the slope of the tangent line to the curve. That’s what to think about.

Where can a slope not exist?

If the tangent line is vertical, the derivative will not exist.

Example 1

Show that \( f(x) = \sqrt[3]{x} = x^{1/3} \) is not differentiable at \( x = 0 \).

From the graph, we can see that the tangent line to this curve at \( x = 0 \) is vertical with undefined slope, which is why the derivative does not exist at \( x = 0 \).
Where can a tangent line not exist?

If there is a sharp corner (cusp) in the graph, the derivative will not exist at that point because there is no well-defined tangent line (a teetering tangent if you will). If there is a jump in the graph, the tangent line will be different on either side and the derivative can’t exist.

Example 2

Show that \( f(x) = |x| \) is not differentiable at \( x = 0 \).

On the left side of the graph, the slope of the line is -1. On the right side of the graph, the slope is +1. Since the average rates of change from both sides do NOT approach the same value, there is no well-defined tangent line at the sharp corner at \( x = 0 \), so the function is not differentiable at that point.
Exercises for Section 2.2

Exercise 2.2.1: Suppose that $P(q)$ is company profit when producing and selling items. In this case, $P(q)$ is profit in thousands of dollars, and $q$ is number of items sold.

(a) If $P(0) = -16.7$. Write a sentence to interpret the meaning of these numbers in context.

(b) Suppose $\frac{dP}{dq}|_{q=450} = 0.0072$. Write a sentence to interpret the meaning of these numbers in context.

(c) Suppose $\frac{dP}{dq}|_{q=972} = -0.0034$. Write a sentence to interpret the meaning of these numbers in context.

(d) Suppose that $P(620) = 958.6$ and $P'(620) = 0$. Write a sentence to interpret the meaning of these numbers in context.

(e) Sketch a possible graph of $P(q)$ based on the information above. Label all the points we know are on the graph.
Exercise 2.2.2: The function \( R(p) = 130,850p \cdot (0.92)^p \) gives the total revenue earned (in dollars) from selling widgets when the widgets are priced at \( p \) dollars each.

(a) Using Excel, find \( R(6.7) \), and numerically estimate \( \frac{dR}{dp}\big|_{p=6.7} \). Round your answers to 2 decimal places. Write a sentence to interpret the meaning of the number found in context.

(b) Using Excel, find \( R(28.2) \), and numerically estimate \( R'(28.2) \). Round your answers to 2 decimal places. Write a sentence to interpret the meaning of the number found in context.

Exercise 2.2.3: The average weekly sales for Abercrombie and Fitch between 2004 and 2008 are given below.

<table>
<thead>
<tr>
<th>year</th>
<th>thousand dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>38.87</td>
</tr>
<tr>
<td>2005</td>
<td>53.56</td>
</tr>
<tr>
<td>2006</td>
<td>63.81</td>
</tr>
<tr>
<td>2007</td>
<td>72.12</td>
</tr>
<tr>
<td>2008</td>
<td>68.08</td>
</tr>
</tbody>
</table>

(a) Using Excel regression, and one of the six functions we study in Business Calculus, find an appropriate model for the function where \( x \) is the number of years since 2000. (2004 = \( t = 4 \)). Name the function \( f(t) \).

(b) Using the function model you found in part a), use Excel to find \( f(7) \). Then use Excel to numerically estimate \( f'(7) \). Round appropriately to be accurate to give an answer to the nearest dollar. Write a sentence to interpret the meaning of the number found in context.
**Exercise 2.2.4**

Looking at the Derivative Graphically and some Review Concepts:

The graph of a function $f$ is given below.

(a) Draw short line segments at all points where the graph would have horizontal tangent lines.

(b) Where does $f$ have its largest value? Mark that point on the graph. What do you call this point?

(c) Where does $f$ have its smallest value? Mark that point on the graph. What do you call this point?

(d) What do you look for to figure out where $f'$ has its largest value? Mark that point (s) on the graph. What do you call this point?

(e) What do you look for to figure out where $f'$ has its smallest value? Mark that point(s) on the graph. What do you call this point?
Exercise 2.2.5

Given the graph of $f(x)$ below, place points A, B, C, D, E, F, and G on the graph. There is more than one correct answer that can be given for some points.

(a) Point A is a point on the curve where the derivative is negative.
(b) Point B is a point on the curve where the value of the function is negative.
(c) Point C is a point on the curve where the derivative is the largest.
(d) Point D is a point on the curve where the value of the function is zero.
(e) Point E is a point on the curve where the derivative is zero.
(f) Points F and G are two different points on the curve where the derivative is approximate the same value, but is NOT zero.
Exercise 2.2.6: Suppose that, for sales between 100 items and 900 items, the function below

\[ R(x) = -x^3 + 1,400x^2 - 450,000x + 100,000,000 \]

gives the revenue earned at a certain company (in dollars).

(a) Use Excel to numerically estimate the marginal revenue when they sell 107 items. Write a complete sentence to interpret the contextual meaning of this value.

(b) Use Excel to numerically estimate the marginal revenue when they sell 570 items. Write a complete sentence to interpret the contextual meaning of this value.

Exercises 2.2.7: Suppose the profit a company earns, in dollars, when selling \( x \) items is given by the function \( P(x) = 33,738 \ln(x) - 197,698 \)

Use Excel to find the profit when selling 200 items, and numerically estimate the marginal profit when selling 200 items. Write complete sentences to interpret the contextual meaning of these values.
Exercise 2.2.8: Answer the following questions related to revenue, cost, marginal revenue, and marginal cost.

(a) Suppose cost is higher than revenue. What do you know about the profit? Explain your answer.

(b) Suppose marginal cost is higher than marginal revenue. What do you know about the profit? Explain your answer.

(c) Suppose, when selling 137 items, the revenue is higher than costs and marginal revenue is lower than marginal costs. What do you know about the profit? Explain your answer.
Unit 2 Section 3: Derivative As A Function

Unit 2 Section 2: Learning Outcomes:
- When given a graph of $f(x)$ or information about $f(x)$:
  - correctly identify where $f'(x) = 0$ and where $f'(x)$ is undefined (critical points)
  - correctly identify the sign of $f'(x)$ and how this relates to $f(x)$
  - correctly identify if $f'(x)$ is increasing/decreasing and how this relates to $f(x)$
  - correctly identify where $f'(x)$ has maximum/minimum points and how this relates to $f(x)$
  - sketch a well-labeled graph of $f'(x)$ including appropriate meaning of both axis (including units)
- Vocabulary: A critical point on the function $f(x)$ is a point, $x = a$, on the function $f(x)$ where:
  - $f'(a) = 0$ or where $f'(x)$ is undefined at $x = a$.
- Understand how sharp points on $f(x)$ or points where $f(x)$ is vertical for an instant affect $f'(x)$.
- Interpret real-world, contextual meaning of points on the graph of $f'(x)$, including "important" points like x-intercepts, maximum/minimum points, and inflection points.
- Interpret real-world, contextual meaning of intervals on the graph of $f'(x)$

Excel Outcomes:
- a) Use Excel to numerically estimate a derivative function with a table and graph

Recall From Earlier Sections
The Derivative of a curve at a point is:
- b) Instantaneous rate of change
- c) Slope of tangent line
- d) Slope of the curve

A function is called differentiable at $(x, f(x))$ if the derivative exists at this point.

Inflection Points: An inflection point is a point on the graph of a function where the concavity of the function changes, from concave up to down or from concave down to up.

The Derivative as a Function
We now know how to find (or at least approximate) the derivative of a function for any x-value; this means we can think of the derivative as a function, too. The inputs are the same x’s; the output is the value of the derivative at that x value.
Concavity and First Derivative

Remember, we discussed concavity of a function in section 1.2. You should go back and review that information now.

While we were talking about rates of change in that section, and hadn’t yet discussed the derivative, we can add that piece in now!

a) If the derivative is increasing on an interval, that is the value of the derivative is moving to the right on a number line, then the original function is concave up over that interval. \( f'(x) \uparrow, f(x) \cup \)

b) If the derivative is decreasing on an interval, that is the value of the derivative is moving to the left on a number line, then the original function is concave down over that interval. \( f'(x) \downarrow, f(x) \cap \)

Let’s put all of this together to consider the graph of a function and determine what we can say about the derivative of that function.

First Derivative Information About Shape (Part 1)

You must be extremely well versed with the following information! Please make sure that you understand why each of the statements is true!

Summary of First Derivative Information about Shape - Part 1

For a function \( f(x) \) which is differentiable on an interval \( (a, b) \):

(a) If \( f(x) \) is increasing on \( (a, b) \), then \( f'(x) \geq 0 \) for all \( x \) in \( (a, b) \).
(b) If \( f(x) \) is decreasing on \( (a, b) \), then \( f'(x) \leq 0 \) for all \( x \) in \( (a, b) \).
(c) If \( f(x) \) is constant (horizontal) on \( (a, b) \), then \( f'(x) = 0 \) for all \( x \) in \( (a, b) \).
(d) If \( f(x) \) is linear (non-horizontal) on \( (a, b) \), then \( f'(x) = \text{a constant} \) for all \( x \) in \( (a, b) \).
(e) If \( f(x) \) is concave down on \( (a, b) \), then \( f'(x) \) must be decreasing.
(f) If \( f(x) \) is concave up on \( (a, b) \), then \( f'(x) \) must be increasing.
(g) If \( f(x) \) is linear on \( (a, b) \), then there is no concavity and \( f'(x) \) is constant for all \( x \) in \( (a, b) \).

The next two examples examine some of the interplay between the shape of the graph of \( f(x) \) and the behavior of \( f'(x) \). If we have a graph of \( f(x) \), we will see what we can conclude about the derivative, \( f'(x) \). Before we begin to work with those examples, let’s set up a system that we can use to approach this work.
Interplay Between a Graph of a Function and the Derivative

The 5 steps below will help ensure that you have considered the full scope of ideas

<table>
<thead>
<tr>
<th>Steps to Analyze a Function to Help Determine Derivative Behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step 1:</strong> Consider intervals for any Linear Parts if applicable.</td>
</tr>
<tr>
<td><strong>Step 2:</strong> Consider input values where the function turns. <strong>NOTE:</strong> If the function has a sharp turning point, the derivative will not exist at that input value. If the function has a smooth turning point, the derivative will be 0 at that input value.</td>
</tr>
<tr>
<td><strong>Step 3:</strong> Use the input values from Steps 1 and 2 to consider intervals where the function is Increasing, Decreasing or constant. Keep in mind that you want to cover all intervals for the domain of the function.</td>
</tr>
<tr>
<td><strong>Step 4:</strong> Consider input values in the domain where the concavity changes. <strong>NOTE:</strong> If the concavity changes to no concavity at a point, there will not be a point of inflection at that input value. Inflection points only occur where the function changes from concave up to down or vice versa.</td>
</tr>
<tr>
<td><strong>Step 5:</strong> Use these input values to consider intervals for concavity. Keep in mind that you want to cover all intervals for the graph.</td>
</tr>
</tbody>
</table>

**Example 1**

Below is the graph of a function $y = f(x)$. Assume that the graph is a curve over the entire domain. Also assume that the graph shown displays all critical behavior of the function, so no further turning points occur. We can use the information in the graph to fill in a table showing values of $f'(x)$:

<table>
<thead>
<tr>
<th>Domain Interval or Values</th>
<th>Behavior of $f(x)$</th>
<th>What this tells us about derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step 1:</strong> Consider any Linear Parts if applicable</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td><strong>Step 2:</strong> Consider input values where the function has a turning point.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Looking at the graph, we want to consider what information we can distill about the derivative function.

We will always consider this in steps. DO NOT try to do this all at once. You need to consider things in a systematic way.
**Step 3:**
Use these input values to consider intervals where the function is Increasing or Decreasing.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Increasing</th>
<th>Decreasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 1))</td>
<td>Derivative is positive – Graph of derivative is above the x axis.</td>
<td></td>
</tr>
<tr>
<td>((1, 3))</td>
<td>Derivative is negative – Graph of derivative is below the x axis.</td>
<td></td>
</tr>
<tr>
<td>((3, \infty))</td>
<td>Derivative is positive – Graph of derivative is above the x axis.</td>
<td></td>
</tr>
</tbody>
</table>

**Step 4:**
Consider input values where the concavity changes

<table>
<thead>
<tr>
<th>Input Values</th>
<th>Concavity</th>
<th>Derivative Behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x = 2) and (3.5)</td>
<td>Points of Inflection</td>
<td>Derivative will turn.</td>
</tr>
</tbody>
</table>

**Step 5:**
Use these input values to consider intervals for concavity

<table>
<thead>
<tr>
<th>Interval</th>
<th>Concavity</th>
<th>Derivative Behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 2))</td>
<td>Concave down</td>
<td>Derivative is decreasing</td>
</tr>
<tr>
<td>((2, 3.5))</td>
<td>Concave up</td>
<td>Derivative is increasing</td>
</tr>
<tr>
<td>((3.5, \infty))</td>
<td>Concave Down</td>
<td>Derivative is decreasing</td>
</tr>
</tbody>
</table>

Based on this information, we can sketch a **rough** picture of what we think the derivative looks like.

**Related Exercises You Should Complete Now**

Work on exercises 2.3.1, 2.3.2, and 2.3.3. Remember, you have written solutions and videos for these exercises in your course.

**Sketching a Graph of the Derivative From Information About a Function**

You can use the steps below to help you use the information from the function to sketch a graph of the derivative function.

**Sketching the Graph of a Derivative From Information About a Function**

**Step 1:** Put a point on the x axis at any point where the derivative is 0.

**Step 2:** Use these points to break the domain into subintervals. Make a mark above the x axis if the derivative is positive on that interval, and a mark below the x axis is the derivative is negative on that interval. This is just to help guide you as you draw the curve.

**Step 3:** Use the input values where concavity changes and draw a vertical dotted line at this point. If the curve changed from concave up to down or vice versa, then somewhere along this line, the derivative will turn since the concavity changed. **SPECIAL NOTE:** You will NOT have a point of inflection if the graph changes to a linear segment at a point. Points of inflection ONLY occur where the curve changes from concave up to concave down or vice versa.
Step 4: Use information about concavity to sketch the derivative graph as increasing, decreasing or constant.

Following this approach, we could create the rough sketch of the graph shown below. While we can approximate the values of the derivative at various points to help create an appropriate vertical scale, we will oftentimes skip that part, instead focusing on the key features of the derivative without worrying about the vertical scale.

Example 2

The graph shows the height of a helicopter during a period of time. We will only consider the domain of \([0, 7]\). Sketch the graph of the upward velocity of the helicopter, \(\frac{dh}{dt} = v(t)\). Keep in mind that \(v(t) = h'(t)\)

YOU TRY: Try this on your own before you look at the information in the table!

<table>
<thead>
<tr>
<th>Domain Interval or Values</th>
<th>Behavior of (f(x))</th>
<th>What this tells us about derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1: Consider any linear Parts if applicable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>None</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Step 2: Consider input values where the function turns</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x = 2, 3) and 5</td>
<td>Turning points</td>
<td>Derivative is 0 – Graph of derivative crosses the x axis.</td>
</tr>
<tr>
<td>Step 3: Use these input values to consider intervals where the function is Increasing or Decreasing.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((0, 2))</td>
<td>Increasing</td>
<td>Derivative is positive – Graph of derivative is above the x axis.</td>
</tr>
<tr>
<td>((2, 3))</td>
<td>Decreasing</td>
<td>Derivative is negative – Graph of derivative is below the x axis.</td>
</tr>
</tbody>
</table>
### Step 4: Consider input values where the concavity changes

<table>
<thead>
<tr>
<th>x = 1, 2.5, 4 and 6</th>
<th>Points of Inflection</th>
<th>Derivative is turning at these points since the curve changes from concave up to down or vice versa.</th>
</tr>
</thead>
</table>

### Step 5: Use these input values to consider intervals for concavity

<table>
<thead>
<tr>
<th>Interval</th>
<th>Concavity</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 1)</td>
<td>Concave up</td>
<td>Increasing</td>
</tr>
<tr>
<td>(1, 2.5)</td>
<td>Concave down</td>
<td>Decreasing</td>
</tr>
<tr>
<td>(2.5, 4)</td>
<td>Concave up</td>
<td>Increasing</td>
</tr>
<tr>
<td>(4, 6)</td>
<td>Concave down</td>
<td>Decreasing</td>
</tr>
<tr>
<td>(6, 7)</td>
<td>Concave up</td>
<td>Increasing</td>
</tr>
</tbody>
</table>

Using this information, we can create a rough sketch of a graph of \( v(t) = \frac{dh}{dt} = h'(t) \).

![Graph showing concavity](image.png)

### Differentiability:

Most of the time, in Business Calculus, we will be considering functions which have derivatives that are defined at all points of the domain we are considering. But it should be noted that it is possible for a function to be continuous but have points at which the derivative is undefined.

1) To be differentiable, a function must be continuous and smooth. A function will NOT have a derivative at certain points on a curve. Both continuity and differentiability are desirable properties for a function to have.

2) If \( f(x) \) has a derivative at a point \( x = a \), then \( f(x) \) must be continuous at \( x = a \).

3) Since a function must be continuous to have a derivative, if it has a derivative then it is continuous.

4) However, the converse is false; that is, there are functions that are continuous but not differentiable.
Derivatives will fail to exist at:

- corner
- cusp
- vertical tangent
- discontinuity

**Definition:**

A point in the domain of a function $f(x)$ at which the derivative is equal to zero ($f'(x) = 0$), or does not exist ($f'(x) DNE$), is called a **critical point** of the function.

**Note:** Maximum and minimum points in the **interior** of a function always occur at critical points, but critical points are not always maximum or minimum values.

**Critical points are not always extremes!**

- $y = x^3$
- $y = x^{1/3}$

- $f' = 0$ (not an extreme)
- $f'$ is undefined. (not an extreme)

**Related Exercises You Should Complete Now**

**Work on exercises 2.3.14 and 2.3.15.** Remember, you have written solutions and videos for these exercises in your course.

**Example 3**

Shown is the graph of the height $h(t)$ of a rocket at time $t$. Sketch the graph of the **velocity** of the rocket at time $t$. (Velocity is the name for the **derivative** of the height function, so it
is the slope of the tangent to the graph of position or height.) We will only consider this function on the domain from [0, 10].

![Graph of height vs. time]

Again, we can consider the behavior of the function over intervals.

Notice from above that rather than trying to do too many things at once, we approach this very systematically, making it very clear what we are considering at each step.

<table>
<thead>
<tr>
<th>Domain Interval or Values</th>
<th>Behavior of f(x)</th>
<th>What this tells us about derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step 1:</strong> Consider any linear Parts if applicable</td>
<td>(8,10)</td>
<td>Linear</td>
</tr>
<tr>
<td><strong>Step 2:</strong> Consider input values where the function turns</td>
<td>x = 5</td>
<td>Smooth turning point</td>
</tr>
<tr>
<td><strong>Step 3:</strong> Use these input values to consider intervals where the function is Increasing or Decreasing.</td>
<td>(0, 5)</td>
<td>Increasing</td>
</tr>
<tr>
<td></td>
<td>(5, 10)</td>
<td>Decreasing</td>
</tr>
<tr>
<td><strong>Step 4:</strong> Consider input values where the concavity changes</td>
<td>x = 8</td>
<td>( f(x) ) changes from curve to linear.</td>
</tr>
<tr>
<td><strong>Step 5:</strong> Use these input values to consider intervals for concavity</td>
<td>(0,8)</td>
<td>Concave down</td>
</tr>
<tr>
<td></td>
<td>(8, 10)</td>
<td>Neither concave up or down since function is linear.</td>
</tr>
</tbody>
</table>

Again, we use this information and the steps indicated previously to sketch a graph of the derivative. Notice that we did not include a vertical scale on the graph of the derivative, but we did indicate the units!

The lower graph below shows the velocity of the rocket. This is \( v(t) = h'(t) \).
Example 4

List the intervals on which the function shown is increasing or decreasing and state what this tells you about the derivative on those intervals.

**YOU TRY:** Try this on your own before checking with the table shown.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Behavior of f(x)</th>
<th>What this tells us about derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step 1:</strong> Consider any linear Parts if applicable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3, 4)</td>
<td>Linear</td>
<td>Derivative is constant, and since the function is constant (graph is horizontal) the derivative is 0 for this interval.</td>
</tr>
<tr>
<td><strong>Step 2:</strong> Consider input values where the function turns</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x = .5, x = 2, x = 6</td>
<td>Smooth turning points</td>
<td>Derivative is 0</td>
</tr>
<tr>
<td><strong>Step 3:</strong> Use the input values from Step 1 and 2 to consider intervals where the function is Increasing or Decreasing or Constant</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, .5)</td>
<td>Increasing</td>
<td>Derivative is positive – Graph of the derivative is above the x axis.</td>
</tr>
<tr>
<td>(.5, 2)</td>
<td>Decreasing</td>
<td>Derivative is negative – Graph of the derivative is below the x axis.</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>Increasing</td>
<td>Derivative is positive – Graph of the derivative is above the x axis.</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>Constant</td>
<td>Derivative is 0 – Graph of the derivative runs along the x axis.</td>
</tr>
</tbody>
</table>
(4, 6) | Increasing | Derivative is positive – Graph of the derivative is above the x axis.
---|---|---
(6, 8) | Decreasing | Derivative is negative – Graph of the derivative is below the x axis.

### Step 4: Consider input values where the concavity changes

<table>
<thead>
<tr>
<th>x = 1, x = 3, x = 4</th>
<th>At x = 1, we have a point of inflection.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>At 3 and 4 the concavity does not change from down to up or vice versa. These are NOT points of inflection.</td>
</tr>
<tr>
<td></td>
<td>Derivative turns at x = 1.</td>
</tr>
<tr>
<td></td>
<td>Derivative is undefined at x = 3 and x = 4. There will be a jump in the derivative graph at these points.</td>
</tr>
</tbody>
</table>

### Step 5: Use these input values to consider intervals for concavity

<table>
<thead>
<tr>
<th>(0, 1)</th>
<th>Concave down</th>
<th>Derivative is decreasing.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 3)</td>
<td>Concave up</td>
<td>Derivative is increasing.</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>Neither concave up or concave down</td>
<td>Notice that the slope just to the left of x = 3 is larger than the slope just to the right of x = 3. Since the function is linear on this interval, the derivative is constant. Since the graph is horizontal, the derivative is 0</td>
</tr>
<tr>
<td>(4, 8)</td>
<td>Concave down</td>
<td>Derivative is decreasing. Notice that the slope just to the left of x = 4 is 0 and the slope just to the right of x = 4 is larger.</td>
</tr>
</tbody>
</table>

**On your own:** You should try your hand at sketching a graph of the derivative of the function. Check with a classmate, or your teacher to see if you agree with what the derivative should look like.

**Related Exercises You Should Complete Now**

**Work on exercises 2.3.4 through 2.3.10.** Remember, you have written solutions and videos for these exercises in your course.
First Derivative Information About Shape (part 2)

The next theorem is almost the converse of the First Shape Theorem and explains the relationship between the values of the derivative and the graph of a function from a different perspective.

It says that if we know something about the values of \( f'(x) \), then we can draw some conclusions about the shape of the graph of \( f(x) \).

**First Derivative Information about Shape (part 2)**

For a function \( f(x) \) which is differentiable on an interval \( I \);

(a) If \( f'(x) > 0 \) for all \( x \) in the interval \( I \), then \( f(x) \) is increasing on \( I \).

(b) If \( f'(x) < 0 \) for all \( x \) in the interval \( I \), then \( f(x) \) is decreasing on \( I \).

(c) If \( f'(x) = 0 \) for all \( x \) in the interval \( I \), then \( f(x) \) is constant on \( I \).

(d) If \( f'(x) = \text{a nonzero constant} \) for all \( x \) in the interval \( I \), then \( f(x) \) is Linear on \( I \).

(e) If \( f'(x) \) is increasing for all \( x \) in the interval \( I \), then \( f(x) \) is concave up on \( I \).

(f) If \( f'(x) \) is decreasing for all \( x \) in the interval \( I \), then \( f(x) \) is concave down on \( I \).

Again, we want a systematic way to consider a graph of a derivative or table of derivative values \( (f'(x)) \) to make statements about the function itself \( f(x) \).

**Sketch a Function Using Information About the Derivative Given as a Table or Graph**

Again, just like when we went from \( f(x) \) to \( f'(x) \), we want a systematic way to consider information about \( f'(x) \) which is given either in table form, or as a graph to discuss key behavior of \( f(x) \).

Following the steps below will help you move through identifying the distinct features that you need about the derivative function \( f'(x) \) in order to graph a function \( f(x) \).

**Steps to Create a Graph of \( f(x) \) given information about \( f'(x) \)**

- **Step 1**: Find input values where the Derivative = 0 or is undefined.
- **Step 2**: Use these values to break the domain axis into intervals.
- **Step 3**: Determine whether the derivative output values are positive or negative on each of these intervals. This tells you whether the original function is increasing or decreasing.
- **Step 4**: Determine where the derivative has turning points and use these points to break the domain axis into intervals.
- **Step 5**: Determine whether the derivative output values are decreasing or increasing in these intervals. This tells you whether the original function is concave up or concave down, and helps you determine where points of inflection occur.
Example 5

Use information about the values of \( f' \) to help graph \( f(x) \).

\[ f' = 3x^2 - 12x + 9 = 3(x - 1)(x - 3) \]

Let’s first create a table of values on the interval from -3 to 8 and graph this function in Excel. NOTE: We want to use enough values in our table so that all the key behavior of the function is shown. We know that the function is quadratic, so the graph will be a parabola. We expect to see a graph that is opening up, with one turning point.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>72</td>
</tr>
<tr>
<td>-2</td>
<td>45</td>
</tr>
<tr>
<td>-1</td>
<td>24</td>
</tr>
<tr>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>24</td>
</tr>
<tr>
<td>6</td>
<td>45</td>
</tr>
<tr>
<td>7</td>
<td>72</td>
</tr>
<tr>
<td>8</td>
<td>105</td>
</tr>
</tbody>
</table>

Now let’s follow the steps above to help determine key behavior of \( f(x) \).

