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Section 1.1 Functions and Function Notation

What is a Function?
The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask, “If I know one quantity, can I then determine the other?” This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Function (* Video link - Tables as Functions)

Function: A rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say “the output is a function of the input.”

Example 1
In the height and age example above, is height a function of age? Is age a function of height?

Solution:
In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. For example, on my 18th birthday, I had exactly one height of 69 inches.

However, age is not a function of height, since one height input might correspond with more than one output age. For example, for an input height of 70 inches, there is more than one output of age since I was 70 inches at the age of 20 and 21.

Example 2
At a coffee shop, the menu consists of items and their prices. Is price a function of the item? Is the item a function of the price?

Solution:
We could say that price is a function of the item, since each input of an item has one output of a price corresponding to it. We could not say that item is a function of price, since two items might have the same price.
Example 3

In many classes the overall percentage you earn in the course corresponds to a letter grade. Is the letter grade a function of percentage? Is percentage a function of letter grade?

Solution:
For any percentage earned, there would be a letter grade associated, so we could say that the letter grade is a function of percentage earned. That is, if you input the percentage earned, your output would be a letter grade. Percentage may or may not be a function of the letter grade, depending upon the teacher’s grading scheme. With some grading systems, there are a range of percentages that correspond to the same letter grade. (i.e. an input of 94% will be associated with an output of a letter grade of A, but every input of a letter grade of A may NOT be associated with an output of 94%, as the percentage grade could be 99%, 97%, etc.)

One-to-One Function

Sometimes in a relationship each input corresponds to exactly one output, and every output corresponds to exactly one input. We call this kind of relationship a **one-to-one function**.

Typically, at a college, each student has exactly one student id number, and no two students have the same student id number. This would be an example of a one-to-one function. Each student corresponds to one student id number, and each student id number corresponds to one specific student.

Try it Now

Let’s consider bank account information.

1. Is your balance a function of your bank account number?
   *(if you input a bank account number does it make sense that the output is your balance?)*

2. Is your bank account number a function of your balance?
   *(if you input a balance does it make sense that the output is your bank account number?)*
Function Notation

To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write “height is a function of age”, we could use the descriptive variable $h$ to represent height and we could use the descriptive variable $a$ to represent age.

“height is a function of age” if we name the function $f$ we write
“$h$ is $f$ of $a$” or more simply
$h = f(a)$ we could instead name the function $h$ and write
$h(a)$ which is read “$h$ of $a$”

Remember we can use any variable to name the function; the notation $h(a)$ shows us that $h$ depends on $a$. The value “$a$” must be put into the function “$h$” to get a result. Be careful - the parentheses indicate that age is input into the function (Note: do not confuse these parentheses with multiplication!).

---

**Example 4**

Introduce function notation to represent a function that takes as input the name of a month, and gives as output the number of days in that month.

**Solution:**
The number of days in a month is a function of the name of the month, so if we name the function $f$, we could write “days = $f$ (month)” or $d = f(m)$. If we simply name the function $d$, we could write $d(m)$.

For example, $d$ (March) = 31, since March has 31 days. The notation $d(m)$ reminds us that the number of days, $d$ (the output) is dependent on the name of the month, $m$ (the input).

---

**Example 5**

A function $N = f(y)$ gives the number of police officers, $N$, in a town in year $y$. What does $f(2005) = 300$ tell us?

**Solution:**
When we read $f(2005) = 300$, we see the input quantity is 2005, which is a value for the input quantity of the function, the year, $y$. The output value is 300, the number of police officers ($N$), a value for the output quantity. Remember $N = f(y)$.

**Interpretation:** This tells us that in the year 2005 there were 300 police officers in the town.
Tables as Functions

Functions can be represented in many ways: Words (as we did in the last few examples), tables of values (numerically), graphs, or formulas. Represented as a table, we are presented with a list of input and output values.