**Step 1:** Find input values where the Derivative = 0 or is undefined.

We can see that \( f' = 0 \) only when \( x = 1 \) or \( x = 3 \). \( f' \) is a polynomial so it is always defined.

**Step 2:** Use these values to break the domain axis into intervals.

The only critical numbers for \( f(x) \) are \( x = 1 \) and \( x = 3 \), and they divide the real number line into three intervals: \((-\infty, 1), (1,3) \) and \((3, \infty)\).

**Step 3:** Determine whether the derivative is positive or negative on each of these intervals. This tells you whether the original function is increasing or decreasing.

**Step 4:** Determine where the derivative has turning points and use these points to break the domain axis into intervals.

We can easily see from the graph that the derivative function seems to turn at \( x = 2 \). That means we will want to consider what happens for \( x < 2 \) or the interval \((-\infty, 2)\) and \( x > 2 \) or the interval \((2, \infty)\).

**IMPORTANT NOTE:** If we didn’t have a graph, we can also glean this information from the table, as we see that the output values are decreasing until 2, and then increasing after that.
**Step 5:** Determine whether the derivative output values are decreasing or increasing in these intervals. This tells you whether the original function is concave up or concave down, and helps you determine where points of inflection occur.

<table>
<thead>
<tr>
<th>Where is the Derivative 0 or Undefined?</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>The Derivative is 0 at input values of 1 and 3</td>
<td></td>
</tr>
<tr>
<td><strong>Interval of input values</strong></td>
<td><strong>Information about $f'(x)$</strong></td>
</tr>
<tr>
<td>For $x &lt; 1$ or the interval $(-\infty, 1)$</td>
<td>The output values of the derivative function are positive. The table shows positive outputs. The graph is above the x axis.</td>
</tr>
<tr>
<td>For $1 &lt; x &lt; 3$ or the interval $(1, 3)$</td>
<td>The output values of the derivative are negative. The table shows negative outputs. The graph is below the x axis.</td>
</tr>
<tr>
<td>For $x &gt; 3$ or the interval $(3, \infty)$</td>
<td>The output values of the derivative are positive. The table shows positive outputs. The graph is above the x axis.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Where does the derivative have turning points?</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>The derivative turns at $x = 2$.</td>
<td></td>
</tr>
<tr>
<td><strong>Interval of input values</strong></td>
<td><strong>Information about $f'(x)$</strong></td>
</tr>
<tr>
<td>For $x &lt; 2$ or the interval $(-\infty, 2)$</td>
<td>Both the graph and the table of values show that the derivative output values are decreasing</td>
</tr>
<tr>
<td>For $x &gt; 2$ or the interval $(2, \infty)$</td>
<td>Both the graph and the table of values show that the derivative output values are increasing.</td>
</tr>
<tr>
<td>At $x = 2$</td>
<td>Both the graph and the table of values show that the derivative output values change from decreasing to increasing at $x = 2$.</td>
</tr>
</tbody>
</table>

Even though we don't know the value of $f(x)$ anywhere yet, we do know a lot about the shape of the graph of $f(x)$:

As we move from left to right along the x–axis, the graph of $f(x)$ increases until $x = 1$, then the graph decreases until $x = 3$, and then the graph increases again.

This also tells us that we have a local maximum at $x = 1$, and a local minimum at $x = 3$. The graph of $f(x)$ is concave down until $x = 2$, and then becomes concave up after that.
This makes a point of inflection at \( x = 2 \).

**Notice:** We do NOT know where \( f(x) \) is positive or negative!

That means that there are many possibilities for \( f(x) \), with just where they exist vertically being the deciding feature of a particular function.

If we knew some other information about \( f(x) \) we could graph a particular function. Let’s suppose that we know that \( f(1) = 5 \), and \((1,5)\) is a local maximum of \( f(x) \). Suppose we also know that \( f(3) = 1 \), and \((3,1)\) is a local minimum of \( f(x) \). The resulting graph of \( f(x) \) is shown here.

If we did not know that information, we could shift the graph above up and down into an infinite number of vertical positions. The SHAPE of the graph will not change, however where it is situated vertically can.

---

**Related Exercises You Should Complete Now**

**Work on exercises 2.3.11 through 2.3.13 and 2.3.16.** Remember, you have written solutions and videos for these exercises in your course.
Key Quick Reference Information

YOU MUST KNOW THESE BASIC FACTS QUICKLY AND EASILY!!!
PRACTICE, PRACTICE, PRACTICE!!!

When:
- $f(x)$ is always constant (flat) $\iff f'(x) = 0$.
- When $f(x)$ is horizontal, $f'(x)$ function (curve) is on $x$–axis.
- $f(x)$ flat for an INSTANT at $x = a \iff f'(x) = 0$ at that INSTANT ($x = a$ is critical point on $f(x)$). This may or may not be a turning point!
- $f(x)$ has a sharp transition at $x = a \iff f'(x)$ is undefined at $x = a$ ($x = a$ is critical point on $f(x)$).
- $f(x)$ increasing $\iff f'(x)$ positive. $f'(x)$ function (curve) is above $x$–axis.
- $f(x)$ decreasing $\iff f'(x)$ negative. $f'(x)$ function (curve) below $x$–axis.
- $f(x)$ concave up $\iff f'(x)$ increasing. $f'(x)$ function (curve) is going up.
- $f(x)$ concave down $\iff f'(x)$ decreasing. $f'(x)$ function (curve) is going down.
- $f(x)$ inflection point $\iff f'(x)$ function (curve) changes from increasing to decreasing (or vice versa).

Related Exercises You Should Complete Now

Work on exercises 2.3.17 through 2.3.20. Remember, you have written solutions and videos for these exercises in your course.
Exercises for Unit 2 Section 3

Exercise 2.3.1: Consider the graph below.

Identify the interval(s) where the function has a positive derivative.

- Identify the interval(s) where the function has a negative derivative.

- Identify where the function has a derivative value of 0.

- Identify the intervals(s) where the derivative is increasing (the derivative value is moving to the right on a number line).

- Identify the interval(s) where the derivative is decreasing (the derivative value is moving to the left on a number line).
Exercise 2.3.2: Given the function $g(x)$, below and your understanding of the relationship between a function and its derivative, fill in the table below as directed. Using complete sentences, explain your answers.

Fill in the table with “0”, “+” or “−”

<table>
<thead>
<tr>
<th>x</th>
<th>$g(x)$</th>
<th>$g'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Explain how you know these are true.

Explanation:
**Exercise 2.3.3:** Fill in the table below.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$ is concave up</td>
<td></td>
</tr>
<tr>
<td>$f(x)$ is decreasing</td>
<td></td>
</tr>
<tr>
<td>$f(x)$ is increasing and concave down</td>
<td></td>
</tr>
<tr>
<td>$f(x)$ is concave down</td>
<td>$f'(x)$ is positive</td>
</tr>
<tr>
<td></td>
<td>$f'(x)$ is increasing</td>
</tr>
<tr>
<td></td>
<td>$f'(x)$ is negative and increasing</td>
</tr>
</tbody>
</table>

**Exercise 2.3.4:** First explain what you know about the derivative of the function shown below. Then sketch a well-labeled graph of the derivative function, $f'(x)$, for the function $f(x)$ given below. Identify any critical points of the function $f(x)$. 

![Graph of function f(x)](image-url)
Exercise 2.3.5: First explain what you know about the derivative of the function shown by the graph below. Then sketch a well-labeled graph of the derivative function, $f'(x)$, for the function $f(x)$ given below. Identify any critical points of the function $f(x)$. The position of key points must be clear.

![Graph of f(x)](image)

Exercise 2.3.6: First state what you know about the derivative based on the function shown below. Then sketch a well-labeled graph of the derivative function, $f'(x)$. Identify any critical points of the function $f(x)$. The position of key points must be clear.

![Graph of f(x)](image)
**Exercise 2.3.7:** First state what you know about the derivative based on the graph of the function shown below. Then sketch a well-labeled graph of the derivative function, $f'(x)$, for the function $f(x)$ shown. Identify any critical points of the function $f(x)$. The position of key points must be clear.

**Exercise 2.3.8:** First state what you know about the derivative based on the graph of the function shown below. Then sketch a well-labeled graph of the derivative function, $f'(x)$, for the function $f(x)$ shown. Identify any critical points of the function $f(x)$. The position of key points must be clear.
**Exercise 2.3.9:** First state what you know about the derivative of the function shown in the graph below. Then sketch a well-labeled graph of the derivative function, $f'(x)$, for the function $f(x)$ shown. Identify any critical points of the function $f(x)$. The position of key points must be clear.

**Exercise 2.3.10:** The graph below shows the profit function for a company where $x$ is the dollars spent on advertising, and $f(x)$ is the profit earned in dollars. First state what you know about the derivative based on the graph shown. Then sketch a graph of the rate of change graph (the derivative graph). Label all important points to the best of your ability from the information given and label the meaning of both axis with units.
**Exercise 2.3.11:** The table below gives some value of the derivative of $f'(x)$. Use the given table of derivative values to make conclusions about $f(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>15246</td>
</tr>
<tr>
<td>1</td>
<td>4800</td>
</tr>
<tr>
<td>2</td>
<td>810</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>294</td>
</tr>
<tr>
<td>5</td>
<td>576</td>
</tr>
<tr>
<td>6</td>
<td>450</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>-450</td>
</tr>
<tr>
<td>9</td>
<td>-576</td>
</tr>
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<td>10</td>
<td>-294</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
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<tr>
<td>12</td>
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</tr>
<tr>
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<td>-4800</td>
</tr>
<tr>
<td>14</td>
<td>-15246</td>
</tr>
<tr>
<td>15</td>
<td>-36864</td>
</tr>
</tbody>
</table>
Numerically Estimating the Derivative Function With the Help of Excel:

Use Excel to create approximate derivative values (i.e. the slope from the x-value to a VERY CLOSE x-value)

**Exercise 2.3.12:** Suppose \( f(x) = 2x^2 - 5x + 7 \)

(a) Graph the function \( f(x) \) in Excel from \( x = -5 \) to \( x = 5 \). *Sketch* the graph below.

(b) *Sketch* a graph of what you think \( f'(x) \) should look like for the graph you sketched above.

(c) Use Excel to find numerical estimates for \( f'(x) \) from \( x = -5 \) to \( x = 5 \). (Use Excel to numerically determine the derivative in this interval). Then have Excel sketch a graph of \( f'(x) \) using your numerical estimate values. Check that the graph you created looks like the graph in part (b)!
**Exercise 2.3.13:** Suppose you are given the function:

\[
g(t) = \frac{50}{1 + 0.05e^{0.65t}}
\]

(a) Graph the function \(g(t)\) in Excel from \(t = -5\) to \(t = 15\). Sketch the graph below.

(b) Sketch a graph of what you think \(g'(t)\) should look like on or near the graph you sketched above.

(c) Use Excel to find numerical estimates for \(g'(t)\) from \(t = -5\) to \(t = 15\). Then have Excel sketch a graph of \(g'(t)\) using your numerical estimate values. Check that the graph you created looks like the graph in part (b)!
Exercise 2.3.14: The function $f(x) = |x|$ is a continuous function which has a sharp point at $x = 0$.

Sketch $f(x)$ below, and then think about what $f'(x)$ would look like for this function.

What would be the value of $f'(x)$ be at $x = 0$?

Exercise 2.3.15: The function $f(x) = \sqrt[3]{x}$ is a continuous function which is vertical for an instant at $x = 0$. Sketch $f(x)$ below, and then think about what $f'(x)$ would look like for this function. What would be the value of $f'(x)$ at $x = 0$?
Exercise 2.3.16: The graph shown represents the rate of change of a function \( f \) with respect to \( x \) on the domain interval \([-5, 12]\). Use interval notation in your answers. Explain using complete sentences. [Note: the graph is a graph of the derivative, \( f'(x) \).]

- On what interval(s) of \( x \) is \( f(x) \) increasing? How do you know?

- On what interval(s) of \( x \) is \( f(x) \) decreasing? How do you know?

- On what interval(s) of \( x \) is the graph of \( f(x) \) concave up? How do you know?

- At what values of \( x \) (approximately) does \( f(x) \) have point(s) of inflection? How do you know?

- Where (x-values) does \( f(x) \) have a local minimum or local maximum values? Which locate minimums; which locate maximums? Why?
Exercise 2.3.17: Explain your reasoning.

Which of the graphs below could be the graph of the derivative of the graph in Figure 1?

![Graphs A, B, C]
Exercise 2.3.18: Explain your reasoning.

Match the graphs of the three functions below with the graphs of their derivatives.

Exercise 2.3.19: Explain your reasoning.

Below are six graphs, three of which are derivatives of the other three. Match the functions with their derivatives.
Exercise 2.3.20
More Practice with graphs and the derivative

Match the function \( f(x) \) on this page to the corresponding derivative \( f'(x) \) on the next page.
Unit 2 Section 4: First Shortcut Rules for Derivatives – Power, Exponential, Logs

Unit 2 Section 4 Learning Outcomes:
• Use differentiation rules to find the derivative of a function.
• Use Excel and differentiation rules to interpret the values of the derivative.

Initial Rules for Derivatives ( # 1 – 9)
In this section, we’ll get the derivative rules that will let us find formulas for derivatives when our function comes to us as a formula. This is a very algebraic section, and you should get lots of practice. When you tell someone you have studied calculus, this is the one skill they will expect you to have. There’s not a lot of deep meaning here – these are strictly algebraic rules.

Building Blocks
These are the simplest rules – rules for the basic functions. We won’t prove these rules; we’ll just use them. But first, let’s look at a few so that we can see they make sense.

Rule 1: Linear Function
Find the derivative of  \( y = f(x) = mx + b \)

This is a linear function, so its graph is its own tangent line! The slope of the tangent line, the derivative, is the slope of the line:  \( f'(x) = m \)

Rule: The derivative of a linear function is its slope

Rule 2: Constant Function
Find the derivative of  \( f(x) = 135 \).

Think about this one graphically, too. The graph of \( f(x) \) is a horizontal line. So its slope is zero.
\( f'(x) = 0 \)

Rule: The derivative of a constant is zero

Rule 3: Power Rule
We can derive the formula to find derivative of  \( f(x) = x^n \) using the formal limit definition of derivatives, but we do not spend time on that in this course. Instead, we focus on understanding and using the rule below.

Power Rule: The derivative  \( f(x) = x^n \) is  \( f'(x) = nx^{n-1} \)

Example 1:
Find derivative of  \( f(x) = x^2 \). Using the power rule, we would find that  \( f'(x) = 2x^{2-1} = 2x \)
Rule 4: Constant Multiple Rule

Find the derivative of \( g(x) = 4x^3 \).

Using the power rule, we know that if \( f(x) = x^3 \), then \( f'(x) = 3x^2 \). Notice that \( g \) is 4 times the function \( f \).

\[
\frac{d}{dx}(4x^3) = 4 \cdot \frac{d}{dx}(x^3) = 4 \cdot 3x^2 = 12x^2
\]

Rule: Constants come along for the ride; \( \frac{d}{dx}(kf) = kf' \)

Rule 5 & 6: Sum and Difference Rule

\[
\frac{d}{dx}(f + g) = f' + g'
\]

\[
\frac{d}{dx}(f - g) = f' - g'
\]

Example 2
\[
f(x) = x^4 + 12x \\
\Rightarrow f'(x) = 4x^3 + 12
\]

Example 3
\[
f(x) = x^4 - 2x^2 + 2 \\
\Rightarrow f'(x) = 4x^3 - 4x
\]

Rule 7: General Exponential Function Rule

\[
\frac{d}{dx}(a^x) = \ln a \cdot a^x
\]

Example 4
\[
f(x) = 2^x \\
\Rightarrow f'(x) = 2^x \cdot \ln 2
\]

Example 5
\[
f(x) = \left(\frac{1}{3}\right)^x \\
\Rightarrow f'(x) = \left(\frac{1}{3}\right)^x \cdot \ln \left(\frac{1}{3}\right)
\]

Rule 8: Natural Exponential Rule

\[
\frac{d}{dx}(e^x) = e^x
\]
Rule 9: Natural Log Rule

\[
\frac{d}{dx} (\ln x) = \frac{1}{x}
\]

The sum, difference, and constant multiple rule combined with the power rule allow us to easily find the derivative of any polynomial.

**Example 6:**

Find the derivative of \( p(x) = 17x^{10} + 13x^8 - 1.8x + 1003 \)

\[
\frac{d}{dx} \left( 17x^{10} + 13x^8 - 1.8x + 1003 \right)
\]

\[
= \frac{d}{dx} \left( 17x^{10} \right) + \frac{d}{dx} \left( 13x^8 \right) - \frac{d}{dx} (1.8x) + \frac{d}{dx} (1003)
\]

\[
= 17 \frac{d}{dx} (x^{10}) + 13 \frac{d}{dx} (x^8) - 1.8 \frac{d}{dx} (x) + \frac{d}{dx} (1003)
\]

\[
= 17(10x^9) + 13(8x^7) - 1.8(1) + 0
\]

\[
= 170x^9 + 104x^7 - 1.8
\]

**You don’t have to show every single step.** Do be careful when you’re first working with the rules, but soon you’ll be able to just write down the derivative directly:

**Example 7**

Find \( \frac{d}{dx} (17x^2 - 33x + 12) \)

Writing out the rules, we'd write

\[
\frac{d}{dx} \left( 17x^2 - 33x + 12 \right) = 17(2x) - 33(1) + 0 = 34x - 33
\]

Once you're familiar with the rules, you can, in your head, multiply the 2 times the 17 and the 33 times 1, and just write

\[
\frac{d}{dx} \left( 17x^2 - 33x + 12 \right) = 34x - 33
\]

The power rule works even if the power is negative or a fraction. In order to apply it, first translate all roots and basic rational expressions into exponents:

**Example 8**

Find the derivative of \( y = 3\sqrt{t} - \frac{4}{t^4} + 5e^t \)

First step – translate into exponents:
y = 3\sqrt{t} - \frac{4}{t^4} + 5e^t = 3t^{1/2} - 4t^{-4} + 5e^t

Now you can take the derivative:
\[
\frac{d}{dt} \left( 3\sqrt{t} - \frac{4}{t^4} + 5e^t \right) = \frac{d}{dt} \left( 3t^{1/2} - 4t^{-4} + 5e^t \right)
\]
\[
= 3 \left( \frac{1}{2} t^{-1/2} \right) - 4(-4t^{-5}) + 5(e^t) = \frac{3}{2} t^{-1/2} + 16t^{-5} + 5e^t.
\]

If there is a reason to, you can rewrite the answer with radicals and positive exponents:
\[
\frac{3}{2} t^{-1/2} + 16t^{-5} + 5e^t = \frac{3}{2} \sqrt{t} + 16t^5 + 5e^t
\]

Be careful when finding the derivatives with negative exponents.

**Example 9**

Find the equation of the line tangent to \( g(t) = 10 - t^2 \) when \( t = 2 \).

1) The slope of the tangent line is the value of the derivative. We can compute \( g'(t) = -2t \).
   To find the slope of the tangent line when \( t = 3 \), evaluate the derivative at that point.
   \[
g'(2) = -2(2) = -4. \text{ The slope of the tangent line is } -4.
\]

2) To find the equation of the tangent line, we also need a point on the tangent line. Since the tangent line touches the original function at \( t = 2 \), we can find the point by evaluating the original function: \( g(3) = 10 - 2^2 = 6 \). The tangent line must pass through the point \((2, 6)\).

3) Using the point-slope equation of a line, the tangent line will have equation \( y - 6 = -4(t - 2) \).

4) Simplifying to slope-intercept form, the equation is \( y = -4t + 14 \).

5) Graphing, we can verify this line is indeed tangent to the curve.

**Example 10**

The cost to produce \( x \) items is \( \sqrt{x} \) hundred dollars.

(a) What is the cost for producing 100 items? 101 items? What is cost of the 101st item?
(b) For \( f(x) = \sqrt{x} \), calculate \( f'(x) \) and evaluate \( f'(100) \) at \( x = 100 \). How does \( f'(100) \) compare with the last answer in part (a)?

(a) Put \( f(x) = \sqrt{x} = \frac{x^1}{2} \) hundred dollars, the cost for \( x \) items. Then \( f(100) = 1000 \) and \( f(101) = 1004.99 \), so it costs $4.99 for that 101st item. Using this definition, the marginal cost is $4.99.

(b) \( f''(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2 \sqrt{x}} \) so \( f''(100) = \frac{1}{2 \sqrt{100}} = \frac{1}{20} \) hundred dollars = $5.00.

Note how close these answers are! This shows (again) why it’s OK that we use both definitions for marginal cost.

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**Related Exercises You Should Complete Now**

Make sure you work through ALL the exercises in this section. Some are mechanical, others are contextual in nature. You MUST make sure you are comfortable with ALL the ideas in the exercises. Remember, you have written solutions and videos for these exercises in your course.
Exercises for Unit 2 Section 4

**Exercise 2.4.1:** Use the short-cut rules to find the equation of the derivative function for each function below.

a. $f(x) = x^2 + 3^x + e^x + \sqrt{x} + \ln(x) + 4x^5$

b. $h(x) = -7(1.04)^x - 2e^x + 9 + 50 \ln(x)$

c. $g(x) = 175(0.92)^2 - 8 + 4x - \frac{2}{x} + \frac{5}{x^3} + \frac{3}{\sqrt{x}}$

d. $P(t) = 7 - 15e^t + 4t - 8t^3 + \frac{9}{t} + 14t - 85 \ln(t)$
**Exercise 2.4.2:** A company produces and then sells widgets. From 0 widgets up to 25 thousand widgets (their maximum possible production), the following models work to predict revenue and cost: The function \( R(n) = -29n^2 + 1450n + 3000 \) gives the revenue earned (in dollars) when selling widgets \( (n) \) is in thousands of widgets). \( C(n) = -2n^2 + 152n + 199 \) gives the cost (in dollars) of producing \( n \) thousand widgets.

(a) Find a model for the profit, \( P(n) \). Simplify the equation.

(b) Find the functions \( R'(n) \) and \( C'(n) \) and \( P'(n) \).

(c) Note that:

\[
C(24.512) = 2,723.15 \quad \text{and} \quad C' (24.512) = 53.952
\]

\[
R (24.512) = 21,118.09 \quad \text{and} \quad R' (24.512) = 28.304
\]

\[
P(24.512) = 18,394.95 \quad \text{and} \quad P' (24.512) = -25.648
\]

Interpret these values in context using complete sentences.

(d) Below is a graph of \( P(n) \). Label the point where \( n = 24.512 \).

Draw and label the tangent line at \( n = 24.512 \).
Exercise 2.4.3: A publishing company estimates that when a new book by a best-selling author is introduced, its sales will be modeled as \( N(x) = 68.95\sqrt{x} + 0.125x \) where \( x \) is the number of weeks after the release, and \( N(x) \) is the number of books sold that week in thousands.

a) Find the function \( N'(x) \).

b) Use Excel to find the values of \( N(52) \) and \( N'(52) \). Using a complete sentence, interpret the real-world, practical meaning of each of these values making sure to include proper units.

Exercise 2.4.4: A lump sum of $1000 is invested in an account which grows at 4.3% each year.

a) Find a mathematical model, \( A(t) \), for the total in the account after \( t \) years.

b) Find an expression for \( A'(t) \).

c) Use Excel to find the values of \( A(13) \) and \( A'(13) \). Using a complete sentence, interpret the real-world, practical meaning of each of these values making sure to include proper units.
Exercise 2.4.5: The function $P(w) = 99.79 - 27.21 \ln(w)$ gives the thousands of pounds of apples left in storage after a harvest, where $w$ is the number of weeks after harvest. Use Excel to find the following.

a) According to this model, when will there be 0 apples remaining in storage? Round to the nearest week.

b) Find the expression for the function $P'(w)$.

c) Use Excel to find the values of $P(5)$ and $P'(5)$. Using a complete sentence, interpret the real-world, practical meaning of each of these values making sure to include proper units.

d) Use Excel to find the values of $P(30)$ and $P'(30)$. Using a complete sentence, interpret the real-world, practical meaning of each of these values making sure to include proper units.

e) Find the equation of the tangent line to $P(w)$ when $w = 5$. Sketch the line on the graph below. Label the line clearly.
**Exercise 2.4.6:** Suppose the total cost in hundreds of dollars to produce \( x \) thousand barrels of a beverage is given by: \( C(x) = 4x^2 + 100x + 500 \).

a) Find the general formula for the marginal cost.

b) Use Excel to find the marginal cost of producing 5000 barrels of the beverage. Using a complete sentence, interpret the real-world, practical meaning of this value making sure to include proper units.

c) Use Excel to find the **actual cost** to produce one thousand more barrels after 5000 are produced.

d) Use Excel to find the marginal cost of producing 30,000 barrels of the beverage. Using a complete sentence, interpret the real-world, practical meaning of this value making sure to include proper units.

**Exercise 2.4.7:** The demand function, defined by \( p = D(q) \), relates the number of units \( q \) of an item that consumers are willing to purchase at the price \( p \). The total revenue \( R(q) \) is related to price per unit and the amount demanded (or sold) by the equation: \( R(q) = q \cdot p = q \cdot D(q) \).

The demand function in dollars for a certain product is given by, \( D(q) = \frac{50000 - q}{25000} \) for \( 0 \leq q \leq 50000 \)

Find the marginal revenue when \( q = 10000 \) units.
Exercise 2.4.8: The cost in dollars to manufacture $x$ graphing calculators is given by

$$C(x) = -0.005x^2 + 20x + 150$$

when the maximum number of calculators manufactured is 2000.

a) Find the rate of change of cost with respect to the number of manufactured when 100 calculators are made and when 1000 calculators are made.

b) Interpret your results using a complete sentence.

c) Find the equation of the tangent line to the cost curve when 100 calculators are manufactured.

Exercise 2.4.9: Suppose the demand for a certain item is given by

$$D(p) = -2p^2 - 4p + 300$$

where $p$ represents the price of the item in dollars.

a) Find the formula for the rate of change of demand with respect to price.

b) Find and interpret the rate of change of demand when the price is $10 per item.
**Exercise 2.4.10**: The revenue in dollars generated from the sale of $x$ picnic tables is given by

$$R(x) = 20x - \frac{x^2}{500}$$

a) Find the marginal revenue when 1000 tables are sold.

b) Estimate the revenue from the sale of the 1001st table by finding $R'(1000)$.

c) Determine the actual revenue from the sale of the 1001st table.

d) Compare your answers for parts (b) and (c). What do you find?
Exercise 2.4.11: For each of the following find the derivative, \( f'(x) \), the value of \( f'(1) \), and the values of \( x \) where \( f'(x) = 0 \) (the critical values).

a) \( f(x) = x^2 - 5x + 13 \)

b) \( f(x) = 5x^2 - 40x + 73 \)

c) \( f(x) = x^3 + 9x^2 + 6 \)

d) \( f(x) = x^3 + 3x^2 + 3x - 1 \)
Unit 2 Section 5: More Derivative Rules - Product and Quotient Rules

Section 2.5 Learning Outcomes:
• Use differentiation rules to find the derivative of a function.
• Use Excel and differentiation rules to interpret the values of the derivative.

We will learn 15 rules to find the derivative of a function that is given to us as an algebraic mathematical expression. You must learn, practice, and KNOW these rules in order are two more rules.

The basic rules will let us tackle simple functions. But what happens if we need the derivative of a combination of these functions?

Example 1

Find the derivative of \( g(x) = (4x^3 - 11)(x + 3) \)

This function is not a simple sum or difference of polynomials. It’s a product of polynomials. We can simply multiply it out to find its derivative:

\[
g(x) = (4x^3 - 11)(x + 3) = 4x^4 - 11x + 12x^3 - 33
\]

\[
g'(x) = 16x^3 - 11 + 36x^2
\]

Now suppose we wanted to find the derivative of 

\[
f(x) = (4x^5 + x^3 - 1.5x^2 - 11)(x^7 - 7.25x^5 + 120x + 3)
\]

This function is not a simple sum or difference of polynomials. It’s a product of polynomials. We could simply multiply it out to find its derivative as before – who wants to volunteer? Nobody?

We’ll need a rule for finding the derivative of a product so we don’t have to multiply everything out.

It would be great if we can just take the derivatives of the factors and multiply them, but unfortunately that won’t give the right answer. to see that, consider finding derivative of 

\[
g(x) = (4x^3 - 11)(x + 3)
\]

We already worked out the derivative. It’s \( g'(x) = 16x^3 - 11 + 36x^2 \). What if we try differentiating the factors and multiplying them? We’d get \( 12x^2 \), which is totally different from the correct answer.

The rules for finding derivatives of products and quotients are a little complicated, but they save us the much more complicated algebra we might face if we were to try to multiply things out. They also let us deal with products where the factors are not polynomials. We can use these rules, together with the basic rules, to find derivatives of many complicated looking functions.
Rule # 10: Derivative of a Product

\[
\frac{d}{dx}(fg) = f'g + fg' \]

The derivative of the first factor times the second left alone, plus the first left alone times the derivative of the second.

The product rule can extend to a product of several functions; the pattern continues – take the derivative of each factor in turn, multiplied by all the other factors left alone, and add them up.

Examples 2 and 3

\[f(x) = x^2 \cdot e^x\]
\[f'(x) = x^2 \cdot e^x + e^x \cdot 2x\]
\[g(x) = (x^3 + 1) \cdot \ln(x)\]
\[g'(x) = (x^3 \cdot +1) \cdot \frac{1}{x} + \ln(x) \cdot (3x^2)\]

Rule # 11: Quotient Rule

\[
\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} \]

The numerator of the result resembles the product rule, but there is a minus instead of a plus; the minus sign goes with the g’. The denominator is simply the square of the original denominator – no derivatives there.

Examples 4 and 5

\[f(x) = \frac{2x^2 + 5x}{x^2 + 3}\]
\[f'(x) = \frac{(x^2 + 3)(6x^2 + 5) - (2x^2 + 5x)(2x)}{(x^2 + 3)^2}\]

This derivative expression can of course be simplified, if necessary. \[f'(x) = \frac{(2x^4 + 13x^2 + 15)}{(x^2 + 3)^2}\]
\[g(x) = \frac{(3^x)}{x^3}\]
\[g'(x) = \frac{(3^x)(\ln(3)) - (3^x)(3x^2)}{(x^3)^2}\]

Example 6

Find the derivative of \(F(t) = e^t \ln t\)

This is a product, so we need to use the product rule. I like to put down empty parentheses to remind myself of the pattern; that way I don’t forget anything.

\[F'(t) = (\ ) + (\ )\]
Then I fill in the parentheses – the first set gets the derivative of $e'$, the second gets $\ln t$ left alone, the third gets $e'$ left alone, and the fourth gets the derivative of $\ln t$.

$$F'(t) = (e')\ln t + (e')\left(\frac{1}{t}\right) = e' \ln t + \frac{e'}{t}$$

Notice that this was one we couldn’t have done by “multiplying out.”

**Example 7**

Find the derivative of $y = \frac{x^4 + 4^x}{3 + 16x^3}$

This is a quotient, so we need to use the quotient rule. Again, you find it helpful to put down the empty parentheses as a template:

$$y' = \frac{}{()}$$

Then fill in all the pieces:

$$y' = \frac{(4x^3 + \ln 4 \cdot 4^x)(3 + 16x^3) - (x^4 + 4^x)(48x^2)}{(3 + 16x^3)^2}$$

Now for goodness’ sakes don’t try to simplify that! Remember that “simple” depends on what you will do next; in this case, we were asked to find the derivative, and we’ve done that. Please STOP, unless there is a reason to simplify further.

**Related Exercises You Should Complete Now**

Make sure you work through ALL the exercises in this section. Some are mechanical, others are contextual in nature. You MUST make sure you are comfortable with ALL the ideas in the exercises. Remember, you have written solutions and videos for these exercises in your course.
Exercises for Unit 2 Section 5

Exercise 2.5.1: Find the derivative of \( h(x) = (5x^2 + 3x - 4)(7^x) \)

Exercise 2.5.2: Find the derivative of \( h(x) = \frac{25(0.8)^x}{7x^2 + 8x + 1} \)

Exercise 2.5.3: A store has determined that the number of Blu-ray movies sold monthly is approximately \( n(x) = 8600(0.932)^x \) where \( x \) is the average price of the movies in dollars per movie.

(a) Find a formula for the revenue function \( R(x) \) where \( x \) is the average price of the movies in dollars per movie.

(b) Find the revenue and find the derivative of revenue when \( x = 10 \). Interpret in context.

(c) Find the revenue and find the derivative of revenue when \( x = 25 \). Interpret in context.
**Exercise 2.5.4:** The number of private donations, in thousands, received by nongovernment disaster relief organizations during the $x^{th}$ hour can be modeled as $f(x) = 0.3x \cdot (0.9704)^x$, where $x$ is the number of hours since a major disaster has struck.

(a) Find a formula for $f'(x)$.

(b) Find $f(20)$ and $f'(20)$. USE EXCEL TO EVALUATE. Interpret the practical, real-world meaning of each.

**Exercise 2.5.5:** During the first 8 months of last year, a grocery store raised the price of a certain brand of tissue paper from $1.14 per package at a rate of 4 cents per month. Consequently, sales declined. The sales of tissue can be modeled as $s(m) = -0.95m^2 + 0.24m + 279.91$ packages during the $m^{th}$ month of the year.

(a) Find a formula for the function $p(m)$ that gives the price of the tissue paper during the $m^{th}$ month.

(b) Find a formula for $R(m)$, the revenue earned on tissue paper during the $m^{th}$ month. Remember how we find revenue!

(c) Find a formula for $R'(m)$. 

Compiled and Modified by M. Johnson and S. Taylor – Dutchess Community College
(d) Find $R(8)$ and $R'(8)$. USE EXCEL TO EVALUATE!! Interpret the practical, real-world meaning.

**Exercise 2.5.6:** The number of trinkets sold when the price of the trinkets is set at $p$ dollars each is given in the following table.

<table>
<thead>
<tr>
<th>price of each trinket</th>
<th>Number sold</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.50</td>
<td>9315</td>
</tr>
<tr>
<td>$5.00</td>
<td>4740</td>
</tr>
<tr>
<td>$5.50</td>
<td>3790</td>
</tr>
<tr>
<td>$6.00</td>
<td>3018</td>
</tr>
</tbody>
</table>

(a) Find a best-fit model for this data, $T(p)$. Assume that there IS a price that would cause sales to drop to 0. Round the parameters to 3 decimal places and be sure to clearly explain the meaning of your variables together with your formula.

(b) Find a formula for the revenue earned when the trinkets are sold at $p$ dollars each.

(c) Find the revenue and find the derivative of revenue when they charge $1.50 per trinket. Interpret.
(d) Find the revenue and find the derivative of revenue when they charge $4.50 per sprocket.
Interpret.

**Exercise 2.5.7:** Total costs when producing $t$ Trinkets are given by the function

$$C(t) = t^3 - 360t^2 + 43,200t + 63,272,000$$

where costs are in dollars. Find the total cost, and find the marginal cost when producing 119 Trinkets. Write a complete sentence to interpret in context.

**Exercise 2.5.8: Additional Practice:** Find each of the following derivatives by hand. You DO NOT need to algebraically simplify your answers.

(a) $f(x) = -7x^{100} + 4x^4 - \frac{3}{2}x^2 - 99x + 12345$
(b) \( f(x) = \frac{3}{7x^2} + x \)  
HINT: Rewrite \( f(x) \) using negative exponents first, then take the derivative.

Rewrite \( f(x) \):

(c) \( f(x) = \frac{x}{6} + \frac{3}{x^7} - \frac{1}{x^{0.4}} \)  
HINT: Rewrite \( f(x) \) using negative exponents first, then take the derivative.

Rewrite \( f(x) \):

(d) \( f(x) = \frac{9}{\sqrt[3]{x^{11}}} + \sqrt{x} + \sqrt[4]{x^4} + \frac{3}{\sqrt{x}} - \frac{2}{\sqrt[3]{x^3}} \)  
HINT: Rewrite \( f(x) \) using negative and fractional exponents first, then take the derivative.

Rewrite \( f(x) \):
(e) \( f(x) = \left( 6x^7 + \frac{3}{8}x^{11} + \frac{4}{9} \right)(2x^{21} + x + 21) \)

(f) \( f(x) = \frac{\sqrt[4]{x} + 6x^2 + \frac{1}{3}}{5x^3 + 14x - 2} \) Rewrite using fractional exponents, then use the quotient rule.

Rewrite \( f(x) \):
Unit 2 Section 6: The Chain Rule

Unit 2 Section 6 Learning Outcomes:
- Find the formula for the derivative function using the chain rule.
- Interpret the real-world, contextual meaning of the derivative values.

Rule # 12: The Chain Rule

There is one more type of complicated function that we will want to know how to differentiate: composition. The Chain Rule will let us find the derivative of a composition. (This is the last derivative rule we will learn!)

Example 1

Find the derivative of \( y = (4x^3 + 15x)^2 \).

This is not a simple polynomial, so we can’t use the basic building block rules yet. It is a product, so we could write it as \( y = (4x^3 + 15x)^2 = (4x^3 + 15x)(4x^3 + 15x) \) and use the product rule. Or we could multiply it out and simply differentiate the resulting polynomial. I’ll do it the second way:

\[
y = (4x^3 + 15x)^2 = 16x^6 + 120x^4 + 225x^2
\]

\[
y' = 64x^5 + 480x^3 + 450x
\]

Now suppose we want to find the derivative of \( y = (4x^3 + 15x)^{20} \). We could write it as a product with 20 factors and use the product rule, or we could multiply it out. But I don’t want to do that, do you?

We need an easier way, a rule that will handle a composition like this. The Chain Rule is a little complicated, but it saves us the much more complicated algebra of multiplying something like this out. It will also handle compositions where it wouldn’t be possible to “multiply it out.”

The Chain Rule is the most common place for students to make mistakes. Part of the reason is that the notation takes a little getting used to. And part of the reason is that students often forget to use it when they should. When should you use the Chain Rule? Almost every time you take a derivative.


**Derivative Rules: Chain Rule**

In what follows, \( f \) and \( g \) are differentiable functions with \( y = f(u) \) and \( u = g(x) \)

(h) Chain Rule (Leibniz notation):

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

Notice that the du’s seem to cancel. This is one advantage of the Leibniz notation; it can remind you of how the chain rule chains together.

(h) Chain Rule (using prime notation):

\[ f'(x) = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x) \]

(h) Chain Rule (in words):

The derivative of a composition is the derivative of the outside, with the inside staying the same, TIMES the derivative of what’s inside.

---

I recite the version in words each time I take a derivative, especially if the function is complicated.

**Example 2**

Find the derivative of \( y = (4x^3 + 15x)^2 \).

This is the same one we did before by multiplying out. This time let’s use the Chain Rule:

The inside function is what appears inside the parentheses: \( 4x^3 + 15x \). The outside function is the first thing we find as we come in from the outside – it’s the square function, \((\text{inside})^2\).

The derivative of this outside function is \((2*\text{inside})\). Now using the chain rule, the derivative of our original function is:

\( (2*\text{inside}) \) TIMES the derivative of what’s inside (which is \(12x^2 + 15\)):

\[
y = (4x^3 + 15x)^2 \\
y' = 2(4x^3 + 15x) \cdot (12x^2 + 15)
\]

If you multiply this out, you get the same answer we got before. Hurray! Algebra works!

**Example 3**

Find the derivative of \( y = (4x^3 + 15x)^{20} \)

Now we have a way to handle this one. It’s the derivative of the outside TIMES the derivative of what’s inside.

The outside function is \((\text{inside})^{20}\), which has the derivative 20(inside)\(^{19}\).

\[
y = (4x^3 + 15x)^{20} \\
y' = 20(4x^3 + 15x)^{19} \cdot (12x^2 + 15)
\]
Rule # 13: The “Hidden” Chain Rule

Notice that often a function using $e^x$ where the exponent is not just “$x$”. We can use the chain rule to see this “hidden” inside function and come up with another shortcut rule.

$$f(x) = e^{3x} \quad \text{Outside function } f(t) = e^t \quad \text{Inside function } t(x) = 3x$$

$$f'(x) = f'(t) \cdot t'(x) = e^t \cdot 3 = 3e^{3x}$$

OR

$$g(x) = e^{-0.54x} \quad \text{Outside function } f(t) = e^t \quad \text{Inside function } t(x) = -0.54x$$

$$f'(x) = -0.54e^{-0.54x}$$

The “Hidden” Chain Rule of $e^{cx}$:

$$f(x) = e^{cx} \quad f'(x) = c \cdot e^{cx}$$

Example 4

Differentiate $e^{x^2+5}$.

This isn’t a simple exponential function; it’s a composition. Typical calculator or computer syntax can help you see what the “inside” function is here.

On a TI calculator, for example, when you push the $e^x$ key, it opens up parentheses: $e^{(x)}$

This tells you that the “inside” of the exponential function is the exponent. Here, the inside is the exponent $x^2 + 5$.

Now we can use the Chain Rule: We want the derivative of the outside TIMES the derivative of what’s inside. The outside is the “$e$ to the something” function, so its derivative is the same thing. The derivative of what’s inside is $2x$. So

$$\frac{d}{dx} \left( e^{x^2+5} \right) = \left( e^{x^2+5} \right) \cdot (2x)$$

Example 5

The table gives values for $f(x), f'(x), g(x)$ and $g'(x)$ at a number of points. Use these values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -1$ and 0.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$g(x)$</th>
<th>$f'(x)$</th>
<th>$g'(x)$</th>
<th>$(f \circ g)(x)$</th>
<th>$(f \circ g)'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(f \circ g)(-1) = f(g(-1)) = f(3) = 0
(f \circ g)(0) = f(g(0)) = f(1) = 1.

(f \circ g)'(-1) = f'(g(-1))g'(-1) = f'(3)(0) = 2(0) = 0 and
(f \circ g)'(0) = f'(g(0))g'(0) = f'(1)(2) = (-1)(2) = -2.

Derivatives of Complicated Functions

You’re now ready to take the derivative of some mighty complicated functions. But how do you tell what rule applies first? Come in from the outside – what do you encounter first? That’s the first rule you need. Use the Product, Quotient, and Chain Rules to peel off the layers, one at a time, until you’re all the way inside.

Example 6

Find \( \frac{d}{dx} \left( e^{3x} \cdot \ln(5x + 7) \right) \)

Coming in from the outside, I see that this is a product of two (complicated) functions. So I’ll need the Product Rule first. I’ll fill in the pieces I know, and then I can figure the rest as separate steps and substitute in at the end:

\[
\frac{d}{dx} \left( e^{3x} \cdot \ln(5x + 7) \right) = \left( \frac{d}{dx} \left( e^{3x} \right) \right) \ln(5x + 7) + e^{3x} \left( \frac{d}{dx} \ln(5x + 7) \right)
\]

Now as separate steps, I’ll find

\[
\frac{d}{dx} \left( e^{3x} \right) = 3e^{3x} \quad \text{(using the Chain Rule)}
\]

and

\[
\frac{d}{dx} \ln(5x + 7) = \frac{1}{5x + 7} \cdot 5 \quad \text{(also using the Chain Rule)}.
\]

Finally, to substitute these in their places:

\[
\frac{d}{dx} \left( e^{3x} \cdot \ln(5x + 7) \right) = 3e^{3x} \ln(5x + 7) + e^{3x} \left( \frac{1}{5x + 7} \cdot 5 \right)
\]

(And please don’t try to simplify that!)

Example 7

Differentiate \( z = \left( \frac{3t^3}{e^{(t-1)}} \right)^4 \)

Don’t panic! As you come in from the outside, what’s the first thing you encounter? It’s that 4th power. That tells you that this is a composition, a (complicated) function raised to the 4th power.

Step One: Use the Chain Rule. The derivative of the outside TIMES the derivative of what’s inside.
\[ \frac{dz}{dt} = \frac{d}{dt} \left( \frac{3t^3}{e'(t-1)} \right)^4 = 4 \left( \frac{3t^3}{e'(t-1)} \right)^3 \cdot \frac{d}{dt} \left( \frac{3t^3}{e'(t-1)} \right) \]

Now we’re one step inside, and we can concentrate on just the \( \frac{d}{dt} \left( \frac{3t^3}{e'(t-1)} \right) \) part. Now, as you come in from the outside, the first thing you encounter is a quotient – this is the quotient of two (complicated) functions.

**Step Two:** Use the Quotient Rule. The derivative of the numerator is straightforward, so we can just calculate it. The derivative of the denominator is a bit trickier, so we’ll leave it for now.

\[ \frac{d}{dt} \left( \frac{3t^3}{e'(t-1)} \right) = \frac{\left( 9t^2 \right) (e'(t-1)) - \left( 3t^3 \right) (e'(t-1))}{(e'(t-1))^2} \]

Now we’ve gone one more step inside, and we can concentrate on just the \( \frac{d}{dt} (e'(t-1)) \) part. Now we have a product.

**Step Three:** Use the Product Rule:

\[ \frac{d}{dt} (e'(t-1)) = (e'(t-1)) + (e')(1) \]

And now we’re all the way in – no more derivatives to take.

**Step Four:** Now it’s just a question of substituting back – be careful now!

\[ \frac{d}{dt} (e'(t-1)) = (e'(t-1)) + (e')(1) \), so

\[ \frac{d}{dt} \left( \frac{3t^3}{e'(t-1)} \right) = \frac{\left( 9t^2 \right) (e'(t-1)) - \left( 3t^3 \right) (e'(t-1)) + (e')(1)}{(e'(t-1))^2} \], so

\[ \frac{dz}{dt} = \frac{d}{dt} \left( \frac{3t^3}{e'(t-1)} \right)^4 = 4 \left( \frac{3t^3}{e'(t-1)} \right)^3 \cdot \left( \frac{\left( 9t^2 \right) (e'(t-1)) - \left( 3t^3 \right) (e'(t-1)) + (e')(1)}{(e'(t-1))^2} \right) \]

Phew! OK….. You are ready to go work on all the Exercises!
Exercises for Unit 2 Section 6

**Exercise 2.6.1:** Suppose \( f(x) = 3(-8x^2 - 7x)^3 + 5(-8x^2 - 7x)^2 + 4. \)

- outer function \( f(t) = \)
- inner function \( t(x) = \)

- derivative of outer function \( \frac{df}{dt} = \)
- derivative of inner function \( \frac{dt}{dx} = \)

And So … \( f'(x) = \frac{df}{dx} = \)

**Exercise 2.6.2:** Suppose that \( f(x) = 30 e^{4x} \)

- outer function \( f(t) = \)
- inner function \( t(x) = \)

- derivative of outer function \( \frac{df}{dt} = \)
- derivative of inner function \( \frac{dt}{dx} = \)

And So … \( f'(x) = \frac{df}{dx} = \)

**Exercise 2.6.3:** Suppose that \( f(x) = \ln(7 - 11x^3) \)

- outer function \( f(t) = \)
- inner function \( t(x) = \)

- derivative of outer function \( \frac{df}{dt} = \)
- derivative of inner function \( \frac{dt}{dx} = \)

And So … \( f'(x) = \frac{df}{dx} = \)
Exercises for 2.6

**Exercise 2.6.4:** Suppose that \( f(x) = \sqrt{7 + 5e^{0.04x}} \)

- outer function \( f(t) = \)
- inner function \( t(x) = \)
- derivative of outer function
- derivative of inner function
- CAREFUL!!! There’s a hidden chain here!

And So … \( f'(x) = \frac{df}{dx} = \)

**Exercise 2.6.5** Suppose that \( f(x) = \frac{130}{1 + 12e^{-0.5x}} \)

- outer function \( f(t) = \)
- inner function \( t(x) = \)
- derivative of outer function
- derivative of inner function
- CAREFUL!!! There’s a hidden chain here!

And So … \( f'(x) = \frac{df}{dx} = \)

**Exercise 2.6.6:** Suppose that \( f(x) = 275 (1.04)^{x^2} \)

- outer function \( f(t) = \)
- inner function \( t(x) = \)
- derivative of outer function
- derivative of inner function
- CAREFUL!!! There’s a hidden chain here!

And So … \( f'(x) = \frac{df}{dx} = \)

Sometimes it is not as obvious that the **chain rule needs** to be used. Finding the inside function is not always clear.

Suppose that \( f(x) = e^{2x^3 + x} \)
Exercise 2.6.7: A balloon begins with a radius of 4 inches, and then the radius decreases by 0.5 inches per second. So, the function \( r(t) = 4 - 0.5t \) gives the radius of the balloon (in inches) after \( t \) seconds have passed.

The balloon's volume is given by the function \( V(r) = \frac{4}{3} \pi r^3 \) where \( r \) is the radius of the balloon in inches, and \( V \) is the volume of the balloon in cubic inches.

(a) Find a formula for \( V(t) \). [Hint: This would be a composite function].

(b) Find a formula for \( \frac{dV}{dt} \).

(c) Evaluate \( V'(t) \) at \( t = 1 \). Interpret the real-world, contextual meaning.
**Exercise 2.6.8:** The number of visitors to Wonderland theme park when the daily high temperature is $t$ degrees Fahrenheit is expected to be $p(t) = -0.0064t^2 + 1.059t - 38.46$ where $p(t)$ is the number of expected visitors in thousands of people.

Revenue from concessions sales at the park can be modeled as $r(p) = -1.9p^2 + 21p - 8.5$ where $p$ is the number of people in attendance, in thousands, and $r(p)$ is the revenue in thousands of dollars.

(a) Find the function $r(t)$. Clearly identify the meaning of $t$ and meaning of $r(t)$ in this function.

(b) Find $\frac{dr}{dt}$

(c) Find the value of $r(t)$ when $t = 80$ and find $\frac{dr}{dt} \big|_{t=80}$. Interpret.

**Exercise 2.6.9:** It costs $C(x) = \sqrt{2x + 1}$ dollars to produce $x$ golf balls.

(a) What is the marginal production cost to make a golf ball?

(b) Use Excel to find the marginal production cost when $x = 24$? when $x = 40$? Include units.

(c) Use a complete sentence to interpret the contextual meaning of the values you found in part b.
Exercise 2.6.10 MORE PRACTICE with the Chain Rule

Find the first derivative of each of the following functions.