In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1: This table represents the input, number of the month (January = 1, February = 2, and so on) while the output is the number of days in that month. This represents everything we know about the months & days for a given year (that is not a leap year), \( D = f(m) \).

<table>
<thead>
<tr>
<th>(input) Month number, ( m )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(output) Days in month, ( D )</td>
<td>31</td>
<td>28</td>
<td>31</td>
<td>30</td>
<td>31</td>
<td>30</td>
<td>31</td>
<td>31</td>
<td>30</td>
<td>31</td>
<td>30</td>
<td>31</td>
</tr>
</tbody>
</table>

Table 2: The table below defines a function \( Q = g(n) \). Remember this notation tells us \( g \) is the name of the function that takes the input \( n \) and gives the output \( Q \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q )</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 3: This table represents the age of children in years and their corresponding heights. This represents just some of the data available for height and ages of children. Here the table does not represent a function because a given input does NOT give a unique output. (i.e. input of 5 gives two different output values).

<table>
<thead>
<tr>
<th>(input) ( a ), age in years</th>
<th>5</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(output) ( h ), height inches</td>
<td>40</td>
<td>42</td>
<td>44</td>
<td>47</td>
<td>50</td>
<td>54</td>
</tr>
</tbody>
</table>
Example 6 (* Video Example Here)

Which of these tables define a function (if any)? Are any of them one-to-one?

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

**Solution:**
The first and second tables define functions. In both, each input corresponds to exactly one output. The third table does not define a function since the input value of 5 corresponds with two different output values.

Only the first table is one-to-one; it is both a function, and each output corresponds to exactly one input. Although table 2 is a function, because each input corresponds to exactly one output, each output does not correspond to exactly one input so this function is not one-to-one. Table 3 is not even a function and so we don’t even need to consider if it is a one-to-one function.

**Try it Now**
3. If a tax table inputs your income and outputs your tax owed, would this be a function? Is it one-to-one?

**Solving and Evaluating Functions:**

When we work with functions, there are two typical things we do: **evaluate and solve.** Evaluating a function is what we do when we know an input, and use the function to determine the corresponding output. Evaluating will always produce one result, since each input of a function corresponds to exactly one output.

Solving equations involving a function is what we do when we know an output, and use the function to determine the inputs that would produce that output. Solving a function could produce more than one solution, since different inputs can produce the same output.
Example 7

Using the table shown, where $Q=g(n)$

a) Evaluate $g(3)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q$</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

Solution:
Evaluating $g(3)$ (read: “g of 3”) means that we need to determine the output value, $Q$, of the function $g$ given the input value of $n=3$. Looking at the table, we see the output corresponding to $n=3$ is $Q=7$, allowing us to conclude $g(3) = 7$.

b) Solve $g(n) = 6$

Solution:
Solving $g(n) = 6$ means we need to determine what input values, $n$, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$.

When we input 2 into the function $g$, our output is $Q = 6$

When we input 4 into the function $g$, our output is also $Q = 6$

Try it Now
4. Using the function in Example 7, evaluate $g(4)$

Graphs as Functions

Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.

The most common graph has $y$ on the vertical axis and $x$ on the horizontal axis, and we say $y$ is a function of $x$, or $y = f(x)$ when the function is named $f$. 
Example 8 (* Video Example Here)

Which of these graphs defines a function \( y = f(x) \)? Which of these graphs defines a one-to-one function?

![Graphs](image)

Solution:
Looking at the three graphs above, the first two define a function \( y = f(x) \), since for each input value along the horizontal axis there is exactly one output value corresponding, determined by the \( y \)-value of the graph. The 3rd graph does not define a function \( y = f(x) \) since some input values, such as \( x = 2 \), correspond with more than one output value.

Graph 1 is not a one-to-one function. For example, the output value 3 has two corresponding input values, -1 and 2.3

Graph 2 is a one-to-one function; each input corresponds to exactly one output, and every output corresponds to exactly one input.

Graph 3 is not even a function so there is no reason to even check to see if it is a one-to-one function.

### Vertical Line Test

The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define only one vertical output for each horizontal input and would therefore not be a function.

### Horizontal Line Test

Once you have determined that a graph defines a function, an easy way to determine if the function is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line crosses the graph more than once, then the graph does not define a one-to-one function.

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.
Example 9

Given the graph of \( f(x) \)

a) Evaluate \( f(2) \)
b) Solve \( f(x) = 4 \)

Solutions:
a) To evaluate \( f(2) \), we find the input of \( x = 2 \) on the horizontal axis. Moving up to the graph gives the point \( (2, 1) \), giving an output of \( y = 1 \). \( f(2) = 1 \).

b) To solve \( f(x) = 4 \), we find the value 4 on the vertical axis because if \( f(x) = 4 \) then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: (1,4) and (3,4). These give the two solutions to \( f(x) = 4 \):
\[
x = -1 \text{ or } x = 3.
\]
This means \( f(-1) = 4 \) and \( f(3) = 4 \), or when the input is -1 or 3, the output is 4.

Notice that while the graph in the previous example is a function, getting two input values for the output value of 4 shows us that this function is not one-to-one.

Try it Now
5. Using the graph from example 9, solve \( f(x)=1 \).

Formulas as Functions

When possible, it is very convenient to define relationships using formulas. If it is possible to express the output as a formula involving the input quantity, then we can define a function.
Example 10

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$ if possible.

Solution:
To express the relationship in this form, we need to be able to write the relationship where $p$ is a function of $n$, which means writing it as $p = [\text{something involving } n]$.

$2n + 6p = 12$  \hspace{0.5cm} \text{Subtract } 2n \text{ from both sides.}$

$6p = 12 - 2n$  \hspace{0.5cm} \text{Divide both sides by 6 and simplify.}$

$p = \frac{12 - 2n}{6} = \frac{12}{6} - \frac{2n}{6} = 2 - \frac{1}{3}n$

Having rewritten the formula as $p = \ldots$, we can now express $p$ as a function:
$p = f(n) = 2 - \frac{1}{3}n$

It is important to note that not every relationship can be expressed as a function with a formula. The important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations and no further solving is required. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed into machines, whenever solving things is much more challenging.

Example 11

Express the relationship $x = y^2$ as a function $y = f(x)$ if possible.

Solution:
If we try to solve for $y$ in this equation, we would take the square root of both sides. This results in the following: $\pm\sqrt{x} = y \hspace{0.5cm} \text{OR } y = \pm\sqrt{x}$.

Remember when we take the square root of both sides here, we get both the positive and negative square root as potential answers. We end up with two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$. Therefore, we cannot write this equation as a function $f(x)$. In this case, $y$ is not a function of $x$ because we will not get a unique y-value for each x-input.

Notice, for example, if $x = 9$, then the y-value could be either 3 or $-3$. 

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As with tables and graphs, it is common to evaluate and solve functions involving formulas. Evaluating will require replacing the input variable in the formula with the value provided and calculating. Solving will require replacing the output variable in the formula with the value provided, and solving for the input(s) that would produce that output.

**Example 12** (*Video Example Here*)

Given the function \( k(t) = t^3 + 2 \)

a) Evaluate \( k(2) \)

b) Solve \( k(t) = 1 \)

**Solutions:**

a) To evaluate \( k(2) \), we plug in the input value 2 into the formula wherever we see the input variable \( t \), then simplify

\[
k(2) = 2^3 + 2 \quad \text{Plug 2 in the equation for } t.
\]

\[
k(2) = 8 + 2 \quad \text{Simplify}
\]

So, \( k(2) = 10 \)

b) To solve \( k(t) = 1 \), we set the formula for \( k(t) \) equal to 1, and solve for the input value that will produce that output

\[
k(t) = 1 \quad \text{Substitute the original formula } k(t) = t^3 + 2
\]

\[
t^3 + 2 = 1 \quad \text{Subtract 2 from each side.}
\]

\[
t^3 = -1 \quad \text{Take the cube root of each side.}
\]

\[
t = -1
\]