(a) \( f(x) = \ln(x^3 + 2x) \) where \( x > 0 \)

(b) \( g(x) = -\sqrt{4x^3 - 1} + 2e^{2x+7} \)

(c) \( h(x) = 2 \cdot 3^{x^2-x} - \frac{(2x+5)^2}{7-x} \)

(d) \( j(x) = 2(1.0792)^x \cdot \ln(3x + 15) \)

(e) \( k(x) = -\frac{3}{8} (x - 6)^4 \cdot (0.62)^x \)

(f) \( m(x) = \frac{3e^{2x-12} \ln(4x-1)}{4x^4} \)
Unit 3 – Section 1: Optimization and the First Derivative Test

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Unit 3 – Section 1: Optimization and the First Derivative Test

Learning Outcomes:
• Define a critical point of a function.
• State the First Derivative Test and use it to interpret the local extremes of a function.

Optimization: Maximum and Minimum Points

In Real-Life applications, we often want to maximize or minimize some quantity.
• An engineer may want to maximize the speed of a new computer or minimize the heat produced by an appliance.
• A manufacturer may want to maximize profits and market share or minimize costs or waste.
• A student may want to maximize a grade in calculus or minimize the hours of study needed to earn a particular grade.

The best way we have, without calculus, to find extremes is to examine the graph of the function, perhaps using technology. But our view depends on the viewing window we choose – we might miss something important or only get an approximation. (In some cases, that will be good enough.)

Recall from section 1.2:

Absolute extreme values are either maximum or minimum points on a curve. They are sometimes called global extremes.

Local Extreme Values:
A local maximum is the maximum value within some open interval.
A local minimum is the minimum value within some open interval.

Recall:
Calculus provides ways of drastically narrowing the number of points we need to examine to find the exact locations of maximums and minimums, while at the same time ensuring that we haven’t missed anything important.

**Local Maxima and Minima**

You should go back and review all the material in the textbook in section 1.2. Remember, we defined local maxima and minima, as well as global maxima and minima in that section, as well as gave some graphical examples of each.

The local and global extremes of the function in the picture below are labeled. You should notice that every global extreme is also a local extreme, but there are local extremes that are not global extremes.

If \( h(x) \) is the height of the earth above sea level at the location \( x \), then the global maximum of \( h \) (\( x \)) is \( h(\text{summit of Mt. Everest}) = 29,028 \text{ feet} \). The local maximum of \( h(x) \) for the United States is \( h(\text{summit of Mt. McKinley}) = 20,320 \text{ feet} \). The local minimum of \( h(x) \) for the United States is \( h(\text{Death Valley}) = -282 \text{ feet} \).

**Example 1**

The table shows the annual calculus enrollments at a large university. Which years had local maximum or minimum calculus enrollments? What were the global maximum and minimum enrollments in calculus?
There were local maxima in 2002 and 2007; the global maximum was 1582 students in 2007. There were local minima in 2003 and 2009; the global minimum was 1336 students in 2003. I choose not to think of 2000 as a local minimum or 2010 as a local maximum. However, some books would include the endpoints.

**Finding Maxima and Minima of a Function**

What must the tangent line look like at a local max or min? Look at these two graphs again—you’ll see that at all the extreme points, the tangent line is horizontal (so $f'(x) = 0$). There is one cusp in the blue graph—so $f'$ is undefined at that point.

That gives us the clue how to find extreme values.

![Graph showing local maxima and minima](image)

**Definition:** A critical number for a function $f(x)$ is a value $x = a$ in the domain of $f(x)$ where either $f'(a) = 0$ or $f'(a)$ is undefined.

**Definition:** A critical point for a function $f(x)$ is a point $(a, f(a))$ where $a$ is a critical number of $f(x)$.

**Useful Fact:** A local max or min of $f(x)$ can only occur at a critical point.

**Example 2**

Find the critical points of $f(x) = x^3 - 6x^2 + 9x + 2$.

A critical number of f can occur only where $f'(x) = 0$ or where $f'(x)$ does not exist.

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$$
\[ f'(x) = 0 \text{ only at } x = 1 \text{ and } x = 3. \] There are no places where \( f'(x) \) is undefined.

The critical numbers are \( x = 1 \) and \( x = 3 \). So the critical points are \((1, 6)\) and \((3, 2)\).

These are the only possible locations of local extremes of \( f(x) \). We haven’t discussed yet how to tell whether either of these points is actually a local extreme of \( f(x) \), or which kind it might be. But we can be certain that no other point is a local extreme.

The graph of \( f(x) \) shown below shows that \((1, f(1)) = (1, 6)\) is a local maximum and \((3, f(3)) = (3, 2)\) is a local minimum. This function does not have a global maximum or minimum over its entire domain.

![Graph of f(x) showing local extrema](image)

**Example 3**

Find all local extremes of \( f(x) = x^3 \).

\( f(x) = x^3 \) is differentiable for all \( x \), and \( f'(x) = 3x^2 \). The only place where \( f'(x) = 0 \) is at \( x = 0 \), so the only candidate is the critical point \((0,0)\).

If \( x > 0 \) then \( f(x) > 0 \), so \( f(0) \) is not a local maximum.
Similarly, if \( x < 0 \) then \( f(x) < 0 \) so \( f(0) \) is not a local minimum.

The critical point \((0,0)\) is the only candidate to be a local extreme of \( f(x) \), and this candidate did not turn out to be a local extreme of \( f \). The function \( f(x) = x^3 \) does not have any local extremes.

![Graph of y=x^3](image)

**Remember this example!** It is not enough to find the critical points -- we can only say that \( f(x) \) might have a local extreme at the critical points.
Related Exercises You Should Complete Now

Make sure you work on Exercises 3.1.1 and 3.1.2. Remember, you have written solutions and videos for these exercises in your course.

First Derivative Test

Is that critical point a Maximum or Minimum (or Neither)?

Once we have found the critical points of \( f \), we still have the problem of determining whether these points are maxima, minima or neither.

All of the graphs in Fig. 25 have a critical point at (2, 3). It is clear from the graphs that the point (2,3) is a local maximum in (a) and (d), (2,3) is a local minimum in (b) and (e), and (2,3) is not a local extreme in (c) and (f).

The critical numbers only give the possible locations of extremes, and some critical numbers are not the locations of extremes. The critical numbers are the candidates for the locations of maxima and minima.

\[ f'(x) \text{ and Extreme Values of } f(x) \]

Four possible shapes of graphs are shown here – in each graph, the point marked by an arrow is a critical point, where \( f'(x) = 0 \). What happens to the derivative near the critical point?
At a local max, such as in the graph on the left,
- the function increases on the left of the local max,
- then decreases on the right.
- The derivative is first positive,
- then negative at a local max.

At a local min,
- the function decreases to the left and
- increases to the right, so
- the derivative is first negative, t
- hen positive.

When there isn’t a local extreme, the function continues to increase (or decrease) right past the critical point – the derivative doesn’t change sign.

The First Derivative Test for Extremes:

<table>
<thead>
<tr>
<th>The First Derivative Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Find the critical points of ( f(x) ).</td>
</tr>
<tr>
<td>• For each critical number ( c ), examine the sign of ( f'(x) ) to the left and to the right of ( c ). What happens to the sign as you move from left to right?</td>
</tr>
<tr>
<td>• If ( f'(x) ) changes from positive to negative around ( x = c ), then ( f(x) ) has a local maximum at ((c, f(c))).</td>
</tr>
<tr>
<td>• If ( f'(x) ) changes from negative to positive around ( x = c ), then ( f(x) ) has a local minimum at ((c, f(c))).</td>
</tr>
<tr>
<td>• If ( f'(x) ) does not change sign around ( x = c ), then ((c, f(c))) is neither a local max nor a local min.</td>
</tr>
</tbody>
</table>

Example 4

Find the critical points of \( f(x) = x^3 - 6x^2 + 9x + 2 \) and classify them as local max, local min, or neither.

We already found the critical points; they are \((1, 6)\) and \((3, 2)\).

Now we can use the first derivative test to classify each. Recall that:

\[
f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3).
\]

We already know the derivative is zero or undefined at the critical numbers. On each interval between these values, the derivative will stay the same sign.
1) To determine the sign in each of these intervals, we can use Excel, making sure that we have each critical point in the table, with at least one value on either side of that critical point for each interval. We can then consider the sign of the derivative in each interval.

2) Notice that we have created a table that shows the critical points of $x = 1$ and $x = 3$. We have also created values on either side of each critical point.

3) We can see that for a value to the left of the critical value at $x = 1$, our derivative is positive, and for a value to the right of the critical value, the derivative is negative. This means that $f(x)$ changes from increasing to decreasing at $x = 1$. Therefore, the point $(1, 6)$ on the graph of $f(x)$ is a local maximum.

4) We see that for a value to the left of the critical value at $x = 3$ (but after our previous critical value) our derivative is negative, and for a value to the right of the critical value, the derivative is positive. This means that $f(x)$ changes from decreasing to increasing at $x = 3$. Therefore, the point $(3, 2)$ on the graph of $f(x)$ is a local minimum.

This confirms what we saw before in the graph.

\[
 f(x) = x^2 - 6x^2 + 9x + 2
\]
When trying to maximize their revenue, businesses also face the constraint of consumer demand. While a business would love to see lots of products at a very high price, typically demand decreases as the price of goods increases. In simple cases, we can construct that demand curve to allow us to maximize revenue.

Example 5

A concert promoter has found that if she sells tickets for $50 each, she can sell 1200 tickets, but for each $5 she raises the price, 50 less people attend. What price should she sell the tickets at to maximize her revenue?

1) We are trying to maximize revenue, and we know that \( R = pq \), where \( p \) is the price per ticket, and \( q \) is the quantity of tickets sold.

2) The problem provides information about the demand relationship between price and quantity - as price increases, demand decreases. We need to find a formula for this relationship. To investigate, let's calculate what will happen to attendance if we raise the price:

<table>
<thead>
<tr>
<th>Price, ( p )</th>
<th>50</th>
<th>55</th>
<th>60</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantity, ( q )</td>
<td>1200</td>
<td>1150</td>
<td>1100</td>
<td>1050</td>
</tr>
</tbody>
</table>

1) You might recognize this as a linear relationship. We can find the slope for the relationship by using two points: \( m = \frac{1150 - 1200}{55 - 50} = -\frac{50}{5} = -10 \). You may notice that the second step in that calculation corresponds directly to the statement of the problem: the attendance drops 50 people for every $5 the price increases.

2) Using the point-slope form of the line, we can write the equation relating price and quantity:

\[ q - 1200 = -10(p - 50) \]

3) Simplifying to slope-intercept form gives the demand equation \( q = 1700 - 10p \)

4) Substituting this into our revenue equation, we get an equation for revenue involving only one variable:

\[ R = pq = p(1700 - 10p) = 1700p - 10p^2 \]

5) Now, we can find the maximum of this function by finding critical numbers. \( R' = 1700 - 20p \), so \( R' = 0 \) when \( p = 85 \).

6) Using the first derivative test, we can pick values for \( p \) on either side of 85 and determine the sign of the first derivative. Use Excel to create a table of convenient values!
7) At \( p = 80 \), we see that \( R'(80) \) is positive, so that means that revenue is increasing. At \( p = 90 \) we see that \( R'(90) \) is negative, so that means that revenue is decreasing.

Therefore, the critical number is a local maximum. Since it's the only critical number, we can also conclude it's the global maximum.

The promoter will be able to maximize revenue by charging $85 per ticket. At this price, she will sell \( q = 1700 - 10(85) = 850 \) tickets, generating $72,250 in revenue.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( R(p) )</th>
<th>( R'(p) = 1700 - 20*p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>71250</td>
<td>200</td>
</tr>
<tr>
<td>80</td>
<td>72000</td>
<td>100</td>
</tr>
<tr>
<td>85</td>
<td>72250</td>
<td>0</td>
</tr>
<tr>
<td>90</td>
<td>72000</td>
<td>-100</td>
</tr>
<tr>
<td>95</td>
<td>71250</td>
<td>-200</td>
</tr>
</tbody>
</table>

**Related Exercises You Should Complete Now**

Make sure you work with exercises 3.1.3 through 3.1.11.

You MUST make sure you are comfortable with ALL the ideas in the exercises.

Remember, you have written solutions and videos for these exercises in your course.
Exercises for Unit 3 Section 1

Exercise 3.1.1
First some review:

The graph of a function $f$ is shown below.

Find all the values of $x$ that answer each of the following AND explain your answer.

(a) Where is $f$ zero?

(b) Where is $f'(x)$ zero?

(c) Where is $f$ positive?

(d) Where is $f'(x)$ increasing?

(e) Where is $f'(x)$ decreasing?

(f) Where is $f(x)$ decreasing?
Exercise 3.1.2: The table shows the annual calculus enrollments at a large university.

<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Enrollment</td>
<td>1257</td>
<td>1324</td>
<td>1378</td>
<td>1336</td>
<td>1389</td>
<td>1450</td>
<td>1523</td>
<td>1582</td>
<td>1567</td>
<td>1545</td>
<td>1571</td>
</tr>
</tbody>
</table>

(a) Which years had local maximum or minimum calculus enrollments?

(b) What were the global maximum and minimum enrollments in calculus?

NOTE: We will use Excel to help us with the next problems. Solver can be used to find when \( f'(x) = 0 \), and we can describe the behavior of \( f'(x) \) and \( f(x) \) on the intervals created by the critical points.

Exercise 3.1.3: Find all local maximum and minimum points (if they exist) of the function
\[
f(x) = 2x^3 - 13x^2 + 18x + 48
\]
using calculus. Show all work/clearly explain how to find these points using calculus. Note that this function has exactly two critical points, and when you get this in Excel, your Excel table should look as follows:

(a) Use the given \( f(x) \) function to find \( f'(x) \), enter the functions in Excel in order to re-create this table, and use solver to find the critical points as shown in this table.

(b) Then use the first derivative test to clearly explain how to identify the local maximum and local minimum points on \( f(x) \) if they exist. Write the explanation in a big textbox near the table in Excel.
Exercise 3.1.4: Use calculus to find all local maximum and minimum points (if they exist) of the function
\[ g(x) = \frac{1}{4}x^4 - \frac{209}{30}x^3 + \frac{1644}{25}x^2 - \frac{32544}{125}x + 2000 \]

Show all work/clearly explain how to find these points using calculus. Note that f(x) has exactly two critical points.

(a) Use the given \( f(x) \) function to find \( f'(x) \), enter the functions in Excel in order to re-create this table and use solver to find the critical points as shown in this table.

(b) Then use the first derivative test to clearly explain how to identify the local maximum and local minimum points on \( f(x) \) if they exist. Write the explanation in a big textbox near the table in Excel.

Exercise 3.1.5 Find all local maximum and minimum points (if they exist) of the function
\[ h(x) = x^3 - 7.5x^2 + 33x + 36 \]
using calculus. Show all work/clearly explain how to find these points using calculus by using Excel to create the necessary table and then writing your explanation in a textbox next to the table. Note that this function does not have any critical points because the discriminant is negative.

Exercise 3.1.6: Find all local maximum and minimum points (if they exist) of the function
\[ f(x) = 7x\ln(0.05x) + 500 \]
using calculus. Show all work/clearly explain how to find these points using calculus by using Excel to create the necessary table and then writing your explanation in a textbox next to the table. Hint: this function has exactly one critical point somewhere on the interval 0<x<20
Exercise 3.1.7: Use calculus to find all local maximum and minimum points (if they exist) of the function

\[ f(x) = 25xe^{-0.04x} \]

(a) Use the given f(x) function to find f'(x), enter the functions in Excel in order to re-create this table, and use solver to find the critical points as shown in this table.

(b) Then use the first derivative test to clearly explain how to identify the local maximum and local minimum points on f(x) if they exist. Write the explanation in a big textbox near the table in Excel.
Find the best solution to each of the following. Show your work and defend each answer analytically, graphically, numerically or with a written sentence. An Excel spreadsheet MAY BE used, however intermediate steps and process MUST be shown. PLEASE use COMPLETE sentences to answer each question.

**Exercise 3.1.8:** In the following cost functions where \( x \) is the number of items produced (in hundreds), determine the average cost function, \( C(x) = \frac{c(x)}{x} \), then find the minimum average cost by using methods of calculus and Excel.

(a) \( C(x) = \frac{1}{2} x^3 + 2 x^2 - 3x + 35 \)

(b) \( C(x) = 10 + 20 x^{1/2} + 16 x^{3/2} \)

**Exercise 3.1.9:** If the price charged for a candy bar is \( p(x) \) cents, when \( x \) thousand candy bars are sold in a certain city the price function is given by \( p(x) = 160 - \frac{x}{10} \).

(a) Find an expression \( R(x) \), for the Total Revenue from the sale of \( x \) thousand candy bars.

(b) Find the value of \( x \) that leads to maximum revenue.

(c) Find the maximum revenue.

**Exercise 3.1.10:** The sale of compact discs of “lesser” performers is very sensitive to price. If a CD manufacturer charges \( p(x) \) dollars per CD, where, \( p(x) = 12 - \frac{x}{8} \) then \( x \) thousand CD’s will be sold.

(a) Find an expression \( R(x) \), for the Total Revenue from the sale of \( x \) thousand CD’s.

(b) Find the value of \( x \) that leads to maximum revenue.

(c) Find the maximum revenue.
Exercise 3.1.11: The cost, in dollars, to produce \( x \) designer dog leashes is \( C(x) = x + 5 \). The price-demand function, in dollars per leash, is \( p(x) = 115 - 3x \).

(a) First find the revenue function \( R(x) \). [Hint: Revenue = quantity times price] Now using the revenue and cost functions find and simplify the profit function.

(b) Use Excel to find the number of leashes which need to be sold to maximize the profit. Write your answer in a full contextual sentence.

(c) Use Excel to find the maximum profit. Write your answer in a full contextual sentence.

(d) Use Excel to find the price to charge per leash to maximize profit. Write your answer in a full contextual sentence.

Exercise 3.1.12: Stephanie makes and sells backpack patches. The total cost in dollars for her to make \( q \) patches is given by \( C(q) = 75 + 2q + .015q^2 \). Find the quantity that minimizes her average cost for making patches.
Unit 3 Section 2: Optimization and Finding All Extremes

Global Maxima and Minima

You should go back and review the information where we defined absolute and local maxima and minima before you work on this section.

In applications, we often want to find the global extreme; knowing that a critical point is a local extreme is not enough.

For example, if we want to make the greatest profit, we want to make the absolutely greatest profit of all. How do we find global max and min?

There are just a few additional things to think about.

Endpoint Extremes

The local extremes of a function occur at critical points – these are points in the function that we can find by thinking about the shape (and using the derivative to help us). But if we’re looking at a function on a closed interval, the endpoints could be extremes.

These endpoint extremes are not related to the shape of the function; they have to do with the interval, the window through which we’re viewing the function.

In the graph above, it appears that there are three critical points – one local min, one local max, and one that is neither one. But the global max, the highest point of all, is at the left endpoint. The global min, the lowest point of all, is at the right endpoint.

How do we decide if endpoints are global max or min? It’s easier than you expected – simply plug in the endpoints, along with all the critical numbers, and compare y-values.

Example 1

Find the global max and min of \( f(x) = x^3 - 3x^2 - 9x + 5 \) for \(-2 \leq x \leq 6\).

\[ f'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3). \] We need to find critical points, and we need to check the endpoints.

\[ f'(x) = 3(x + 1)(x - 3) = 0 \] when \( x = -1 \) and \( x = 3 \).
The endpoints of the interval are \( x = -2 \) and \( x = 6 \).

Now we simply compare the values of \( f(x) \) at these 4 values of \( x \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>-22</td>
</tr>
<tr>
<td>6</td>
<td>59</td>
</tr>
</tbody>
</table>

The global minimum of \( f(x) \) on \([-2, 6]\) is \(-22\), when \( x = 3 \), and the global maximum of \( f(x) \) on \([-2, 6]\) is \( 59 \), when \( x = 6 \).

**If there’s only one critical point**

If the function has only one critical point and it’s a local max (or min), then it must be the global max (or min). To see this, think about the geometry. Look at the graph on the left – there is a local max, and the graph goes down on either side of the critical point. Suppose there was some other point that was higher – then the graph would have to turn around. But that turning point would have shown up as another critical point. If there’s only one critical point, then the graph can never turn back around.

![Graph showing local max and min](image)

**When in doubt, graph it and look.**

If you are trying to find a global max or min on an open interval (or the whole real line), and there is more than one critical point, then you need to look at the graph to decide whether there is a global max or min. Be sure that all your critical points show in your graph, and that you go a little beyond – that will tell you what you want to know.

**Example 2**

Find the global max and min of \( f(x) = x^3 - 6x^2 + 9x + 2 \).

In Example 4 of section 3.1, we found that \((1, 6)\) is a local max and \((3, 2)\) is a local min. This is not a closed interval, and there are two critical points, so we must turn to the graph of the function to find global max and min.

The graph of \( f(x) \) shows that points to the left of \( x = 4 \) have \( y \)-values greater than 6, so \((1, 6)\) is not a global max. Likewise, if \( x \) is negative, \( y \) is less than 2, so \((3, 2)\) is not a global min. There are no endpoints, so we’ve exhausted all the possibilities. This function does not have a global maximum or minimum.
To find Global Extremes:

The only places where a function can have a global extreme are critical points or endpoints.

(a) If the function has only one critical point, and it’s a local extreme, then it is also the global extreme.

(b) If there are endpoints, find the global extremes by comparing y-values at all the critical points and at the endpoints.

(c) When in doubt, graph the function to be sure.

Extreme Value Theorem:

On a closed interval \( x, [a, b] \) a continuous function \( f(x) \) will attain a global maximum and will also attain a global minimum.

Note: The absolute max and absolute min will occur at critical points OR at the endpoints. So, to find the absolute max and min, we should determine how high/low the function \( f(x) \) is at each endpoint and at each critical point.

How to find the absolute maximum/minimum on a closed interval:

1. Find all critical points.
2. Evaluate \( f(x) \) at each critical point and evaluate \( f(x) \) at each end point. The highest y-value is the absolute max, and the lowest is the absolute min.

Related Exercises You Should Complete Now

Make sure you work through ALL the exercises in this section. Some are mechanical, others are contextual in nature. You MUST make sure you are comfortable with ALL the ideas in the exercises. Remember, you have written solutions and videos for these exercises in your course.
Exercises for Unit 3 Section 2

Exercise 3.2.1: Find all critical points and global extremes of each function on the given intervals.
\( f(x) = x - e^x \) on the interval [-2, 2].

1) Find all critical points.

2) Evaluate \( f(x) \) at each critical point and evaluate \( f(x) \) at each end point. The highest y-value is the absolute max, and the lowest is the absolute min.

Exercise 3.2.2: The revenue earned from selling X-bots is modeled using the formula
\( R(x) = 0.2x^3 - 2.4x^2 + 9.6x \) where \( x \) is the number of items sold in thousands, and \( R(x) \) is the revenue in millions.

(a) Use Excel to find the value of \( R' \) (4.283) and interpret the real-world, contextual meaning.

(b) Assuming the company is only capable of producing at most 10,000 items, use Excel to find the number of items that will produce the highest revenue and the number that produces the lowest revenue, and identify the revenue at each of those times.

(c) Using Excel, produce a graph of the function. Using calculus, identify the significant features (critical point(s), absolute minimum, absolute maximum) in the given interval.
Exercise 3.2.3: Andy’s Store experienced some bad publicity and their profit was adversely affected. The profit they earned, in thousands of dollars, \( x \) weeks after the incident is modeled by the function \( P(x) = x \ln(0.01x) + 500 \).

(a) Use Excel to find the value of \( P'(22) \) and interpret the real-world, contextual meaning.

(b) From week 0.1 up to week 60, use Excel to find the week in which profit was lowest and the week in which profit was highest, and identify the profit at each of those times. (Why can’t you start at \( x = 0 \) ?)

(c) Using Excel, produce a graph of the function. Using calculus, identify the significant features (critical point(s), absolute minimum, absolute maximum) in the given interval.
**Exercise 3.2.4:** The function $f(x)$ gives the profit a company earns, in millions of dollars, when charging $x$ dollars per item for their product. **The company has been directed that they are not allowed to charge more than $2.00 per item.** The function $f(x)$ has exactly one critical point. Use the table of values for $f(x)$, and $f'(x)$ given below to answer some questions about the company profit.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>45</td>
<td>13</td>
<td>30</td>
</tr>
<tr>
<td>0.5</td>
<td>54.25</td>
<td>22</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>65</td>
<td>19</td>
<td>-18</td>
</tr>
<tr>
<td>1.5</td>
<td>71.25</td>
<td>4</td>
<td>-42</td>
</tr>
<tr>
<td>2</td>
<td>67</td>
<td>-23</td>
<td>-66</td>
</tr>
</tbody>
</table>

(a) When will profit be the absolute lowest? Identify the correct statement below. Make sure you can explain why you chose the answer you did.

- The profit will be lowest when charging a price of $0.01 per item.
- The profit will be lowest when charging a price somewhere between a price of $0.50 per item and $1.00 per item.
- The profit will be lowest when charging a price somewhere between a price of $1.50 per item and $2.00 per item.
- The profit will be lowest when charging a price of $1.50 per item.
- The profit will be lowest when charging a price of $2.00 per item.

(b) When will profit be the absolute highest? Identify the correct statement below. Make sure you can explain why you chose the answer you did.

- The profit will be highest when charging a price of $0.01 per item.
- The profit will be highest when charging a price somewhere between a price of $0.50 per item and $1.00 per item.
- The profit will be highest when charging a price somewhere between a price of $1.50 per item and $2.00 per item.
- The profit will be highest when charging a price of $1.50 per item.
- The profit will be highest when charging a price of $2.00 per item.
Unit 3 Section 3: Second Derivative and Concavity

Recall from section 1.2:

Graphically, a function is **concave up** if its graph is curved with the opening upward (a in the figure). Similarly, a function is **concave down** if its graph opens downward (b in the figure).