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

We want to know if \( k(t) = 1 \) is true when \( t = -1 \).

\[
k(-1) = (-1)^3 + 2 \quad \text{Plug -1 in the equation for } t.
\]

\[
k(-1) = -1 + 2 \quad \text{Simplify}
\]

\[
k(-1) = 1 \quad \text{This shows that our solving above in part (b) is accurate.}
\]
Example 13
Given the function \( h(p) = p^2 + 2p \)
a) Evaluate \( h(4) \)
b) Solve \( h(p) = 3 \)

Solutions:
To evaluate \( h(4) \) we substitute the value 4 for the input variable \( p \) in the given function.

a) \( h(4) = (4)^2 + 2(4) \quad \text{Plug 4 in the equation for } p. \\
= 16 + 8 \quad \text{Simplify.} \\
= 24 \)

b) \( h(p) = 3 \quad \text{Substitute the original function } h(p) = p^2 + 2p. \\
p^2 + 2p = 3 \quad \text{This is quadratic, so we can rearrange the equation to get it } = 0. \\
p^2 + 2p - 3 = 0 \quad \text{Subtract 3 from each side.} \\
p^2 + 2p - 3 = 0 \quad \text{This expression is factorable, so we factor it.} \\
(p + 3)(p - 1) = 0 \quad \text{Factor.} \\
By the zero factor theorem since \( (p + 3)(p - 1) = 0 \), either \( p + 3 = 0 \) or \( p - 1 = 0 \) (or both of them equal 0) and so we solve both equations for \( p \), finding \( p = -3 \) from the first equation and \( p = 1 \) from the second equation.

This gives us the solution: \( h(p) = 3 \) when \( p = 1 \) or \( p = -3 \).

We found two solutions to \( h(p) = 3 \) in this case, which tells us this function is not one-to-one.

Try it Now
6. Given the function \( g(m) = \sqrt{m - 4} \)
a. Evaluate \( g(5) \)
b. Solve \( g(m) = 2 \)
**Basic Toolkit Functions**

In this text, we will be exploring functions – the shapes of their graphs, their unique features, their equations, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of elements to build from. We call these our “toolkit of functions” – a set of basic named functions for which we know the graph, equation, and special features.

For these definitions we will use $x$ as the input variable and $f(x)$ as the output variable.

<table>
<thead>
<tr>
<th>Toolkit Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Linear</strong></td>
</tr>
<tr>
<td>Constant:</td>
</tr>
<tr>
<td>Identity:</td>
</tr>
<tr>
<td>Absolute Value:</td>
</tr>
<tr>
<td>Power</td>
</tr>
<tr>
<td><strong>Quadratic:</strong></td>
</tr>
<tr>
<td><strong>Cubic:</strong></td>
</tr>
<tr>
<td>Reciprocal:</td>
</tr>
<tr>
<td>Reciprocal squared</td>
</tr>
<tr>
<td><strong>Square root:</strong></td>
</tr>
<tr>
<td><strong>Cube root:</strong></td>
</tr>
</tbody>
</table>

You will see these toolkit functions, combinations of toolkit functions, their graphs and their transformations frequently throughout this book. In order to successfully follow along later in the book, it will be very helpful if you can recognize these toolkit functions and their features quickly by name, equation, graph, and basic table values.
Graphs of the Toolkit Functions

Constant Function: $f(x) = 2$  
Identity: $f(x) = x$  
Absolute Value: $f(x) = |x|$

Quadratic: $f(x) = x^2$  
Cubic: $f(x) = x^3$  
Square root: $f(x) = \sqrt{x}$

Cube root: $f(x) = \sqrt[3]{x}$  
Reciprocal: $f(x) = \frac{1}{x}$  
Reciprocal squared: $f(x) = \frac{1}{x^2}$
### Important Topics of this Section