This figure shows the concavity of a function at several points. Notice that a function can be concave up regardless of whether it is increasing or decreasing.

Remember the example we considered in Unit 1 Section 2: **An Epidemic:** Suppose an epidemic has started, and you, as a member of congress, must decide whether the current methods are effectively fighting the spread of the disease or whether more drastic measures and more money are needed. In the figure below, $f(x)$ is the number of people who have the disease at time $x$, and two different situations are shown. In both (a) and (b), the number of people with the disease, $f(now)$, and the rate at which new people are getting sick, $f'(now)$, are the same. The difference in the two situations is the concavity of $f(x)$, and that difference in concavity might have a big effect on your decision.

In (a), $f(x)$ is concave down at "now", the slopes are decreasing, and it looks as if it’s tailing off. We can say “$f(x)$ is increasing more slowly.” It appears that the current methods are starting to bring the epidemic under control.

In (b), $f(x)$ is concave up, the slopes are increasing, and it looks as if it will keep increasing faster and faster. It appears that the epidemic is still out of control.
The differences between the graphs come from whether the *derivative* is increasing or decreasing.

The derivative of a function $f(x)$ is a function that gives information about the slope of $f(x)$. The derivative tells us if the original function is increasing or decreasing.

Because $f'(x)$ is a function, we can take its derivative. This second derivative also gives us information about our original function $f(x)$. The second derivative gives us a mathematical way to tell how the graph of a function is curved. The second derivative tells us if the original function is concave up or down.

The Second Derivative

**Second Derivative**  \[ y = f(x) \]

The second derivative $f''(x)$ is the derivative of $y' = f'(x)$.

Using prime notation, this is $f''(x)$ or $y''$. You can read this aloud as “y double prime.”

Using Leibniz notation, the second derivative is written $\frac{d^2y}{dx^2}$ or $\frac{d^2f}{dx^2}$. This is read aloud as “the second derivative of $f(x)$.

If $f''(x)$ is positive on an interval, the graph of $y = f(x)$ is **concave up** on that interval.

Remember what we said in Unit 1 Section 2: If a function is increasing and concave up, it is increasing more rapidly. However, if a function is decreasing and concave up, it is decreasing more slowly. Make sure you understand how both cases have the first derivative values increasing as you move from left to right! Therefore, we can technically say that $f(x)$ is increasing (or decreasing) at an *increasing rate*. However, we feel it is clearer and more descriptive to use the language introduced in Unit 1 Section 2.

If $f''(x)$ is negative on an interval, the graph of $y = f(x)$ is **concave down** on that interval.

Remember what we said in Unit 1 Section 2: If a function is increasing and concave down, it is increasing less rapidly or more slowly. However, if a function is decreasing and concave down, it is decreasing more rapidly. Again, make sure you understand how both cases have the first derivative values decreasing as you move from left to right. We can say that $f$ is increasing (or decreasing) at a *decreasing rate*. Again, it is clearer and more descriptive to use the language introduced in Unit 1 Section 2.
Example 1

Find \( f''(x) \) for \( f(x) = 3x^7 \)

First, we need to find the first derivative:
\[
 f'(x) = 21x^6
\]

Then we take the derivative of that function:
\[
 f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}(21x^6) = 126x^5
\]

If \( f(x) \) represents the position of a particle at time \( x \), then \( v(x) = f'(x) \) will represent the velocity (rate of change of the position) of the particle and \( a(x) = v'(x) = f''(x) \) will represent the acceleration (the rate of change of the velocity) of the particle.

You are probably familiar with acceleration from driving or riding in a car. The speedometer tells you your velocity (speed). When you leave from a stop and press down on the accelerator, you are accelerating - increasing your speed.

Example 2

The height (feet) of a particle at time \( t \) seconds is \( f(t) = t^3 - 4t^2 + 8t \). Find the height, velocity and acceleration of the particle when \( t = 0, 1, \) and \( 2 \) seconds.

\[
 f(t) = t^3 - 4t^2 + 8t \quad \text{so} \quad f(0) = 0 \text{ feet}, f(1) = 5 \text{ feet}, \text{ and } f(2) = 8 \text{ feet}.
\]

The velocity is \( v(t) = f'(t) = 3t^2 - 8t + 8 \) so \( v(0) = 8 \text{ ft/s}, v(1) = 3 \text{ ft/s}, \) and \( v(2) = 4 \text{ ft/s}. \) At each of these times the velocity is positive, and the particle is moving upward, increasing in height.

The acceleration is \( a(t) = f''(t) = 6t - 8 \) so \( a(0) = -8 \text{ ft/s}^2, a(1) = -2 \text{ ft/s}^2 \) and \( a(2) = 4 \text{ ft/s}^2. \)

At time 0 and 1, the acceleration is negative, so the particle's velocity would be decreasing at those points - the particle was slowing down. At time 2, the velocity is positive, so the particle was increasing in speed.

Inflection Points

Recall our Definition of Inflection Points in Unit 1 Section 2

Definition: An inflection point is a point on the graph of a function where the concavity of the function changes, from concave up to down or from concave down to up.
Inflection points happen when the concavity changes. Because we know the connection between the concavity of a function and the sign of its second derivative, we can use this to find inflection points.

Which of the labeled points in the graph below are inflection points?

Solution: Point b is a POI because the function changes from being concave up to concave down at this point.

Point g is a POI because the function changes from being concave down to concave up at this point.

Working Definition: An inflection point is a point on the graph where the second derivative changes sign.

In order for the second derivative to change signs, it must either be zero or be undefined. So to find the inflection points of a function we only need to check the points where \( f''(x) \) is 0 or undefined.

Note that it is not enough for the second derivative to be zero or undefined. We still need to check that the sign of \( f''(x) \) changes sign. The functions in the next example illustrate what can happen.

Example 3

Let \( f(x) = x^3 \), \( g(x) = x^4 \) and \( h(x) = x^{\frac{1}{3}} \). For which of these functions is the point (0,0) an inflection point?

Graphically, it is clear that the concavity of \( f(x) = x^3 \) and \( h(x) = x^{1/3} \) changes at (0,0), so (0,0) is an inflection point for \( f(x) \) and \( h(x) \). The function \( g(x) = x^4 \) is concave up everywhere so (0,0) is not an inflection point of \( g(x) \).
We can also compute the second derivatives and check the sign change.

If \( f(x) = x^3 \), then \( f'(x) = 3x^2 \) and \( f''(x) = 6x \).

- The only point at which \( f''(x) = 0 \) or is undefined (meaning that \( f'(x) \) is not differentiable) is at \( x = 0 \).
- If \( x < 0 \), then \( f''(x) < 0 \) so \( f(x) \) is concave down.
- If \( x > 0 \), then \( f''(x) > 0 \) so \( f(x) \) is concave up.
- At \( x = 0 \) the concavity changes so the point \( (0, f(0)) = (0,0) \) is an inflection point of \( f(x) \).

If \( g(x) = x^4 \), then \( g'(x) = 4x^3 \) and \( g''(x) = 12x^2 \).

- The only point at which \( g''(x) = 0 \) or is undefined is at \( x = 0 \).
- If \( x < 0 \), then \( g''(x) > 0 \) so \( g(x) \) is concave up.
- If \( x > 0 \), then \( g''(x) > 0 \) so \( g(x) \) is also concave up.
- At \( x = 0 \) the concavity does not change so the point \( (0, g(0)) = (0,0) \) is not an inflection point of \( g(x) \).
- Keep this example in mind!

If \( h(x) = x^\frac{1}{3} \), then \( h'(x) = \frac{1}{3} x^{-\frac{2}{3}} \) and \( h''(x) = -\frac{2}{9} x^{-\frac{5}{3}} \).

- \( h''(x) \) is not defined if \( x = 0 \), but \( h''(\text{negative number}) > 0 \) and \( h''(\text{positive number}) < 0 \) so \( h \) changes concavity at \((0,0)\) and \((0,0)\) is an inflection point of \( h(x) \).

So how is the second derivative related to the original function (what Does \( f'' \) Say About \( f \) )?

Since \( f''(x) \) is the derivative of \( f'(x) \):

- If \( f''(x) > 0 \) on an interval, then \( f'(x) \) is increasing over that interval.
- If \( f''(x) < 0 \) on an interval, then \( f'(x) \) is decreasing over that interval.

Putting it all together!

\[
\begin{align*}
 f' > 0, & \quad f \uparrow \quad f' < 0, & \quad f \downarrow \\
 f'' > 0, & \quad f' \uparrow, & \quad f \cup \\
 f'' < 0, & \quad f' \downarrow, & \quad f \cap
\end{align*}
\]
Second Derivative Information About Curve Shape

Until now, we've only used first derivative information, but we could also use information from the second derivative to provide more information about the shape of the function.

**Second Derivative Information about Shape**

(i) if \( f(x) \) is concave up on \((a, b)\), then \( f''(x) \geq 0 \) for all \( x \) in \((a, b)\).

(ii) if \( f(x) \) is concave down on \((a, b)\), then \( f''(x) \leq 0 \) for all \( x \) in \((a, b)\).

The converse of both of these are also true:

(i) if \( f''(x) \geq 0 \) for all \( x \) in \((a, b)\), then \( f(x) \) is concave up on \((a, b)\).

(ii) if \( f''(x) \leq 0 \) for all \( x \) in \((a, b)\), then \( f(x) \) is concave down on \((a, b)\).

**Related Exercises You Should Complete Now**

Complete Exercises 3.3.1 through 3.3.4. Remember, you have written solutions and videos for these exercises in your course.

**Example 6**

Use information about the values of \( f''(x) \) to help determine the intervals on which the function \( f(x) = x^3 - 6x^2 + 9x + 1 \) is concave up and concave down.

For concavity, we need the second derivative: \( f''(x) = 3x^2 - 12x + 9 \), so \( f''(x) = 6x - 12 \).

To find possible inflection points, set the second derivative equal to zero.
\[ 6x - 12 = 0 \], so \( x = 2 \). This divides the real number line into two intervals: \((-\infty, 2)\) and \((2, \infty)\).

For \( x < 2 \), we can see from our Excel table that the second derivative is negative so \( f(x) \) is concave down. Furthermore, we can see that the values of \( f''(x) \) are decreasing on the interval \((-\infty, 2)\).

For \( x > 2 \), the second derivative is positive, so \( f(x) \) is concave up. Furthermore, we can see that the values of \( f'(x) \) are increasing on the interval \((2, \infty)\).
Example 7

Use information about the values of $f'(x)$ and $f''(x)$ to help graph $f(x) = x^{2/3}$.

$f'(x) = \frac{2}{3}x^{-1/3}$. This is undefined at $x = 0$. Remember, a negative exponent means that you can think of this being equivalent to $f'(x) = \frac{2}{3x^{1/3}}$. This would create a 0 in the denominator at $x = 0$.

$f''(x) = -\frac{2}{9}x^{-4/3}$. This is also undefined at $x = 0$.

This creates two intervals: $x < 0$, and $x > 0$.

On the interval $x < 0$, we see from our Excel table that both $f'(x)$ and $f''(x)$ are negative values. We can conclude that the function $f(x)$ is decreasing (because the derivative is negative) and concave down (because the second derivative is negative) on this interval. Further note that we see $f'(x)$ decreasing in this interval, another way to see evidence that $f(x)$ is concave down.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-1.259921</td>
<td>-0.2645668</td>
<td>-0.5599649</td>
</tr>
<tr>
<td>-1.5</td>
<td>-1.1447142</td>
<td>-0.2911935</td>
<td>-0.3815714</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-0.2222222</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.79370053</td>
<td>-0.4199737</td>
<td>-0.0881889</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>Not defined</td>
<td>Not defined</td>
</tr>
<tr>
<td>0.5</td>
<td>0.79370053</td>
<td>0.41997368</td>
<td>-0.0881889</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.3333333333</td>
<td>-0.2222222</td>
</tr>
<tr>
<td>1.5</td>
<td>1.14471424</td>
<td>0.29119349</td>
<td>-0.3815714</td>
</tr>
<tr>
<td>2</td>
<td>1.25992105</td>
<td>0.26456684</td>
<td>-0.5599649</td>
</tr>
</tbody>
</table>

On the interval $x > 0$, we see that both $f'(x)$ is positive and $f''(x)$ is negative. We can conclude that the function $f(x)$ is increasing and concave down on this interval. Again, notice that the values of $f'(x)$ are decreasing in this interval, another way to see evidence that $f(x)$ is concave down.

We can also calculate that $f(0) = 0$, giving us a base point for the graph. Using this information, we can conclude the graph must look like the graph shown:

---

**Related Exercises You Should Complete Now**

Complete Exercises 3.3.6 through 3.3.9 and 3.3.11 *NOTE: For 3.3.11 remember the Extreme Value Theorem. Also complete 3.3.12. Remember, you have written solutions and videos for these exercises in your course.*
Sketching without an Equation

Of course, graphing calculators and computers are great at graphing functions. Calculus provides a way to illuminate what may be hidden or out of view when we graph using technology. More importantly, calculus gives us a way to look at the derivatives of functions for which there is no equation given.

We can summarize all the derivative information about shape in a table.

<table>
<thead>
<tr>
<th>Summary of Derivative Information about the Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
</tr>
<tr>
<td>$f'(x)$</td>
</tr>
<tr>
<td>$f''(x)$</td>
</tr>
</tbody>
</table>

When $f'(x) = 0$, the graph of $f(x)$ may have a local max or min.
When $f''(x) = 0$, the graph of $f(x)$ may have an inflection point.

Example 8

A company's bank balance, $B$, in millions of dollars, $t$ weeks after releasing a new product is shown in the graph below. Sketch a graph of the marginal balance - the rate at which the bank balance was changing over time.

![Graph of bank balance over time](image)

We will follow the same type of technique we used in Chapter 2, but will add information about the 2nd derivative.

Notice that since the tangent line will be horizontal at about $t = 0.6$ and $t = 3.2$, the derivative will be 0 at those points.

There appear to be inflection points at about $t = 1.5$ and $t = 5.5$. At these points, the derivative will be changing from increasing to decreasing or vice versa, so the derivative will have a local max or min at those points.

Remember the steps we have taken in Chapter 2 to capture the behavior of the function by using a clear table.
## Input Values

<table>
<thead>
<tr>
<th>Behavior of $B(t)$</th>
<th>What this tells us about derivative $B'(t)$</th>
<th>What this tells us about the second derivative $B''(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smooth turning points</td>
<td>Derivative is 0</td>
<td></td>
</tr>
</tbody>
</table>

### Consider input values where the function turns

- $t = .6$ and $3.2$

### Use the input values from above to consider intervals where the function is Increasing or Decreasing

<table>
<thead>
<tr>
<th>Interval</th>
<th>Behavior of $B(t)$</th>
<th>Derivative is negative–Graph of the derivative is below the t axis.</th>
<th>Derivative is positive–Graph of the derivative is above the t axis.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, .6)$</td>
<td>Decreasing</td>
<td>$t = 1.5$ and $5.5$ we have points of inflection.</td>
<td>$t = 1.5$ and $5.5$.</td>
</tr>
<tr>
<td>$(.6, 3.2)$</td>
<td>Increasing</td>
<td>Derivative turns at $t = 1.5$ and $5.5$.</td>
<td>Second derivative $= 0$ at $1.5$ and $5.5$.</td>
</tr>
<tr>
<td>$(3.2, \infty)$</td>
<td>Decreasing</td>
<td>Derivative is negative–Graph of the derivative is below the t axis.</td>
<td></td>
</tr>
</tbody>
</table>

### Consider input values where the concavity changes

- $t = 1.5$ and $5.5$

### Use these input values to consider intervals for concavity

<table>
<thead>
<tr>
<th>Interval</th>
<th>Behavior of $B(t)$</th>
<th>Derivative is increasing.</th>
<th>Derivative is decreasing.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 1.5)$</td>
<td>Concave up</td>
<td>Second derivative is positive, so graph of the second derivative is above the t axis.</td>
<td>Second derivative is negative, so graph of the second derivative is below the t axis.</td>
</tr>
<tr>
<td>$(1.5, 5.5)$</td>
<td>Concave down</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(5.5, \infty)$</td>
<td>Concave up</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If we wanted a more accurate sketch of the derivative function, we could also estimate the derivative at a few points:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$B'(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-10</td>
</tr>
<tr>
<td>1.5</td>
<td>3</td>
</tr>
<tr>
<td>$t &gt; 5.5$</td>
<td>-1</td>
</tr>
</tbody>
</table>

---

Now we can sketch the derivative. We know a few values for the derivative function, and on each interval we know the shape we need. We can use this to create a rough idea of what the graph of the derivative, \( B'(t) \), should look like.

![Graph of the derivative]

Smoothing this out gives us a good estimate for the graph of the derivative of \( B(t) \).

![Smoothed graph]

**Law of Diminishing Returns**

The law of diminishing returns in economics is related to the idea of concavity.

If the input of this function were advertising costs for some product, the output might be the corresponding revenue from sales.

Here from \( x = 0 \) to the point shown, the curve is increasing and concave upward, so that \( f''(x) \) is positive.

This means that \( f(x) \) is increasing more and more rapidly up to the point shown, or at an increasing rate.

From the point shown to \( x = 6 \), the curve is increasing and concave downward, so that \( f''(x) \) is negative.
This means that $f(x)$ is increasing more and more slowly, or at a decreasing rate.

The inflection point is called the point of diminishing returns because beyond this point there is a smaller and smaller return for each dollar invested on advertising. Keep in mind that this idea will only apply when the function is increasing and changing from concave up to concave down.

**NOTE:** Go back and review previous material that talks about what happens for each point of inflection in the various scenarios that can occur. This was covered in Unit 1, Section 2.

**Related Exercises You Should Complete Now**

**Complete Exercises 3.3.10 and 3.3.12.** Remember you have written solutions and videos for these exercises in your course.
Exercises for Unit 3 Section 3

Exercise 3.3.1: Each quotation is a statement about a quantity of something changing over time. Let $f(t)$ represent the quantity at time $t$. For each quotation, tell what $f$ represents and whether the first and second derivatives of $f$ are positive or negative.

(a) "Unemployment rose again, but the rate of increase is smaller than last month."

(b) "Our profits declined again, but at a slower rate than last month."

(c) "The population is still rising and at a faster rate than last year."

Exercise 3.3.2: On which intervals of $x$ is the function in this graph, a) concave up? b) concave down?

Exercise 3.3.3: Find the derivative and second derivative of each function.

a) $f(x) = (6x - x^2)^{10}$

b) $g(x) = \ln (x^2 + 4)$
Exercise 3.3.4: Fill in the table with “0”, “+” or “-” Explain how you know these are true.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Exercise 3.3.5: Explain and graph what you know about the function $f(x)$ from each given set of information.

(a) We know that $f' (4)$ is negative and we know that $f'' (4)$ is positive. What do you know about the function $f(x)$ specifically? Sketch $f(x)$ at $x = 4$.

(b) We know that $g' (-5)$ is positive and we know that $g'' (-5)$ is positive. What do you know about the function $g(x)$ specifically? Sketch $g(x)$ at $x = -5$. 
(c) We know that \( P'(7) \) is positive and we know that \( P''(7) \) is negative. What do you know about the function \( P(t) \) specifically? Sketch \( P(t) \) at \( t = 7 \).

(d) We know that \( R'(20) \) is negative and we know that \( R''(20) \) is negative. What do you know about the function \( R(Q) \) specifically? Sketch \( R(Q) \) at \( Q = 20 \).

**Exercise 3.3.6:** Suppose that \( P(x) = -5x^3 + 25x^2 - 4x - 8 \) gives the profit of a company in hundreds of dollars, where \( x \) is the number of items sold in thousands.

(a) Use Excel and methods of calculus to identify and justify:
   - all local maximums, and local minimums of the function \( P(x) \).
   - all inflection points on \( P(x) \).

(b) Graph the function in Excel. Sketch the function here. Label the points you found in part (a).

(c) In a text box in Excel, write complete sentences to explain the real-world, practical significance of the maximum/minimum points as well as the inflection point.
**Exercise 3.3.7:** Suppose that \( f(x) = 50x e^{-0.04x} \) gives the profit, in thousands, when a company spends \( x \) dollars on advertising each day.

**(a)** Use Excel and methods of calculus to identify and justify:
- all local maximums, and local minimums of the function \( f(x) \).
  - **Note that there is exactly one critical point.**
- all inflection points on \( f(x) \).

**(b)** Identify the global maximum and global minimum on the interval \( x: [0,200] \).

**(c)** Graph the function in Excel. Make sure you can clearly see the points you identified in parts a and b.

**(d)** In a text box in Excel, write complete sentences to explain the real-world, practical significance of the maximum/minimum points as well as the inflection point.

**Exercise 3.3.8:** The median size of a new single-family house built in the U.S. between 1987 and 2001 can be modeled as

\[
h(x) = 0.359x^3 - 15.198x^2 + 221.738x + 826.514
\]

where \( x \) is the number of years since 1980, and \( h(x) \) is the size in square feet. You should recognize that we want to consider this function on the interval \([7, 21]\).

**(a)** Find \( h'(x) \) and \( h''(x) \).

**(b)** Are there any relative minimum or relative maximum points? Where is the function increasing? Where is it decreasing? Use derivative values to answer the question. Explain/show the process for how to figure this out using Excel and methods of calculus.

**(c)** Find any inflection points. Where is the function concave up? Where is the function concave down? Explain/show the process for how to figure this out using Excel and methods of calculus.

**(d)** In a text box in Excel, write complete sentences to explain the real-world, practical significance of the inflection point.
**Exercise 3.3.9:** Using the graph below complete the chart by indicating whether the function is positive, negative, or zero at the indicated points. Make sure you can explain WHY you give the answers you choose.

<table>
<thead>
<tr>
<th></th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f''(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>e</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>f</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>g</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>i</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Exercise 3.3.10:** In the following exercises find the point of diminishing returns \((x, y)\) for the given functions, where \(R(x)\), represents revenue in thousands of dollars and \(x\) represents the amount spent on advertising in thousands of dollars.

(a) \(R(x) = 10000 - x^3 + 42x^2 + 800x, 0 \leq x \leq 20\)

(b) \(R(x) = \frac{4}{7}(-x^3 + 66x^2 + 1050x - 400), 0 \leq x \leq 25\)
(c) \( R(x) = -0.3x^3 + x^2 + 11.4x, 0 \leq x \leq 6 \)

**Exercise 3.3.11:** The projected year-end assets in the Social Security trust funds, in trillions of dollars, where \( t \) represents the number of years since 2000, can be approximated by:

\[
A(t) = 0.0000329t^3 - 0.00450t^2 + 0.0613t + 2.34, 0 \leq t \leq 50
\]

(a) Use Excel to find the absolute maximum and absolute minimum value of the trust funds and when they occur.

(b) Use Excel, and the second derivative to find the value of \( t \) when the assets will decrease most rapidly. Approximately when does this occur?

**Exercise 3.3.12:** Fill in the blank with the best possible answer.

1. If \( f'(c) = 0 \) or \( f'(c) \) does not exist then \( f(c) \) has __________________________.

2. If \( f(x) \) has a change in concavity, then \( f(x) \) has __________________________.

3. If \( f(x) \) is decreasing, then \( f'(x) \) is __________________________.

4. \( f'(x) \) is positive if \( f(x) \) is __________________________.

5. \( f''(x) \) is positive if \( f(x) \) is __________________________.

6. If \( f(x) \) is concave down, then \( f'(x) \) is __________________________.

7. \( f'(x) \) is decreasing, then \( f(x) \) is __________________________.

8. \( f'(x) \) is increasing, then \( f''(x) \) is __________________________.

9. If \( f'(x) > 0 \) and \( f''(x) > 0 \), then \( f(x) \) is ___________ and ___________.

10. If \( f(x) \) is an exponential growth curve, then \( f'(x) \) is ______________ and ______________.
11. If $f(x)$ has a horizontal tangent at $x = c$, then $f'(c) \underline{\hspace{3cm}}$.

12. If $f(x)$ has a sharp corner at $x = 1$, then $f'(1) \underline{\hspace{3cm}}$.

13. If $f''(x)$ has a change in sign and is always defined, then $f(x)$ has \underline{\hspace{3cm}}.

14. If $f'(b) = 0$ and $f'(x)$ changes sign from positive to negative around $x = b$, then $f(x)$ has \underline{\hspace{3cm}} at \underline{\hspace{3cm}}.

15. If $f''(x) > 0$ for all values of $x$, then $f(x)$ is always \underline{\hspace{3cm}}.

16. If $f''(x) = 0$ for all values of $x$, then $f(x)$ is \underline{\hspace{3cm}}.

17. If $f(x)$ is an exponential decaying function, then $f(x)$ is \underline{\hspace{3cm}} and \underline{\hspace{3cm}}.

18. If $f''(c) = 0$ and $f'''(x) < 0$ for all $x < c$ and $f'''(x) > 0$ for all $x > c$, then $f(c)$ is \underline{\hspace{3cm}}.
Unit 3 Section 4: Local Linearity and Elasticity

Local Linearity

For any function \( f(x) \), the tangent is a close approximation of the function for some small distance from the tangent point.