- Definition of a function
- Input (independent variable)
- Output (dependent variable)
- Definition of a one-to-one function
- Function notation
- Descriptive variables
- Functions in words, tables, graphs & formulas
- Vertical line test
- Horizontal line test
- Evaluating a function at a specific input value
- Solving a function given a specific output value
- Toolkit Functions

### Try it Now Answers

1. Yes: for each bank account, there would be one balance associated
2. No: there could be several bank accounts with the same balance
3. Yes it’s a function; No, it’s not one-to-one (taxes owed many be the same for an interval of incomes.)
4. When \( n = 4 \), \( Q = g(4) = 6 \)
5. There are two points where the output is 1: \( x = 0 \) or \( x = 2 \)
6. a. \( g(5) = \sqrt{5 - 4} = 1 \)
   
   b. \( \sqrt{m - 4} = 2 \). Square both sides to get \( m - 4 = 4 \). \( m = 8 \)
Section 1.1 Exercises

1. The amount of garbage, $G$, produced by a city with population $p$ is given by $G = f(p)$. $G$ is measured in tons per week, and $p$ is measured in thousands of people.
   a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function $f$.
   b. Explain the meaning of the statement $f(5) = 2$.

2. The number of cubic yards of dirt, $D$, needed to cover a garden with area $a$ square feet is given by $D = g(a)$.
   a. A garden with area 5000 ft$^2$ requires 50 cubic yards of dirt. Express this information in terms of the function $g$.
   b. Explain the meaning of the statement $g(100) = 1$.

3. Let $f(t)$ be the number of ducks in a lake $t$ years after 1990. Explain the meaning of each statement:
   a. $f(5) = 30$
   b. $f(10) = 40$

4. Let $h(t)$ be the height above ground, in feet, of a rocket $t$ seconds after launching. Explain the meaning of each statement:
   a. $h(1) = 200$
   b. $h(2) = 350$

5. Select all of the following graphs which represent $y$ as a function of $x$.
   a. 
   b. 
   c. 
   d. 
   e. 
   f. 

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6. Select all of the following graphs which represent \(y\) as a function of \(x\).

a. 

b. 

c. 

d. 

e. 

f. 

7. Select all of the following tables which represent \(y\) as a function of \(x\).

a. 

\[
\begin{array}{ccc}
 x & 5 & 10 \\
 y & 3 & 8 \\
\end{array}
\]

b. 

\[
\begin{array}{ccc}
 x & 5 & 10 \\
 y & 3 & 8 \\
\end{array}
\]

c. 

\[
\begin{array}{ccc}
 x & 5 & 10 \\
 y & 3 & 8 \\
\end{array}
\]

8. Select all of the following tables which represent \(y\) as a function of \(x\).

a. 

\[
\begin{array}{cc}
 x & y \\
 0 & -1 \\
 2 & 3 \\
 3 & 4 \\
 4 & 7 \\
 8 & 11 \\
 3 & 1 \\
\end{array}
\]

b. 

\[
\begin{array}{cc}
 x & y \\
 3 & 5 \\
 6 & 7 \\
\end{array}
\]

c. 

\[
\begin{array}{cc}
 x & y \\
 0 & -5 \\
 3 & 1 \\
 4 & 2 \\
 9 & 8 \\
 16 & 13 \\
\end{array}
\]

d. 

\[
\begin{array}{cc}
 x & y \\
 -1 & -4 \\
 1 & 2 \\
 4 & 2 \\
 9 & 7 \\
 12 & 13 \\
\end{array}
\]

9. Select all of the following tables which represent \(y\) as a function of \(x\) and are one-to-one.

a. 

\[
\begin{array}{ccc}
 x & 3 & 8 \\
 y & 4 & 7 \\
\end{array}
\]

b. 

\[
\begin{array}{ccc}
 x & 3 & 8 \\
 y & 4 & 7 \\
\end{array}
\]

c. 

\[
\begin{array}{ccc}
 x & 3 & 8 \\
 y & 4 & 7 \\
\end{array}
\]

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10. Select all of the following graphs which are one-to-one functions.