The linearization is the equation of the tangent line. Start with the point/slope equation:

\[
 y - y_1 = m(x - x_1)
\]

Solve for \( y \) by adding \( y_1 \) to both sides.

\[
 y = m(x - x_1) + y_1
\]

Replace the slope with the derivative of \( f \) evaluated at \( a \), and the value of \( y_1 \) with \( f(a) \). Then the linearization of a curve \( f(x) \) at \( x = a \) is given by:

\[
 L(x) = f'(a)(x - a) + f(a)
\]

Where \( x_1 = a, \ y_1 = f(a) \quad m = f'(a) \)

\( f(x) \approx L(x) \) is the standard linear approximation of \( f \) at \( a \).

If a function is concave down on an interval, the tangent line to the curve will above the curve, therefore the tangent line approximation (linearization) will be an overestimate of the actual value. If a function is concave up on an interval, the tangent line to the curve will below the curve, therefore the tangent line approximation (linearization) will be an underestimate of the actual value.

Example 1:

Given \( f(10) = 50 \) and \( f'(10) = 2 \), find \( f(10.25) \) and \( f(9.25) \).

We would first find the equation for the linearization of \( L(x) \) at \( x = 10 \). Remember, this is the equation of the line tangent to the curve at \( x = 10 \), and this line will lie close to the curve.
for values of the input near 10.

From above, we know that: \( L(x) = f'(a)(x - a) + f(a) \)

Using the values \( a = 10, f'(10) = 2 \) and \( f(10) = 50 \) we get \( L(x) = 2(x - 10) + 50 \).

We can now use this formula to find the 2 quantities asked for this question.

\[
\begin{align*}
 f(10.25) & \approx L(10.25) = 2(10.25 - 10) + 50 = 2(.25) + 50 = 50.5 \\
 f(9.5) & \approx L(9.5) = 2 (9.5 - 10) + 50 = 2(-.5) + 50 = 49
\end{align*}
\]

Related Exercises You Should Complete Now

Complete Exercise 3.4.1. Remember you have written solutions and videos for these exercises in your course.

Elasticity

We know that demand functions are decreasing, so when the price increases, the quantity demanded goes down. But what about revenue = price \( \times \) quantity? When the price increases will revenue go down because the demand dropped so much? Or will revenue increase because demand didn’t drop very much?

Elasticity of demand is a measure of how demand reacts to price changes. It’s normalized – that means the particular prices and quantities don’t matter, and everything is treated as a percent change. The formula for elasticity of demand involves a derivative, which is why we’re discussing it here.

Elasticity of Demand

Given a demand function that gives \( q \) in terms of \( p \),

\[
\text{The elasticity of demand is } E = \left| \frac{p \frac{dq}{dp}}{q} \right|
\]

(Note that since demand is a decreasing function of \( p \), the derivative is negative. That’s why we have the absolute values – so \( E \) will always be positive.)

If \( E < 1 \), we say demand is inelastic. In this case, raising prices increases revenue.
If \( E > 1 \), we say demand is elastic. In this case, raising prices decreases revenue.
If \( E = 1 \), we say demand is unitary. \( E = 1 \) at critical points of the revenue function.

Interpretation of elasticity:
If the price increases by 1%, the demand will decrease by \( E\% \).

Example 2

A company sells \( q \) ribbon winders per year at \( Sp \) per ribbon winder. The demand function for ribbon winders is given by \( p = 300 - 0.02q \). Find the elasticity of demand when the price is \$70 apiece. Will an increase in price lead to an increase in revenue?

First, we need to solve the demand equation so it gives $q$ in terms of $p$, so that we can find $\frac{dq}{dp}$.

$p = 300 - 0.02q$ so $q = 15000 - 50p$. Then $\frac{dq}{dp} = -50$.

We need to find $q$ when $p = 70$: $q(70) = 15000 - 50 \times 70 = 11500$.

Now compute $E = \left| \frac{p \cdot dq}{q \cdot dp} \right| = \left| \frac{70 - 50}{11500} \right| \approx 0.3$.

$E < 1$, so demand is inelastic. Increasing the price by 1% would only cause a 0.3% drop in demand. Increasing the price would lead to an increase in revenue, so it seems that the company should increase its price.

The demand for products that people have to buy, such as onions, tends to be inelastic. Even if the price goes up, people still have to buy about the same amount of onions, and revenue will not go down. The demand for products that people can do without, or put off buying, such as cars, tends to be elastic. If the price goes up, people will just not buy cars right now, and revenue will drop.

**Example 3**

A company finds the demand $q$, in thousands, for their kites to be $q = 400 - p^2$ at a price of $p$ dollars. Find the elasticity of demand when the price is $5$ and when the price is $15$. Then find the price that will maximize revenue.

Calculating the derivative, $\frac{dq}{dp} = -2p$. Using this information, as well as the formula $E = \left| \frac{p \cdot dq}{q \cdot dp} \right|$, in Excel, we can create the table shown:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$q'(p)$</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>400</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>399</td>
<td>-2</td>
<td>0.00501253</td>
</tr>
<tr>
<td>2</td>
<td>396</td>
<td>-4</td>
<td>0.02020202</td>
</tr>
<tr>
<td>3</td>
<td>391</td>
<td>-6</td>
<td>0.04603581</td>
</tr>
<tr>
<td>4</td>
<td>384</td>
<td>-8</td>
<td>0.08333333</td>
</tr>
<tr>
<td>5</td>
<td>375</td>
<td>-10</td>
<td>0.13333333</td>
</tr>
<tr>
<td>6</td>
<td>364</td>
<td>-12</td>
<td>0.1978022</td>
</tr>
<tr>
<td>7</td>
<td>351</td>
<td>-14</td>
<td>0.27920228</td>
</tr>
<tr>
<td>8</td>
<td>336</td>
<td>-16</td>
<td>0.38095238</td>
</tr>
<tr>
<td>9</td>
<td>319</td>
<td>-18</td>
<td>0.50783699</td>
</tr>
<tr>
<td>10</td>
<td>300</td>
<td>-20</td>
<td>0.66666667</td>
</tr>
<tr>
<td>11.5470061</td>
<td>266.666651</td>
<td>-23.094012</td>
<td>1.00000018</td>
</tr>
<tr>
<td>12</td>
<td>256</td>
<td>-24</td>
<td>1.125</td>
</tr>
<tr>
<td>13</td>
<td>231</td>
<td>-26</td>
<td>1.46320346</td>
</tr>
<tr>
<td>14</td>
<td>204</td>
<td>-28</td>
<td>1.92156863</td>
</tr>
<tr>
<td>15</td>
<td>175</td>
<td>-30</td>
<td>2.57142857</td>
</tr>
<tr>
<td>16</td>
<td>144</td>
<td>-32</td>
<td>3.55555556</td>
</tr>
<tr>
<td>17</td>
<td>111</td>
<td>-34</td>
<td>5.20720721</td>
</tr>
</tbody>
</table>

We can see from this table that $E(5) \approx 0.133$, so the demand is inelastic when the price is $5.

At a price of $5, a 1\%$ increase in price would decrease demand by only $0.133\%$. Revenue could be raised by increasing prices.

$E(15) \approx 2.571$. The demand is elastic when the price is $15.

At a price of $15, a 1\%$ increase in price would decrease demand by $2.571\%$. Revenue could be raised by decreasing prices.

To maximize the revenue, we would use Solver in Excel to find when $E = 1$. We can see that this occurs when $p \approx 11.55$. A price of $11.55 will maximize the revenue.

**Related Exercises You Should Complete Now**

**Complete Exercises 3.4.2 through 3.4.5.** Remember you have written solutions and videos for these exercises in your course.
Exercises for Unit 3 Section 4

**Exercise 3.4.1:** Suppose we know that \( g(20) = 5 \) and \( g'(20) = 1.4 \). Use this information to approximate \( g(23) \) and \( g(18) \).

**Exercise 3.4.2:** Suppose that customer demand for a product is given by the function 
\[
Q(p) = 500,000e^{-0.25p}
\]
where \( p \) is the price per item (dollars per item) and \( Q \) is the number of items sold.

(a) Find a formula for elasticity of demand, \( E(p) \).

(b) Using Excel, find the price elasticity of demand at a price of $2.35 per item. Is the demand elastic or inelastic at the price? In a text box in Excel, explain the exact meaning of the elasticity value.

(c) Using Excel, find the price elasticity of demand at a price of $8.46 per item. Is the demand elastic or inelastic at the price? In a text box in Excel, explain the exact meaning of the elasticity value.

(d) Use Excel and Solver to identify the exact price when demand will be unit elastic. In a text box in Excel, write a sentence to explain the exact meaning of this value in terms of revenue.

(e) Now go back to the original function \( Q(p) \) and use it to identify a formula for the revenue function. Using Excel, the revenue function, and the first derivative test, identify the maximum revenue. Does your answer match the answer you got in part (d)?

**Exercise 3.4.3:** Suppose that customer demand for a product is given by the function 
\[
Q(p) = -250p^2 - 6000p + 50,000
\]
where \( p \) is the price per item (dollars per item) and \( Q \) is the number of items sold.

(a) Find a formula for elasticity of demand, \( E(p) \).
(b) Using Excel, find the price elasticity of demand at a price of $1.75 per item. Is the demand elastic or inelastic at the price? In a text box in Excel, explain the exact meaning of the elasticity value.

(c) Using Excel, find the price elasticity of demand at a price of $4.83 per item. Is the demand elastic or inelastic at the price? In a text box in Excel, explain the exact meaning of the elasticity value.

(d) Use Excel and Solver to identify the exact price when demand will be unit elastic. In a text box in Excel, write a sentence to explain the exact meaning of this value in terms of revenue.

(e) Now go back to the original function $Q(p)$ and use it to identify a formula for the revenue function. Using Excel, the revenue function, and the first derivative test, identify the maximum revenue. Does your answer match the answer you got in part (d)?

**Exercise 3.4.4:** A company sells $q$ chairs per year at $p$ per chair. The demand function for chairs is given by

$$p = 300 - 0.02q$$

(a) Use Excel to find the elasticity of demand when the price is $70 per chair.

(b) Will an increase in price lead to an increase in revenue?

**Exercise 3.4.5:** A company finds the demand $q$, in thousands, for their kites to be

$$q = 400 - p^2$$

at a price of $p$ dollars. Use Excel to find the elasticity of demand when the price is $5$ and when the price is $15$. Then use Excel to find the price that will maximize revenue.
Unit 4 Section 1: The Integral

Precalculus Idea – The Area of a Rectangle

Distance from Velocity

Definition of the Definite Integral

Approximating with Many Rectangles

Signed Area

How is the definite integral interpreted as area?

The Definite Integral and Signed Area:

Accumulation in Real Life

Setting the Stage to Pull It All Together

Approximating in Various Ways

Exercises for Unit 4 Section 1

Unit 4 Section 2: Antidifferentiation and The Fundamental Theorem of Calculus

Going Backwards – From Derivative to General Antiderivative

General Antiderivatives

Building Blocks

Specific Anti-Derivatives

The Fundamental Theorem of Calculus

Antiderivatives Graphically or Numerically

Antiderivatives of Formulas

Very Important Summary

Summary of Key Ideas

Exercises for Unit 4 Section 2
Unit 4 Section 1: The Integral

The previous chapters dealt with Differential Calculus. We started with the "simple" geometrical idea of the slope of a tangent line to a curve, developed it into a combination of theory about derivatives and their properties, techniques for calculating derivatives, and applications of derivatives. This chapter deals with Integral Calculus and starts with the "simple" geometric idea of area. This idea will be developed into another combination of theory, techniques, and applications.

Precalculus Idea – The Area of a Rectangle

If you look on the inside cover of nearly any traditional math book, you’ll find a bunch of area and volume formulas – the area of a square, the area of a trapezoid, the volume of a right circular cone, and so on. Some of these formulas are complicated. But you still won’t find a formula for the area of a jigsaw puzzle piece or the volume of an egg. There are lots of things for which there is no formula. Yet we might still want to find their areas or volumes.

One reason areas are so useful is that they can represent quantities other than simple geometric shapes. If the units for each side of the rectangle are meters, then the area will have the units meters × meters = square meters = m². But if the units of the base of a rectangle are hours and the units of the height are miles/hour, then the units of the area of the rectangle are hours × miles/hour = miles, a measure of distance. Similarly, if the base units are centimeters and the height units are grams, then the area units are gram-centimeters, a measure of work.

The basic shape we will use is the rectangle; the area of a rectangle is base × height. You should also know the area formulas for:

Triangles, \( A = \frac{1}{2}bh \) and

Circles, \( A = \pi r^2 \) and

Trapezoids, \( A = \frac{1}{2} h(b_1 + b_2) \)

Distance from Velocity

Example 1

Suppose a car travels on a straight road at a constant speed of 40 miles per hour for two hours. See the graph of its velocity below. How far has it gone?
We all remember distance = rate \times time, so this one is easy. The car has gone 40 miles per hour \times 2 hours = 80 miles.

The trouble with our old reliable distance = rate \times time relationship is that it only works if the rate is constant. If the rate is changing, there isn’t a good way to use this formula. But look at the graph from the last example again.

Notice that in Example 1 distance = rate \times time also describes the area between the velocity graph and the t-axis, between t = 0 and t = 2 hours. The rate is the height of the rectangle, the time is the length of the rectangle, and the distance is the area of the rectangle. This is the way we can extend our simple formula to handle more complicated velocities.

**Example 2**

Now suppose that a car travels so that its speed increases steadily from 0 to 40 miles per hour, for two hours. (Just be grateful you weren’t stuck behind this car on the highway.) See the graph of its velocity in below. How far has this car gone?

The distance the car travels is the area between its velocity graph, the t-axis, t = 0 and t = 2. This region is a triangle, so its area is \frac{1}{2}bh = \frac{1}{2}(2 \text{ hours})(40 \text{ miles per hour}) = 40 \text{ miles}. So, the car travels 40 miles during its annoying trip.
In our distance/velocity examples, the function represented a rate of travel (miles per hour), and the area represented the total distance traveled. This principle works more generally:

For functions representing other rates such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the total amount of something.

Example 3

Consider the revenue is growing at a constant rate of 300 $/month. If we draw a graph of the marginal revenue, the accumulated amount of change in the revenue is equal to the area under the horizontal line at 300, over the interval for our time span.

If we want the amount the revenue increased in 4 months, we would simply multiply 300 $/month * 4 months = $1200. NOTICE: The months canceled out! That left us with just dollars ($). This makes sense since Revenue is measured in dollars.

Definition of the Definite Integral

Because the area under the curve is so important, it has a special vocabulary and notation.

The definite integral of a positive function \( f(x) \) over an interval \([a, b]\) is the area between \( f(x) \), the \( x \)-axis, \( x = a \) and \( x = b \).

The definite integral of a positive function \( f(x) \) from \( a \) to \( b \) is the area under the curve between \( a \) and \( b \).

If \( f(t) \) represents a positive rate (in y-units per t-units), then the definite integral of \( f(t) \) from \( a \) to \( b \) is the total y-units that accumulate between \( t = a \) and \( t = b \).

Notation for the Definite Integral:

The definite integral of \( f(x) \) from \( a \) to \( b \) is written \( \int_{a}^{b} f(x) \, dx \).

- The \( \int \) symbol is called an integral sign; it’s an elongated letter S, standing for sum. (The \( \int \) corresponds to the \( \Sigma \) from the Riemann sum)
• The \( dx \) on the end must be included. The \( dx \) tells what the variable is – in this example, the variable is \( x \). (The \( dx \) corresponds to the \( \Delta x \) from the Riemann sum)

• The function \( f(x) \) or \( f(t) \) is called the **integrand**. The \( a \) and \( b \) are called the **limits of integration**.

**Verb forms:**

• We **integrate** or **find the definite integral** of a function. This process is called **integration**.

---

**Example 4**

Using the idea of area, determine the value of \( \int_1^3 (1+x) \, dx \).

\[ \int_1^3 (1+x) \, dx \] represents the area between the graph of \( f(x) = 1 + x \), the \( x \)-axis, and the vertical lines at \( x = 1 \) and \( x = 3 \).

Since this area is bound by a trapezoid (quadrilateral with 2 parallel sides), we can use the formula: \( A = \frac{1}{2} h(b_1 + b_2) \) where \( h = 2 \), \( b_1 = 2 \) and \( b_2 = 4 \). Therefore, \( A = \frac{1}{2} \times 2 \times (2 + 4) = 6 \) square units.

---

**Example 5**

If \( f'(x) = 2x \), then the graphical representation of \( \int_4^7 f'(x) \, dx \) is the yellow area below.

Geometrically, we can calculate this area using formulas we already know!

Area of a Trapezoid = \( \frac{1}{2} h \,(b_1 + b_2) = \frac{1}{2} \times (3)(8 + 14) = 33 \) units.

\[ \int_4^7 f'(x) \, dx = 33 \text{ units.} \]
Approximating with Many Rectangles

How do we approximate the area if the rate curve is, well, curvy? We could use geometric shapes like we did in Example 1 through 5. But it turns out to be more useful (and easier) to simply use rectangles. The more rectangles we use, the better our approximation is.

Suppose we want to calculate the area between the graph of a positive function $f(x)$ and the x–axis on the interval $[a, b]$ (graphed above).

The Riemann Sum method is to build rectangles with bases on the interval $[a, b]$ and sides that reach up to the graph of $f(x)$ (see below).

Then the areas of the rectangles can be calculated and added together to get a number called a Riemann Sum of $f(x)$ on $[a, b]$. The area of the region formed by the rectangles approximates the area we want. As we use more and more rectangles, we will get closer and closer to the true area under the curve.

Example 6

Let $A$ be the region bounded by the graph of $f(x) = \frac{1}{x^2}$, the x–axis, and vertical lines at $x = 1$ and $x = 5$. We can’t find the area exactly (with what we know now), but we can approximate it using rectangles.
When we make our rectangles, we have a lot of choices. We could pick any (non-overlapping) rectangles whose bottoms lie within the interval on the x-axis, and whose tops intersect with the curve somewhere. But it’s easiest to choose rectangles that:

(a) have all the same width, and
(b) take their heights from the function at one edge of each sub-interval.

To see this clearly, we will start with a small number of rectangles.

Below you'll see two ways to use four rectangles to approximate this area. In the first graph, we used left endpoints; the height of each rectangle comes from the function value at the left edge of each partition of input, so we use \( x = 1, 2, 3 \) and 4. In the second graph on the next page, we used right-hand endpoints of each partition of input, so we use \( x = 2, 3, 4 \) and 5.

**Left-hand endpoints:** The area is approximately the sum of the areas of the rectangles. Each rectangle gets its height from the function \( f(x) = \frac{1}{x} \) and each rectangle has width = 1.

You can find the area of each rectangle using area = height \( \times \) width. So, the total area of the rectangles, the left-hand estimate of the area under the curve, is:

\[
f(1) \times 1 + f(2) \times 1 + f(3) \times 1 + f(4) \times 1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.08.
\]

Notice that because this function is decreasing, all the left endpoint rectangles stick out above the region we want – using left-hand endpoints will overestimate the area. **We can call this the Left-Hand Rectangular Approximation Method (LRAM).**

**Right Hand endpoints:** The right-hand estimate of the area is:

\[
f(2) \times 1 + f(3) \times 1 + f(4) \times 1 + f(5) \times 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} \approx 1.28
\]
All the right-hand rectangles lie completely under the curve, so this estimate will be an underestimate. **We can call this the Right-Hand Rectangular Approximation Method (RRAM).**

We can see that the true area is actually in between these two estimates. So, we could take their average:

\[
\text{Average: } \frac{25/12 + 77/60}{2} = \frac{101}{60} \approx 1.68
\]

**In general, the average of the left-hand and right-hand estimates will be closer to the real area than either individual estimate.**

My estimate of the area under the curve is about 1.68. (The actual area determined by techniques that we will learn later is about 1.61.)

Now we can use the notation of the definite integral to describe it. Our estimate of \( \int_1^5 \frac{1}{x} \, dx \) was 1.68. The true value of \( \int_1^5 \frac{1}{x} \, dx \) is about 1.61.

If we wanted a better answer, we could use even more, even narrower rectangles. But there’s a limit to how much work we want to do by hand. Hopefully, you can start to envision using Excel for this type of work. In a few keystrokes, we can use many rectangles!

These sums of areas of rectangles are called **Riemann sums.** You may see a shorthand notation used when people talk about sums. We won’t use it much in this book, but you should have at least been exposed to the terminology and notation.

**Riemann sum:** A Riemann sum for a function \( f(x) \) over an interval \([a, b]\) is a sum of areas of rectangles that approximates the area under the curve.

- **Start by dividing the interval \([a, b]\) into \( n \) subintervals; each subinterval will be the base of one rectangle. We usually make all the rectangles the same width.**
- **Our width will be** \( \Delta x = \frac{b-a}{n} \) **where** \( n \) **represents the number of subintervals (rectangles) we want to use.**
- The height of each rectangle comes from the function evaluated at some point in its sub interval. Notice that \( a = x_0 \) and that \( b = x_n \).
- **Further notice that the left-hand sum starts by evaluating** \( f \) **at** \( a = x_0 \) **and ends at the input just to the left of** \( b = x_n \) **which we call** \( x_{n-1} \).
- **The right-hand sum starts by evaluating** \( f \) **at the input just to the right of** \( a = x_0 \), **which we call** \( x_1 \) **and ends at the input** \( b = x_n \).
- We will evaluate both the left-hand sum and the right-hand sum. Look carefully at the notation below.

\[
\text{LRAM} = f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \ldots + f(x_{n-1}) \cdot \Delta x
\]
RRAM = \( f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x \)

- Notice that each term is being multiplied by \( \Delta x \), so we could factor that out and write the right hand sum as those shown below.

RRAM = \( (f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)) \cdot \Delta x \)

LRAM = \( (f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})) \cdot \Delta x \)

**Sigma Notation:** The upper-case Greek letter Sigma \( \Sigma \) is used to stand for Sum. Sigma notation is a way to compactly represent a sum of many similar terms, such as a Riemann sum.

- Using the Sigma notation, the Right-Hand Riemann sum can be written: \( \sum_{i=1}^{n} f(x_i) \cdot \Delta x \)
- This is read aloud as “the sum as \( i = 1 \) to \( n \) of \( f \) of \( x \) sub \( i \) Delta \( x \)” The “\( i \)” is a counter, like you might have seen in a programming class. If we wanted the Left Hand Riemann sum, it would look like: \( \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x \). Notice that we start with \( f(x_0) \) in this case.

Often, we are given **ONLY data** about the rate of change of a function (derivative) and need to **APPROXIMATE the accumulated amount of change** function.

**Example 7**

The table shows rates of population growth for Berrytown for several years. Use this table to estimate the total population growth from 1970 to 2000:

<table>
<thead>
<tr>
<th>Year (t)</th>
<th>1970</th>
<th>1980</th>
<th>1990</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of population growth R(t) (thousands of people per year)</td>
<td>1.5</td>
<td>1.9</td>
<td>2.2</td>
<td>2.4</td>
</tr>
</tbody>
</table>

The definite integral of this rate will give the total change in population over the thirty-year period. We only have a few pieces of information, so we can only estimate. Because we are given data, we don’t need to graph the function to approximate the area under the rate curve using rectangles. How wide are the rectangles? I have information every 10 years, so the rectangles have a width of 10 years. How many rectangles? Be careful here – this is a thirty-year span, so there are three rectangles. Make sure you understand that! It is a crucial piece!

**Using left-hand endpoints (LRAM):** \( (1.5)(10) + (1.9)(10) + (2.2)(10) = 56 \). Notice that since each term is being multiplied by 10. We could therefore simply add up the rates, and then multiply that result by 10. \( (1.5 + 1.9 + 2.2)(10) = 5.6 \cdot 10 = 56 \). Notice that either way, we get the same result.

**Using right-hand endpoints (RRAM):** \( (1.9)(10) + (2.2)(10) + (2.4)(10) = 65 \)

Taking the average of these two: \( \frac{56 + 65}{2} = 60.5 \)

Our best estimate of the total population growth from 1970 to 2000 is 60.5 thousand people.
Example 8

If the marginal revenue is not constant, we might guess that the accumulated change is still equal to the area under the curve. (The units work out.) Let’s use a marginal revenue of $R'(t) = \frac{1}{8}t^2 + 1$ where $t$ represents days and revenue is measured in thousands of dollars. Therefore, the units on $R'(t)$ will be $\frac{\text{thousands of dollars}}{\text{day}}$. Let’s consider what our approximate increase in Revenue would be for the first 4 days, so $t = 0$ to $t = 4$.

Let’s use Excel and use 20 subdivisions. That means our $\Delta t = \frac{4-0}{20} = .2$. We would set up our Excel sheet to look like that shown. Look carefully at the second table shown to see how we entered the formulas in.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
<td>0</td>
<td>t</td>
<td>R'(t)</td>
<td>Left Sum</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>b</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1.005</td>
<td>6.47</td>
</tr>
<tr>
<td>3</td>
<td>n</td>
<td>20</td>
<td>0.2</td>
<td>1.005</td>
<td>6.47</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>increment</td>
<td>0.2</td>
<td>1.02</td>
<td>Right Sum</td>
<td>6.87</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1.125</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.2</td>
<td>1.18</td>
<td></td>
<td></td>
<td>Average</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.4</td>
<td>1.245</td>
<td></td>
<td></td>
<td>6.67</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.6</td>
<td>1.32</td>
<td></td>
<td></td>
<td>=</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.8</td>
<td>1.405</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>1.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>2.2</td>
<td>1.605</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>2.4</td>
<td>1.72</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>2.6</td>
<td>1.845</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>2.8</td>
<td>1.98</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td>2.125</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>3.2</td>
<td>2.28</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>3.4</td>
<td>2.445</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>3.6</td>
<td>2.62</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>3.8</td>
<td>2.805</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Notice that for the left-hand sum, we start with the \( R'(0) \) and end with \( R'(3.8) \) in our sum.