- a. 
- b. 
- c. 
- d. 
- e. 
- f. 

Given each function $f(x)$ graphed, evaluate $f(1)$ and $f(3)$

11. 
12. 

13. Given the function $g(x)$ in the graph below, find/solve $g(2)$ and also $g(x)=3$. 

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14. Given the function f(x) in the graph below, find/solve f(x)=3 and also f(4).

![Graph of function f(x)]

15. Based on the table below,
   a. Evaluate f(3)  
   b. Solve f(x)=1
<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>74</td>
<td>28</td>
<td>1</td>
<td>53</td>
<td>56</td>
<td>3</td>
<td>36</td>
<td>45</td>
<td>14</td>
<td>47</td>
</tr>
</tbody>
</table>

16. Based on the table below,
   a. Evaluate f(8)  
   b. Solve f(x)=7
<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>62</td>
<td>8</td>
<td>7</td>
<td>38</td>
<td>86</td>
<td>73</td>
<td>70</td>
<td>39</td>
<td>75</td>
<td>34</td>
</tr>
</tbody>
</table>

For each of the following functions, evaluate: f(−2), f(−1), f(0), f(1), and f(2)
17. f(x) = 4−2x  
18. f(x) = 8x^2−7x+3  
19. f(x) = −x^3+2x  
20. f(x) = 5x^4+x^2  
21. f(x) = 3+\sqrt{x+3}  
22. f(x) = 4−\sqrt{x−2}  
23. f(x) = (x−2)(x+3)  
24. f(x) = \frac{x−3}{x+1}  
25. f(x) = 2^x  
26. f(x) = 3^x

27. Let f(t) = 3t + 5
   a. Evaluate f(0)  
   b. Solve f(t) = 0

28. Let g(p) = 6−2p
   a. Evaluate g(0)  
   b. Solve g(p) = 0
29. Match each function name with its equation.
   a. \( y = x \)
   b. \( y = x^3 \)
   c. \( y = \sqrt[3]{x} \)
   d. \( y = \frac{1}{x} \)
   e. \( y = x^2 \)
   f. \( y = \sqrt{x} \)
   g. \( y = |x| \)
   h. \( y = \frac{1}{x^2} \)

   i. Cube root
   ii. Reciprocal
   iii. Linear
   iv. Square Root
   v. Absolute Value
   vi. Quadratic
   vii. Reciprocal Squared
   viii. Cubic

30. Match each graph with its equation.

   a. \( y = x \)
   b. \( y = x^3 \)
   c. \( y = \sqrt[3]{x} \)
   d. \( y = \frac{1}{x} \)
   e. \( y = x^2 \)
   f. \( y = \sqrt{x} \)
   g. \( y = |x| \)
   h. \( y = \frac{1}{x^2} \)

31. Sketch a reasonable graph for each of the following functions. [UW]
   a. Height of a person depending on age.
   b. Height of the top of your head as you jump on a pogo stick for 5 seconds.
   c. The amount of postage you must put on a first class letter, depending on the weight of the letter.

32. Sketch a reasonable graph for each of the following functions. [UW]
   a. Distance of your big toe from the ground as you ride your bike for 10 seconds.
   b. Your height above the water level in a swimming pool after you dive off the high board.
   c. The percentage of dates and names you’ll remember for a history test, depending on the time you study.

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33. Using the graph shown,
a. Evaluate \( f(c) \)
b. Solve \( f(x) = p \)
c. Suppose \( f(b) = z \). Find \( f(z) \)
d. What are the coordinates of points \( L \) and \( K \)?
**Section 1.2 Introduction to Excel**

Excel is a computer software tool that will aid in the calculation, evaluation, analysis, and graphing of mathematical expressions that we study. It will take time and require consistent practice to become familiar with this tool and proficient in its use.