Our right-hand sum starts at \( R'(0.2) \) and ends at \( R'(4) \). This is crucial for you to understand.

Furthermore, notice that we added the function values up \textbf{first} and then multiplied by the width. Remember, since each area would have a factor of the width, we can factor it out!

Last, notice that we average the LRAM and RRAM to get out final estimate.

We can now say that over the first 4 days, our revenue increased by approximately 6.67 thousand dollars, or $6,670.

\[
\begin{array}{cccccc}
A & B & C & D & E & F \\
1 & a & 0 & t & R'(t) & \text{Left Sum} \\
2 & b & 4 & 0 & =\frac{1}{8}t^2+2t+1 & \\
3 & n & 20 & 0.2 & =\frac{1}{8}t^3+2t+1 & =\text{SUM(E2:E21)}*B4 \\
4 & \text{increment} & =\frac{(B2-B1)/B3} & 0.4 & =\frac{1}{8}t^4+2t+1 & \\
5 & 0.6 & =\frac{1}{8}t^5+2t+1 & \text{Right Sum} & \\
6 & 0.8 & =\frac{1}{8}t^6+2t+1 & =\text{SUM(E2:E22)}*B4 \\
7 & 1 & =\frac{1}{8}t^7+2t+1 \\
8 & 1.2 & =\frac{1}{8}t^8+2t+1 & \text{Average} & =\frac{F3+F6}{2} \\
9 & 1.4 & =\frac{1}{8}t^9+2t+1 & & \\
10 & 1.6 & =\frac{1}{8}t^{10}+2t+1 & & \\
11 & 1.8 & =\frac{1}{8}t^{11}+2t+1 & & \\
12 & 2 & =\frac{1}{8}t^{12}+2t+1 & & \\
13 & 2.2 & =\frac{1}{8}t^{13}+2t+1 & & \\
14 & 2.4 & =\frac{1}{8}t^{14}+2t+1 & & \\
15 & 2.6 & =\frac{1}{8}t^{15}+2t+1 & & \\
16 & 2.8 & =\frac{1}{8}t^{16}+2t+1 & & \\
17 & 3 & =\frac{1}{8}t^{17}+2t+1 & & \\
18 & 3.2 & =\frac{1}{8}t^{18}+2t+1 & & \\
19 & 3.4 & =\frac{1}{8}t^{19}+2t+1 & & \\
20 & 3.6 & =\frac{1}{8}t^{20}+2t+1 & & \\
21 & 3.8 & =\frac{1}{8}t^{21}+2t+1 & & \\
22 & 4 & =\frac{1}{8}t^{22}+2t+1 & & \\
\end{array}
\]

\textbf{Related Exercises You Should Complete Now}

\textbf{Make sure you work on Exercises 4.1.9 and 4.1.10.} Remember, you have written solutions and videos for these exercises in your course.

\textbf{Signed Area}

You may have noticed that until this point, we’ve insisted that the integrand (the function we’re integrating) be positive. That’s because we’ve been talking about area, which is always positive.

If the function value (which we have been thinking of as a “height”) is a negative number, then multiplying it by the width doesn’t give us actual area, it gives us the area but with a negative sign.

\textbf{Negative rates indicate that the amount is decreasing.} This is true of any rate. For example, if \( f(t) \) is the rate of population change (people/year) for a town, then negative values of \( f(t) \) would indicate that the population of the town was decreasing (remember, if the derivative is negative, the original function must be decreasing!), and the definite integral (now a negative number) would be the change in the population, a decrease, during the time interval.
How is the definite integral interpreted as area?

The Definite Integral of a positive function \( f'(x) \) (that lies above the x-axis) over the interval \([a, b]\) is equivalent to the area between the function and the x-axis, and the vertical lines \( x = a \), and \( x = b \).

We know that if the derivative is positive, the original function is increasing. \((f^+, f \uparrow)\). Therefore, we know that the accumulated amount of change of \( f(x) \) is positive over the interval and \( f(x) \) increased by the amount = to the area shown.

If given a Marginal Cost function \( MC(t) = C'(t) \), the shaded area represents the accumulated change in the cost function over the given interval.

If the units of \( C'(t) \) is thousands of dollars per month, the area in blue represents the accumulated change in the cost \( C(t) \) in thousands of dollars.

What happens when the function is not positive, when \( f'(x) \) is below the x – axis?

If \( f'(x) \) is positive on \([a,b]\) then \( \int_a^b f'(x) \, dx \) = positive value

If \( f'(x) \) is negative on \([a,b]\), then \( \int_a^b f'(x) \, dx \) = negative value
When \( f'(x) \) is positive for some values and negative for others, and \( a < b \):

\[
\int_a^b f'(x) \, dx
\]

is the sum of areas above the x-axis, counted positively, and the areas below the x-axis, counted negatively.

\[
\int_a^c f'(x) \, dx = \int_a^b f'(x) \, dx + \int_b^c f'(x) \, dx
\]

It turns out to be useful to think about the possibility of an area that is counted as a negative value. We’ll expand our idea of a definite integral now to include integrands that might not always be positive. The values from the function, which we have been thinking of as the “heights” of the rectangles, now might not always be positive.

The Definite Integral and Signed Area:

The **definite integral** of a function \( f(x) \) over an interval \([a, b]\) is the **signed area** between \( f(x) \) the \( x \)-axis, \( x = a \) and \( x = b \).

The **definite integral** of a function \( f(x) \) from \( a \) to \( b \) is the **signed area** under the curve between \( a \) and \( b \).

If the function is positive, the area above the x axis is positive, as before (and we can call it area.)

If the function dips below the x-axis, the regions below the x-axis will have a negative sign. In this case, we cannot call it simply “area” since we know that area is always positive. These regions take away from the value of the definite integral. We can still think of area, but we need to consider subtracting the value we produce which is bounded below the x axis, since it will be a negative result.

\[
\int_a^b f(x) \, dx = (\text{Area above x-axis}) - (\text{Area below x-axis}).
\]

If \( f(t) \) represents a positive rate (in y-units per t-units), then the **definite integral** of \( f \) from \( a \) to \( b \) is the **total** y-units that accumulate between \( t = a \) and \( t = b \).

If \( f(t) \) represents any rate (in y-units per t-units), then the **definite integral** of \( f \) from \( a \) to \( b \) is the **net** y-units that accumulate between \( t = a \) and \( t = b \). This is a result of our counting the area above the x axis as a positive value and the area under the x axis as a negative value – hence the sum of the **signed areas**.
Example 9

Use the graph to calculate \( \int_0^2 f(x) \, dx \), \( \int_0^4 f(x) \, dx \), \( \int_4^5 f(x) \, dx \), and \( \int_0^5 f(x) \, dx \).

\[
\begin{array}{cccc}
0 & 2 & 4 & 5 \\
\int f(x) \, dx & 2 & 4 & 5 \\
0 & 2 & 4 & 0
\end{array}
\]

Using the areas shown in the picture above we know that
\[
\int_0^2 f(x) \, dx = 2, \quad \int_0^4 f(x) \, dx = -5,
\]
\[
\int_4^5 f(x) \, dx = 2, \quad \int_0^5 f(x) \, dx = \text{(area above)} - \text{(area below)} = (2+2) - (5) = -1.
\]

Related Exercises You Should Complete Now

Make sure you work on Exercises 4.1.5, 4.1.6, 4.1.7, 4.1.8 and 4.1.14. Remember, you have written solutions and videos for these exercises in your course.

Accumulation in Real Life

We have already seen that the "area" under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. For example, if \( t \) is a measure of time in seconds and \( f(t) \) is a velocity with units feet/second, then the definite integral has units (feet/second) \cdot (seconds) = feet.

Remember that our integrand is a rate function! That means that the output of \( f(x) \) is a rate. When we integrate, we then create units for the definite integral \( \int_a^b f(x) \, dx \) as (units for \( f(x) \)) times \( x \)-units). A quick check of the units can help avoid errors in setting up an applied problem.

For functions representing other rates such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the total amount of something.

Example 10

In 1980 there were 12,000 ducks nesting around a lake, and the rate of population change (in ducks per year) is shown in the picture below. Write a definite integral to represent the total change in the duck population from 1980 to 1990 and estimate the population in 1990.
The change in population = \[ \int_{1980}^{1990} f(t) \, dt = - \left( \text{area between } f \text{ and axis} \right) \]

= \[- \left( 200 \frac{\text{ducks}}{\text{year}} \right) \cdot (10 \text{ years}) = -2000 \text{ ducks} \].

Then (1990 duck population) = (1980 population) + (change from 1980 to 1990)

= (12,000) + (−2000) = 10,000 ducks.

Example 11
Suppose MR(q) is the marginal revenue in dollars/item for selling q items. What does \[ \int_{0}^{150} MR(q) \, dq \] represent? (This could also be considered as \[ \int_{0}^{150} R'(q) \, dq \].)

\[ \int_{0}^{150} MR(q) \, dq \] has units (dollars/item) \cdot (items) = dollars and represents the accumulated dollars for selling from 0 to 150 items.

Setting the Stage to Pull It All Together

Related Exercises You Should Complete Now

Make sure you work on Exercises 4.1.11, 4.1.13, and 4.1.15. Remember, you have written solutions and videos for these exercises in your course.

Example 12
The table below gives some marginal revenue values at a company. We want to use this data to make some statements about revenue based on an interval of items sold.

<table>
<thead>
<tr>
<th>x items sold</th>
<th>Marginal revenue in dollars per item R'(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3500</td>
<td>0.9</td>
</tr>
<tr>
<td>4000</td>
<td>0.5</td>
</tr>
<tr>
<td>4500</td>
<td>0.2</td>
</tr>
<tr>
<td>5000</td>
<td>0</td>
</tr>
<tr>
<td>5500</td>
<td>-0.1</td>
</tr>
<tr>
<td>6000</td>
<td>-0.3</td>
</tr>
</tbody>
</table>
Use the values in the table to estimate the value of $\int_{3500}^{4000} R'(x)dx$

**First Step: Clearly Identify What the Problem is Asking**

- This calculation is asking us to find the value of the definite integral from $x = 3500$ up to $x = 4000$. The answer will tell us how much the revenue function, $R(x)$, increases or decreases from $x = 3500$ to $x = 4000$. $R(x)$ is an anti-derivative of the function $R'(x)$.

**Second Step: Clearly Identify What You Know**

- In this case, we do not have the formula of the $R'(x)$ function. Instead, the only information we have about $R'(x)$ on this interval is that $R'(3500) = 0.9$ and $R'(4000) = 0.5$.
- This means that when they are selling 3500 items, the revenue is increasing at a rate of $0.90$ per item, and when they are selling 4000 items, the revenue is increasing at a rate of $0.50$ per item. So, we want to know how much revenue ($R(x)$) went up or how much $R(x)$ went down from $x = 3500$ to $x = 4000$, but we only know the rate of increase at two points on that interval. We don't know the rate of increase or decrease at any other point on that interval. So, the best we can do is ESTIMATE the amount that $R(x)$ went up or down on that interval.

**Third Step: Use the Given Information And Interpret Your Answers**

- One way to estimate is to use the rate on the left side, $R'(3500) = 0.9$. We could just assume that this rate stayed constant on the entire interval from $x = 3500$ all the way up to $x = 4000$. That would mean that the revenue function increased at a constant rate of $0.90$ per item sold all the way from $x = 3500$ over to $x = 4000$ (so for 500 total items): $(0.90/1 \text{ item}) \times (500 \text{ items}) = 450 \text{ dollars increase}$.
- So, this would give us an estimate that the revenue increased a total of $450$ as they increased from 3500 items sold up to 4000 items sold.

- A second way to estimate is to use the rate on the right side, $R'(4000) = 0.5$. Again, we could assume that this was the constant rate on the entire interval from $x = 3500$ all the way up to $x = 4000$. That would mean that the revenue function increased at a constant rate of $0.50$ per item sold all the way from $x = 3500$ over to $x = 4000$ (so for 500 total items): $(0.50/1 \text{ item}) \times (500 \text{ items}) = 250 \text{ dollars increase}$.
- So, this would give us an estimate that the revenue increased a total of $250$ as they increased from 3500 items sold up to 4000 items sold.

- Overall, to get a better estimate, we would probably average the two of these estimates: $(450+250) / 2 = 350$
- So, overall, we would estimate that the revenue increased a total of $350$ when they increased from 3500 items sold up to 4000 items sold.

**Fourth Step: Make Sure You Can Summarize the Key Idea**

- To create a left-hand estimate of the accumulated increase/decrease on an interval:
  We assume the derivative value on the left-side is true for the entire interval, and then calculate the accumulated increase or decrease in $R(x)$.
To create a right-hand estimate of the accumulated increase/decrease on an interval:
We assume the derivative value on the right-side is true for the entire interval, and then calculate the accumulated increase or decrease in $R(x)$.

We estimated the value using a left-hand estimate, and a right-hand estimate. Then we averaged the two estimates to give an overall estimate.

This estimate told us how much we thought revenue changed as the number of items sold went from 3500 to 4000.

Fifth Step: Think About Extending These Ideas to Include All Data

Notice that the table gives us data starting at 3500 items sold and ending at 6000 items sold, with the breakdown being given every 500 items. This means that we have an interval that is 2500 units long. Since we have data every 500 units, we see that we have 5 subintervals.

That means that we can estimate the net change in revenue using left hand and right-hand estimates based on these 5 sub-intervals.

Stated as you will see it in the Exercises, we can say estimate $\int_{3500}^{6000} R'(x)dx$ using LRAM and RRAM with 5 sub-intervals.

Since some of the rates are positive, and some of the rates are negative, we expect that revenue will be increasing and decreasing over the interval from 3500 to 6000. That means that our final estimate for $\int_{3500}^{6000} R'(x)dx$ will be the net change in revenue over that interval.

### Related Exercises You Should Complete Now

Go back and review your answers for the contextual meaning of 4.1.2, 4.1.6, 4.1.11. Remember, you have written solutions and videos for these exercises in your course.

Approximating in Various Ways

If your function is given as a graph or table, you will have to approximate definite integrals using areas. While there are other areas you can use, we will work with rectangles unless the region can easily be broken up into familiar shapes.

If your function is given as a formula, you can turn to technology to get a better approximate answer by using more subintervals. We will be using Excel. You should watch the videos contained in your course that show how to accomplish this.

### Related Exercises You Should Complete Now

Make sure you work on Exercise 4.1.12. Remember, you have written solutions and videos for these exercises in your course.
Exercises for Section 4.1

Exercises for Unit 4 Section 1

Exercise 4.1.1: Given data of the marginal profit of a company, use the data and the graph to find the profit the for the company in a given interval. Use both LRAM and RRAM and create the average of the two.

<table>
<thead>
<tr>
<th>Time (weeks)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P'(t)$ $\frac{\text{$1000}}{\text{week}}$</td>
<td>25</td>
<td>31</td>
<td>35</td>
<td>43</td>
<td>47</td>
<td>46</td>
<td>41</td>
</tr>
</tbody>
</table>

Exercise 4.1.2:

(a) Use Excel to estimate the value of $\int_{50}^{350} C'(x) \, dx$ in TWO WAYS (i.e. LRAM and RRAM)

<table>
<thead>
<tr>
<th>x DVDs produced</th>
<th>Marginal Cost in dollars per DVD, $C'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>4.55</td>
</tr>
<tr>
<td>100</td>
<td>3.25</td>
</tr>
<tr>
<td>150</td>
<td>2.25</td>
</tr>
<tr>
<td>200</td>
<td>1.55</td>
</tr>
<tr>
<td>250</td>
<td>−1.15</td>
</tr>
<tr>
<td>300</td>
<td>−0.95</td>
</tr>
<tr>
<td>350</td>
<td>−0.87</td>
</tr>
</tbody>
</table>

(b) Sketched below are two plotted graphs of the given $C'(x)$ points. For the first graph create a LABELED GRAPHICAL representation of the left-hand estimate calculation. Similarly, for the second graph, create a LABELED GRAPHICAL representation of the right-hand estimate calculation. Your graphs will be a series of rectangles.
(e) Write a complete sentence to interpret the contextual meaning of the value you find in part (a)

**Exercise 4.1.3:** The marginal cost \( MC(t) = C'(t) \) (in hundreds of dollars per week) in the table below is decreasing on the interval [2, 12]. Using \( n = 5 \) subdivisions (rectangles) to approximate the total cost, finding the LRAM and RRAM values. Write a complete sentence to interpret the contextual meaning of the value you find.

<table>
<thead>
<tr>
<th>Time</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C'(t) )</td>
<td>44</td>
<td>42</td>
<td>41</td>
<td>40</td>
<td>37</td>
<td>35</td>
</tr>
</tbody>
</table>
Exercise 4.1.4: Suppose a population increases at a fixed rate each year (see the graphs below). How much did the population increase overall during the first 6 years?

(a)

(b)

(c)

Note: If you have the “derivative of population with respect to time,” then you do not need the actual starting population in order to determine the overall, accumulated change in the population over time!
Exercise 4.1.5: Suppose a company’s profit decreases at a fixed rate each month (see the graphs below). How much did the profit decrease overall during the 5 month period?

(a) [Graph showing rate of profit with a point at (0, -2000) and another at (5, -2000)].

(b) [Graph showing profit decrease with a point at (0, 10000) and another at (5, 0)].

(c) [Graph showing further decrease in profit with a point at (0, 30000) and another at (5, 20000)].

Note: If you have the “derivative of profit with respect to time,” then you do not need the actual starting profit for the company in order to determine the overall, accumulated change in the profit over time!
Exercise 4.1.6: Suppose a tank of water is draining, and then they plug the hole and begin to fill the pool. The function $v'(t) = 5t - 20$ gives the rate at which the water is draining or filling in the pool. See the graph below. How much water did the pool lose while draining? How much did it gain while filling if it stops at 6 minutes? What was the overall change in the amount of water in the pool during this time?

Note: If you have the “derivative of volume with respect to time,” then you do not need the actual starting amount of water in order to determine the overall, accumulated change in the water over time!
**Exercise 4.1.7:** Suppose \( f(x) = x^2 - x + 5 \). Then, \( f'(x) = 2x - 1 \).
Also suppose \( g(x) = x^2 - x - 10 \). Then, \( g'(x) = 2x - 1 \).

Now, consider the values \( \int_{-2}^{7} f'(x) \, dx \) and \( \int_{-2}^{7} g'(x) \, dx \) and their graphical meaning.

Show the graphical (geometric) meaning of the integral on the graphs of \( f'(x) \) and \( g'(x) \) below.

Show the graphical (geometric) meaning of the integral on the graphs of \( f(x) \) and \( g(x) \) below.
**Exercise 4.1.8:** Suppose \( f(x) = x^3 - 11x^2 + 31x - 21 \). Then, \( f'(x) = 3x^2 - 22x + 31 \).

Now, consider the values \( \int_0^6 f'(x) \, dx \). Show the integral’s **graphical meaning on the graphs of** \( f'(x) \) and \( f(x) \) below.
**Exercise 4.1.9:** Using four rectangles then 8 rectangles, estimate using Excel the value of \( \int_{0}^{8} f'(x) dx \) if we know that \( f'(x) = 1.25x^3 - 17.5x^2 + 40x + 240 \).

Sketch the rectangles below on the graph of \( f'(x) \).

**Left Sum, 4 rectangles:**

![Graph of f'(x) with 4 rectangles on the left side]

**Right Sum, 4 rectangles:**

![Graph of f'(x) with 4 rectangles on the right side]
Exercise 4.1.10: Using Excel, estimate the value of $\int_{0}^{8} f'(x) dx$ if we know that $f'(x)$ is the function shown below. Show your Excel conclusions below. (LRAM, RRAM, Average)

$$f'(x) = 1.25x^3 - 17.5x^2 + 40x + 240$$

a) Use 50 rectangles (too many to draw and picture on the graph). Make it clear what value you are using for your increment $\Delta x$. 
Exercises for Section 4.1

b) Use 100 rectangles (too many to draw and picture on the graph) Make it clear what value you are using for your increment $\Delta x$.

c) Use 500 rectangles (too many to draw and picture on the graph). Make it clear what value you are using for your increment $\Delta x$.

Exercise 4.1.11: The marginal profit for a company is given by the function $p'(x) = -2x + 7500$ where $x$ is the number of items sold, $p'(x)$ is the marginal profit in dollars per item.

(a) Calculate $\int_0^{3750} p'(x)dx$ exactly (using geometry) and show the graphical representation of this calculation on the $p'(x)$ graph. Write a complete sentence to give the contextual meaning of the answer.

(b) Calculate $\int_{3750}^{5000} p'(x)dx$ exactly and show the graphical representation of this calculation on the $p'(x)$ graph. Write a complete sentence to give the contextual meaning of the answer.
(c) Calculate $\int_0^{5000} p'(x)dx$ exactly and show the graphical representation of this calculation on the $p'(x)$ graph. Write a complete sentence to give the contextual meaning of the answer.

**Exercise 4.1.12:** You and a friend start off at noon and walk in the same direction along the same path at the rates shown.

a) Who is walking faster at 2 pm? How do you know?

b) Who is ahead at 2 pm? How do you know?

c) Who is walking faster at 3 pm? How do you know?

d) Who is ahead at 3 pm? How do you know?

e) What will be true of the graphs when you and your friend are walking side by side?
Exercise 4.1.13: Given the units of $f'(x)$ what would be the units of $\int f'(x)\,dx$?

a) $f'(x)$ is in meters per second

b) $f'(x)$ is in gallons per hour

c) $f'(x)$ is in hundreds of dollars per month

d) $f'(x)$ is in dollars per item

e) $f'(x)$ is in miles per gallon

Exercise 4.1.14: Suppose that $f'(x) = 0.3x^3 - 17x^2 + 5x + 5000$.

(a) Below is a sketch of the graph of $f'(x)$ from $x = 0$ to $x = 60$. Shade the region of the graph that represents the definite integral $\int_0^{60} f'(x)\,dx$.

(b) The function $f'(x) = 0.3x^3 - 17x^2 + 5x + 5000$, has x-intercepts (shown) at $x \approx 22.2491$ and $x = 49.5389$. It is also known that

$$\int_0^{22.2491} f'(x)\,dx = 68,450.042$$

And

$$\int_{22.2491}^{49.5389} f'(x)\,dx = -51,840.56$$

And

$$\int_{49.5389}^{60} f'(x)\,dx = 40,390.52$$

Label the graph above with the x-intercepts and the definite integral values of the three regions.
(c) Use the values given in part (b) to give the value of \( \int_0^{60} (0.3x^3 - 17x^2 + 5x + 5000) \, dx \)

(d) Approximate the value of the definite integral \( \int_0^{60} (0.3x^3 - 17x^2 + 5x + 5000) \, dx \) using 600 rectangles using Excel. Use the Excel Left-Sum Right-Sum template to help. Your answer should come close to the value we found in part (c).

Left Sum:

Right Sum:

Overall estimate of the definite integral:

Exercise 4.1.15: On the basis of data obtained from a preliminary report by a geological survey team, it is estimated that for the first 10 years of production, a certain oil well can be expected to produce oil at the rate of \( B'(t) = 3.94 t^{3.55} \cdot e^{-1.35t} \) where \( t \) is the number of years after production begins, and \( B'(t) \) is measured in thousand barrels per year.

(a) Estimate the yield during the first 10 years of production. Use 200 rectangles. Remember to use the “Fill” command in Excel to produce your table.

Left Sum:

Right Sum:

Overall Estimate:

(b) Write the proper definite integral notation for this problem together with your answer from part (a).

(c) Write a complete sentence to interpret the meaning of this integral in context.
Unit 4 Section 2: Antidifferentiation and The Fundamental Theorem of Calculus

Going Backwards – From Derivative to General Antiderivative

Now that we have talked about derivatives and examined the relationship between a function, the first derivative and the second derivative, we will now explore “thinking backward”. Our goal is to find a function \( f(x) \) whose derivative is a known function \( f'(x) \). If such a function \( f(x) \) exists, it is called a **General Antiderivative of** \( f'(x) \).

Now that we know the differentiation formulas, we can find explicit expressions for general antiderivatives. Before we note some definitions, let’s do some analysis.

**Example 1**

Find \( f(x) \) if \( f'(x) = 2x \).

Oooh, I know this one. It’s \( f(x) = x^2 + 3 \). Oh, wait, you were thinking something else? Yes, I guess you’re right -- \( f(x) = x^2 \) works too. So does \( f(x) = x^2 - \pi \), and \( f(x) = x^2 + 104,589.2 \). In fact, there are lots of answers.

**In fact, there are infinitely many functions that all have the same derivative.** And that makes sense – the derivative tells us about the shape of the function, but it doesn’t tell about the location. We could shift the graph up or down and the shape wouldn’t be affected, so the derivative would be the same.

**Example 2**

If we started with the derivative function \( f'(x) = x^2 \) how can we find the **general antiderivative function**??

If we had the function, \( f(x) = x^3 \) the **derivative** would be \( f'(x) = 3x^2 \) but the given function does NOT have “3” in the derivative function!!

If we put a **coefficient** in front of the \( f(x) \) function we can get the desired derivative function.
If \( f(x) = \frac{1}{3}x^3 \) then the derivative would be \( f'(x) = x^2 \) which is the derivative we want!

So our general antiderivative for this derivative function is \( f(x) = \frac{1}{3}x^3 + C \)

By assigning specific values to the constant \( C \), we obtain a **family of functions** whose graphs are vertical translates of one another.
Example 3

Find an antiderivative of $2x$.

I can choose any function I like as long as its derivative is $2x$, so I’ll pick $F(x) = x^2 - 5.2$.

General Antiderivatives

A **general antiderivative** of a function $f(x)$ is any function $F(x)$ where $F'(x) = f(x)$.

The **general antiderivative** of a function $f(x)$ is a **whole family of functions**, written $F(x) + C$, where $F'(x) = f(x)$ and $C$ represents any constant.

The antiderivative is also called the **indefinite integral**.

**Notation for the antiderivative:**
The antiderivative of $f(x)$ is written $\int f(x) \, dx$.

This notation resembles the definite integral, but in this notation, there are no limits of integration.

The $\int$ symbol is still called an **integral sign**; the $dx$ on the end still must be included; you can still think of $\int$ and $dx$ as left and right parentheses. The function $f$ is still called the **integrand**.

**Verb forms:**
We **antidifferentiate**, or **integrate**, or **find the indefinite integral** of a function. This process is called **antidifferentiation** or **integration**.

Example 4

Find the antiderivative of $2x$.

Now I need to write the entire family of functions whose derivatives are $2x$. I can use the notation:

\[
\int 2x \, dx = x^2 + C
\]

Example 5

Find $\int e^x \, dx$.

This is likely one you remember -- $e^x$ is its own derivative, so it is also its own antiderivative. The integral sign tells me that I need to include the entire family of functions, so I need that + $C$ on the end:

\[
\int e^x \, dx = e^x + C
\]
There are no small families in the world of antiderivatives: if a function \( f(x) \) has one antiderivative \( F(x) \), then \( F(x) \) has an infinite number of antiderivatives and every one of them has the form \( F(x) + C \).

You can see how important it is that you KNOW and recognize the derivative functions in order to “think backwards” to antiderivatives!

**Building Blocks**

Antidifferentiation is going backwards through the derivative process. So, the easiest antiderivative rules are simply backwards versions of the easiest derivative rules. Recall from Chapter 2:

<table>
<thead>
<tr>
<th>Derivative Rules: Building Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>In what follows, ( f ) and ( g ) are differentiable functions of ( x ) and ( k ) and ( n ) are constants.</td>
</tr>
<tr>
<td>(a) Constant Multiple Rule: ( \frac{d}{dx} (kf(x)) = kf'(x) )</td>
</tr>
<tr>
<td>(b) Sum (or Difference) Rule: ( \frac{d}{dx} (f + g) = f' + g' ) (or ( \frac{d}{dx} (f - g) = f' - g' ))</td>
</tr>
<tr>
<td>(c) Power Rule: ( \frac{d}{dx} (x^n) = nx^{n-1} )</td>
</tr>
<tr>
<td>Special cases: ( \frac{d}{dx} (k) = 0 ) (because ( k = kx^0 ))</td>
</tr>
<tr>
<td>( \frac{d}{dx} (x) = 1 ) (because ( x = x^1 ))</td>
</tr>
<tr>
<td>(d) Exponential Functions: ( \frac{d}{dx} (e^x) = e^x )</td>
</tr>
<tr>
<td>( \frac{d}{dx} (a^x) = \ln a \cdot a^x )</td>
</tr>
<tr>
<td>(e) Natural Logarithm: ( \frac{d}{dx} (\ln x) = \frac{1}{x} )</td>
</tr>
</tbody>
</table>

Thinking about these basic rules was how we came up with the antiderivatives of \( 2x \) and \( e^x \) before.

The corresponding rules for antiderivatives are next – each of the antiderivative rules is simply rewriting the derivative rule. All of these antiderivatives can be verified by differentiating.

**There is one surprise – the antiderivative of \( 1/x \) is actually not simply \( \ln(x) \), it’s \( \ln|x| \).** This is a good thing – the antiderivative has a domain that matches the domain of \( 1/x \), which is bigger than the domain of \( \ln(x) \), so we don’t have to worry about whether our \( x \)’s are positive or negative. But you must be careful to include those absolute values – otherwise, you could end up with domain problems.
Antiderivative Rules: Building Blocks

In what follows, \( f \) and \( g \) are differentiable functions of \( x \) and \( k, n, \) and \( C \) are constants.

(a) **Constant Multiple Rule:**
\[
\int kf(x)\,dx = k \int f(x)\,dx
\]

(b) **Sum (or Difference) Rule:**
\[
\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx
\]

(c) **Power Rule:**
\[
\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \text{ provided that } n \text{ is NOT } -1
\]

Special case:
\[
\int k\,dx = kx + C \text{ (because } k = kx^0) \]

(d) **Exponential Functions:**
\[
\int e^x\,dx = e^x + C
\]
\[
\int a^x\,dx = \frac{a^x}{\ln a} + C
\]

(e) **Natural Logarithm:**
\[
\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C
\]

**Example 6**

Find the antiderivative of \( 3x^7 - 15\sqrt{x} + \frac{14}{x^2} \)

\[
\int \left(3x^7 - 15\sqrt{x} + \frac{14}{x^2}\right)\,dx = \int \left(3x^7 - 15x^{1/2} + 14x^{-2}\right)\,dx = 3\frac{x^8}{8} - 15\frac{x^{3/2}}{3/2} + 14\frac{x^{-1}}{-1} + C
\]

That’s a little hard to look at, so you might want to simplify a little:

\[
\int \left(3x^7 - 15\sqrt{x} + \frac{14}{x^2}\right)\,dx = \frac{3x^8}{8} - 10x^{3/2} - 14x^{-1} + C.
\]

**Example 7**

Find \( \int \left(e^x + 12 - \frac{16}{x}\right)\,dx \)

\[
\int \left(e^x + 12 - \frac{16}{x}\right)\,dx = e^x + 12x - 16\ln|x| + C
\]
Related Exercises You Should Complete Now

Make sure you work on Exercises 4.2.1, 4.2.4, 4.2.6, 4.2.7 and 4.2.8. Remember, you have written solutions and videos for these exercises in your course.

Specific Anti-Derivatives

Suppose that \( f'(x) \) is the derivative of \( f(x) \). Then a specific anti-derivative of \( f(x) \) is \( f(x) + c \) written \( \int f'(x) \, dx = f(x) + c \) where \( c \) is a KNOWN value. This is often called an initial value problem

An equation containing a derivative is called a differential equation. We can find a specific antiderivative when you are given the initial condition and asked to find the original equation.

Example 8

Consider the function \( f'(x) = 3x^2 \).

We know that \( \frac{df}{dx}(x^3) = 3x^2 \).

The function \( f(x) = x^3 \) is a specific anti-derivative of \( f'(x) = 3x^2 \) where \( f(x) = x^3 \) passes through the point (0,0).

We know that \( \frac{df}{dx}(x^3) = 3x^2 \).

The function \( f(x) = x^3 + 7 \) is a specific anti-derivative of \( f'(x) = 3x^2 \) where \( f(x) = x^3 + 7 \) passes through the point (0, 7).

We know that \( \frac{df}{dx}(x^3) = 3x^2 \).

The function \( f(x) = x^3 - 13 \) is a specific anti-derivative of \( f'(x) = 3x^2 \) where \( f(x) = x^3 - 13 \) passes through the point (0, -13).

We know that \( \frac{df}{dx}(x^3) = 3x^2 \) for any constant value \( C \).

The function \( f(x) = x^3 + C \) is the general anti-derivative of \( f'(x) = 3x^2 \). We do not know any specific point on \( f(x) \). We only know its shape.

Example 9

Find \( F(x) \) so that \( F'(x) = e^x \) and \( F(0) = 10 \).

This time we are looking for a particular antiderivative; we need to find exactly the right constant. Let’s start by finding the antiderivative:

\[
\int e^x \, dx = e^x + C
\]

So we know that \( F(x) = e^x + \text{some constant} \); we just need to find which one. For that, we’ll use the other piece of information (the initial condition):
\[
F(x) = e^x + C \\
F(0) = e^0 + C = 1 + C = 10 \\
C = 9 \\
\]
The particular constant we need is 9; \( F(x) = e^x + 9 \).

**Example 10**

The graph to the right shows \( f'(x) \) - the rate of change of \( f(x) \). Use it sketch a graph of \( f(x) \) that satisfies \( f(0) = 0 \).

Recall from the last chapter the relationships between the function graph and the derivative graph:

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>Increasing</th>
<th>Decreasing</th>
<th>Concave Up</th>
<th>Concave Down</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>Positive</td>
<td>Negative</td>
<td>Increasing</td>
<td>Decreasing</td>
</tr>
</tbody>
</table>

In the graph shown, we can see the derivative is positive on the interval \((0, 1)\) and \((3, \infty)\), so the graph of \( f \) should be increasing on those intervals. Likewise, \( f \) should be decreasing on the interval \((1,3)\).

In the graph, \( f' \) is decreasing on the interval \((0, 2)\), so \( f \) should be concave down on that interval. Likewise, \( f \) should be concave up on the interval \((2, \infty)\).

The derivative itself is not enough information to know where the function \( f(x) \) starts, since there are a family of antiderivatives, but in this case we are given a specific point to start at.

To start the sketch, we might note first the shapes we need:

<table>
<thead>
<tr>
<th>Increasing</th>
<th>Decreasing</th>
<th>Decreasing</th>
<th>Increasing</th>
<th>Increasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>conc down</td>
<td>conc down</td>
<td>conc up</td>
<td>conc up</td>
<td>conc up</td>
</tr>
</tbody>
</table>

then sketch the basic shapes.

Now we can attempt to sketch the graph, starting at the point \((0, 0)\). Notice we are very roughly sketching this, as we don't have much information to work with. We can tell, though, from the graph that the area from \( x = 0 \) to \( x = 1 \) is about the same as the area from \( x = 1 \) to \( x = 3 \), so we would expect the net area from \( x = 0 \) to \( x = 3 \) to be close to 0.
It turns out this graph isn't horribly bad. Smoothing it out would give a graph closer to the actual antiderivative graph, shown below.

Related Exercises You Should Complete Now

Make sure you work on Exercises 4.2.2, 4.2.3, 4.2.5, and 4.2.10. Remember, you have written solutions and videos for these exercises in your course.

The Fundamental Theorem of Calculus

This section contains the most important and most used theorem of calculus, the Fundamental Theorem of Calculus. Discovered independently by Newton and Leibniz in the late 1600s, it establishes the connection between derivatives and integrals, provides a way of easily calculating many (but not all!) integrals, and was a key step in the development of modern mathematics to support the rise of science and technology. Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is a most important brick in that beautiful structure.

The Fundamental Theorem of Calculus:

\[
\int_a^b F'(x) \, dx = F(b) - F(a)
\]

This is actually not new for us; we’ve been using this relationship for some time; we just haven’t written it this way. This says what we’ve said before: the definite integral of a rate from \( a \) to \( b \) is the net y-units, the change in \( y \), that accumulate between \( t = a \) and \( t = b \). Here we’ve just made it plain that the rate is a derivative.
Thinking about the relationship this way gives us the key to finding exact answers for some definite integrals. If the integrand is the derivative of some \( F \), then maybe we could simply find \( F \) and subtract — that would be easier than approximating with rectangles. Going backwards through the differentiation process will help us evaluate definite integrals.

Do keep in mind that there could be functions whose antiderivative we cannot find. In that case, the approximation techniques (we discussed using more and more rectangles) are still essential!

Antiderivatives Graphically or Numerically

Another way to think about the Fundamental Theorem of Calculus is to solve the expression for \( F(b) \):

**The Fundamental Theorem of Calculus (restated)**

\[
\int_a^b F'(x) \, dx = F(b) - F(a)
\]

The definite integral of a derivative from \( a \) to \( b \) gives the net change in the original function.

\[
F(b) = F(a) + \int_a^b F'(x) \, dx
\]

The amount we end up is the amount we start with plus the net change in the function.

This lets us get values for the antiderivative — as long as we have a starting point, and we know something about the area.

**Example 11**

Suppose \( F(t) \) has the derivative \( f(t) \) shown below, and suppose that we know \( F(0) = 5 \). Find values for \( F(1), F(2), F(3), \text{and} F(4) \).

Using the second way to think about the Fundamental Theorem of Calculus,

\[
F(b) = F(a) + \int_a^b F'(x) \, dx -- \text{ we can see that}
\]

\[
F(1) = F(0) + \int_0^1 f(x) \, dx \text{. We know the value of } F(0) \text{, and we can easily find } \int_0^1 f(x) \, dx \text{ from the graph -- it’s just the area of a triangle.}
\]

So  \( F(1) = F(0) + \int_0^1 f(x) \, dx = 5 + 0.5 = 5.5 \)

\[
F(2) = F(0) + \int_0^2 f(x) \, dx = 5 + 1 = 6
\]

Note that we can start from any place we know the value of — now that we know \( F(2) \), we can use that:
Example 12

\[ F'(t) = f(t) \] is shown below. Where does \( F(t) \) have maximum and minimum values on the interval \([0, 4]\)?

Since \( F(b) = F(a) + \int_a^b f(t) \, dt \), we know that \( F \) is increasing as long as the area accumulating under \( F'(t) = f(t) \) is positive (until \( t = 3 \)), and then decreases when the curve dips below the t-axis so that signed area starts accumulating with a negative value. The area between \( t = 3 \) and \( t = 4 \) is much smaller than the positive area that accumulates between 0 and 3, so we know that \( F(4) \) must be larger than \( F(0) \). The maximum value is when \( t = 3 \); the minimum value is when \( t = 0 \).

Antiderivatives of Formulas

Now we can put the ideas of areas and antiderivatives together to get a way of evaluating definite integrals that is exact and often easy. To evaluate a definite integral \( \int_a^b f(t) \, dt \), we can find any antiderivative \( F(t) \) of \( f(t) \) and evaluate \( F(b) - F(a) \). The problem of finding the exact value of a definite integral reduces to finding some (any) antiderivative \( F(t) \) of the integrand and then evaluating \( F(b) - F(a) \). Even finding one antiderivative can be difficult, and we will stick to functions that have easy antiderivatives.

The evaluation \( F(b) - F(a) \) is represented by the symbol \( F(x)\big|_a^b \) or \( F(x)\big|_a^b \).

Example 13

Evaluate \( \int_1^3 x \, dx \) in two ways:

(i) By sketching the graph of \( y = x \) and geometrically finding the area.
(ii) By finding an antiderivative of \( F(x) \) of the integrand and evaluating \( F(3) - F(1) \).
(i) The graph of \( y = x \) is shown to the right, and the shaded region corresponding to the integral has area 4. (Make sure you agree!!)

(ii) One antiderivative of \( x \) is \( F(x) = \frac{1}{2} x^2 \), and

\[
\int_1^3 x \, dx = \frac{1}{2} x^2 \bigg|_1^3 = \left[ \frac{1}{2} (3)^2 \right] - \left[ \frac{1}{2} (1)^2 \right] = \frac{9}{2} - \frac{1}{2} = 4.
\]

Note that this answer agrees with the answer we got geometrically.

If we had used another antiderivative of \( x \), say \( F(x) = \frac{1}{2} x^2 + 7 \), then

\[
\int_1^3 x \, dx = \left( \frac{1}{2} x^2 + 7 \right) \bigg|_1^3 = \left[ \frac{1}{2} (3)^2 + 7 \right] - \left[ \frac{1}{2} (1)^2 + 7 \right] = \frac{9}{2} + 7 - \frac{1}{2} - 7 = 4.
\]

Whatever constant you choose, it gets subtracted away during the evaluation; we might as well always choose the easiest one, where the constant = 0.

Example 14

Find the area between the graph of \( y = 3x^2 \) and the horizontal axis for \( x \) between 1 and 2.

This is \( \int_1^2 3x^2 \, dx = x^3 \bigg|_1^2 = 2^3 - 1^3 = 7. \)

Example 15

A company determines their marginal cost for production, in dollars per item, is \( MC(x) = \frac{4}{\sqrt{x}} + 2 \) when producing \( x \) thousand items. Find the cost of increasing production from 4 thousand items to 5 thousand items.

Remember that marginal cost is the rate of change of cost, and so the fundamental theorem tells us that \( \int_a^b MC(x) \, dx = \int_a^b C'(x) \, dx = C(b) - C(a) \). In other words, the integral of marginal cost will give us a net change in cost. To find the cost of increasing production from 4 thousand items to 5 thousand items, we need to integrate \( \int_4^5 MC(x) \, dx \).

We can write the marginal cost as \( MC(x) = 4x^{-1/2} + 2 \). We can then use the basic rules to find an antiderivative:

\[
C(x) = 4 \left( \frac{x^{1/2}}{1/2} \right) + 2x = 8\sqrt{x} + 2x.
\]
Net change in cost = \( \int_{\frac{5}{4}}^{5} \left( \frac{4}{\sqrt[3]{x}} + 2 \right) dx = \left( 8\sqrt[3]{x} + 2x \right) \bigg|_{\frac{5}{4}}^{5} = \left( 8\sqrt[3]{5} + 2 \cdot 5 \right) - \left( 8\sqrt[3]{4} + 2 \cdot 4 \right) \approx 3.889 \)

Remember, our units are in thousands of dollars, so we can now say: **It will cost $3,889 to increase production from 4 thousand items to 5 thousand items.**

### Related Exercises You Should Complete Now

Make sure you work on Exercises 4.2.9, 4.2.11 and 4.2.12. Remember, you have written solutions and videos for these exercises in your course.

### Very Important Summary

The **definite integral** finds a NUMBER that represents the overall accumulated amount of change in the antiderivative function over the given interval.

You should distinguish carefully between definite and indefinite integrals.

A **definite integral** \( \int_{a}^{b} f(x) \, dx \) is a **number**!

An **indefinite integral** \( \int f(x) \, dx \) is a family of **functions**

The **connection between them** is given by the FTC:

If \( f(x) \) is continuous on \([a, b]\), then \( \int_{a}^{b} f'(x) \, dx = f(b) - f(a) \)

### Summary of Key Ideas

There are 3 types of problems we have worked with.

(a) **Finding general anti-derivatives.**

Example: Find \( \int (2x) \, dx \)

Remember: This **symbol** is called the **indefinite integral** and gives us the general antiderivative.

**Solution:**

This problem is asking us to find the **general anti-derivative**, or indefinite integral, of the function \( f'(x) = 2x \).

Notice that there are not lower and upper limits given on this integral. Your answer will be a general function that contains an unknown constant, \( + C \).

Therefore \( \int (2x) \, dx = x^2 + C \).
(b) Finding specific anti-derivatives.

Example: Find \( f(x) = \int (2x)dx \), if we know that \( f(3) = 16 \).

Solution:
This problem is asking us to find the **specific anti-derivative**, knowing that the anti-derivative function passes through the point \((3, 16)\).

First, find the **general anti-derivative function**. Answer: \( f(x) = x^2 + C \)

\[
\begin{align*}
  f(x) &= x^2 + C \\
  16 &= (3)^2 + C \\
  16 &= 9 + C \\
  7 &= C \\
  f(x) &= x^2 + 7
\end{align*}
\]

(c) Finding definite integrals using the Fundamental Theorem of Calculus.

Example: Find \( \int_{4}^{7} (2x)dx \)

This symbol is called the **definite integral** and gives us the accumulated amount of change. The integral has upper and lower bounds

Solution:
This problem is asking us to find the **accumulated change** in the anti-derivative function between \( x = 4 \) and \( x = 7 \). We want to know how much the anti-derivative function, \( f(x) \), went up, or how much it went down between \( x = 4 \) and \( x = 7 \).

An alternative way to interpret this is to realize that we are finding the area between \( y = 2x \) and the \( x \)-axis (where area below the \( x \)-axis is counted negative).
Exercises for Unit 4 Section 2

**Exercise 4.2.1:** Find the following general anti-derivative functions.

a) $\int x^9 \, dx$  

b) $\int x^6 \, dx$  

c) $\int t^{-11} \, dt$  

d) $\int w^{-6} \, dw$  

e) $\int x^{0.6} \, dx$  

f) $\int x^{-0.8} \, dx$

**Exercise 4.2.2:** Find $f(x)$ if we know that $f'(x) = x^7$ and we know that $f(0) = 12$. This is a TYPE (2) problem: Finding specific anti-derivatives.

**Exercise 4.2.3:** Find $f(x)$ if we know that $f'(x) = x^{-3}$ and we know that $f(1) = 5$. This is a TYPE (2) problem: Finding specific anti-derivatives.
Exercise 4.2.4: Find the following general anti-derivative functions.

(a) \( \int 7x^4 - 8x^{-5} + 9 \, dx = \)

(b) \( \int 8x + 7 \, dx = \)

Exercise 4.2.5: Find \( f(x) \) if we know that \( f'(x) = x^4 + 3x^{-5} - 9 \) and we know that \( f(1) = -3.55 \).

Exercise 4.2.6: Find the following general anti-derivative functions.

(a) \( \int e^{5x} \, dx \)

(b) \( \int 5^x \, dx \)
Exercise 4.2.7: Find the following general anti-derivative functions.

(a) $\int \frac{5}{x} \, dx$

(b) $\int \left[ \frac{1}{x} + 3x^4 - 1 \right] \, dx$

Exercise 4.2.8: Find the anti-derivative for each.

(a) $(x^3 + x^2 + x + 8) \int \, dx$

(b) $\int (e^x + \frac{1}{x^2} + 7^x) \, dx$, where $x > 0$

(c) $\int (5x^3 - 3x^2 + 7x - 5^x) \, dx$

(d) $\int \left( 75 \left( 1.04 \right)^x + \frac{180}{x} + 7 \left( 0.85 \right)^x \right) \, dx$ where $x > 0$

(e) $\int \left( 3x^7 + \frac{14}{x^2} - 15\sqrt{x} \right) \, dx$
(f) \( \int \left( e^x - \frac{16}{x} + 12 \right) dx \)

**Exercise 4.2.9:**

(a) Find the general anti-derivative \( \int (5x^3 - 4x^2 - 7x - 9)dx \)

(b) Find the specific anti-derivative \( f(x) = \int (5x^3 - 4x^2 - 7x - 9)dx \)
   if we know that \( f(5) = 20 \).

(c) Find the definite integral \( \int_{3}^{7} (5x^3 - 4x^2 - 7x - 9) \, dx \) using the Fundamental Theorem of Calculus.
Suppose \( f'(x) = 5x^3 - 4x^2 - 7x - 9 \), and suppose \( f(x) \) is one of the anti-derivative functions. Graphically show what the value in part c represents on the graph of \( f'(x) \) below.

**Exercise 4.2.10:** The rate of change of the weight of a laboratory mouse can be modeled as 
\[
 w'(t) = \frac{7.37}{t}
\]
where \( t \) is the age of the mouse in weeks, and \( w'(t) \) is the weight change in grams per week. The model is valid for \( t > 2 \).

(a) Is the mouse’s weight increasing or decreasing? Explain how you know from the given function.

(b) At the age of 9 weeks the mouse weighed 26 grams. Find the specific anti-derivative function that gives the weight of the mouse. Show all work.
(c) Find the weight of the mouse after 11 weeks.

Exercise 4.2.11: The rate of change of the number of retail prescription drug sales can be modeled as
\[ P'(x) = -0.0084x + 0.166 \] where \( x \) is the number of years since 1995, and \( P'(x) \) is in billions of prescriptions per year. This model is valid \( 0 \leq x \leq 13 \).

(a) Find \( \int_3^{10} P'(x) \, dx \)

(b) Write a sentence to interpret the contextual meaning of the calculation and answer from part (a).

(c) The graph shown is one particular antiderivative of \( P'(x) \). Label the meaning of the calculation you found in part (a) on the graph.
**Exercise 4.2.12.** The marginal revenue at a certain company between 2001 and 2008 can be modeled as \( R'(t) = 5.23 (1.0546)^t \) where \( t \) is the number of items sold in millions, and \( R'(t) \) is the millions of dollars revenue per million sold in revenue.

(a) Find the general anti-derivative \( \int R'(t)\,dt \)

(b) Suppose we know that the total revenue was $51.811 million when sales were 13 million. Use that information to find the formula for revenue, \( R(t) \), where \( t \) is the millions sold and \( R(t) \) is the millions earned in revenue. Show all work.

(c) Find \( \int_6^9 R'(t)\,dt \). Interpret the meaning of the answer in real-world, contextual terms.
ADDITIONAL PRACTICE USING THE FTC

If \( f \) is continuous on the interval \([a, b]\), then \( \int_{a}^{b} f(x) \, dx = F(b) - F(a) \) where \( F \) is any antiderivative of \( f \), that is \( F' = f \). Use the FTC to find each of the following EXACTLY.

1. \( \int_{-1}^{2} x^2 \, dx = \)

2. \( \int_{1}^{2} \frac{3}{x^5} \, dx = \)

3. \( \int_{0}^{4} (9t^4 - 4t + 5) \, dt = \)

4. \( \int_{0}^{4} [(4 - t)\sqrt{t}] \, dt = \) [HINT: Distribute first!]

5. \( \int_{2}^{5} 8 \, dx = \)

6. \( \int_{0}^{3} \frac{1}{8} e^{4x} \, dx = \)

7. \( \int_{0}^{1} (5x^4 - 8x^2 - 2x + 1) \, dx = \)

8. \( \int_{1}^{4} \sqrt{x} \left( x + \frac{1}{x} \right) \, dx = \) [HINT: Distribute first!]
9. \( \int_1^8 \frac{2x+7}{x^{4/3}} \, dx = \)

[Hint: Simplify, Distribute first!]

10. \( \int_1^4 \left( 2e^{x/2} + \frac{4}{x} \right) \, dx = \)