**Downloading Excel:** (*Video - How to Download Excel to Your Home Computer*)

DCC students are given free access to downloading the Office suite, which includes Microsoft Excel. To Install Office 365 Apps, follow the instructions below. Make sure you do this from the computer you want to install the programs on!

1. Access your email from myDCC. Once in your email, click the “grid” icon in the upper left corner.

2. This will open a side menu. At the top of the side menu, click “Office 365.”

3. Clicking this link will open a new window. To the far right of this new window is a link to install office on your computer.

4. Clicking the “Install Office” button will give you two options. For most cases, the top one is what you want.

5. Follow the instructions given on the screen from here.
What we see in Excel to start:

Microsoft Excel is a commonly used spreadsheet program. In this section we will learn some basic Excel skills.

When we first open Excel, we see a worksheet labeled as “Sheet 1.”

We are able to add additional worksheets by clicking the + symbol next to “Sheet 1.”

Within each Excel worksheet, Excel labels the columns with letters (A, B, C, …), and Excel labels the rows with numbers (1, 2, 3, …). We refer to the little boxes in an Excel worksheet as Cells.

We reference particular Cells within Excel by using the Column-row designation. For example, in the picture above, the grey Cell is referred to as Cell C4.
Creating a Textbox in Excel  (*Video Link - Creating a Textbox in Excel)

Most people have experience creating textboxes in Microsoft Word. Microsoft Excel also allows users to create textboxes within worksheets. Students can use textboxes to take notes for themselves right there next to the work in Excel, and in general textboxes are a nice way for people to create summaries and comments regarding the Excel work that we see in a worksheet.

To create a textbox in Excel:

1. Click the Insert tab, and then select “Text Box.”

2. At this point, your cursor will change to look like this:  
   To create the textbox, hold down the left-mouse-button and drag your cursor down and to the right in order to draw the height and width of the textbox. Release the mouse and the textbox will appear.

3. You can adjust the size of the textbox after creating it. You can also drag the textbox and move it to wherever you like in the worksheet.
Basic Excel evaluation skills:

Example 1: Calculations in Excel  (*Video Example Here*)

Excel can work as any calculator to find the value of any arithmetic expression.

(a) Use Excel to calculate $2+5$ and also $7/9$, rounding your answers to two decimal places.

Solution: Open an Excel Blank Workbook.

In Cell A1 type the mathematical expression, $2+5$, and then hit the enter key.

In cell A2 type the mathematical expression, $7/9$, and then hit the enter key.

Notice that Excel DOES NOT calculate the sum of $2+5$. Excel will simply refer to the first expression, in cell A1, as text and Excel will refer to the second expression, in cell A2, as a date on the calendar.

In order to use Excel as a calculator, we must always begin our expression with an equal sign.

Retry these expressions again, using the “=” sign.

Notice the first value, $=2+5$, evaluates just fine as 7.

But the second calculation, $=7/9$, does not. The $7/9$ calculation still returns a date.

In order to have Excel calculate numerical values, we must format the column properly so that it returns numbers instead of dates.
Click your cursor at the top of column A in order to select all cells in column A.

Then right click to bring up a menu of options (the Format Cells box shown on the right).

Click the “Number” tab, and click “Number” from the list of options.

Choose the number of decimal places that you would like displayed in your answers. The default is 2 decimal places.

You can also choose here how you would like negative answers to be displayed.

Click “OK” to apply your changes.

Your answers in column A are now displayed accurately to two decimal places.

(b) Let’s try a new expression. Evaluate the numerical value of the expression: $(14.5 - 11.31)(9.2 - 17^2)$ using cell A3 in your Excel Workbook.

**Solution:**

This expression requires many operations, and just as we would with a calculator, we will use the “*” symbol for multiplication, and the “^” for raising to an exponent. Remember to start your expression in Excel with an “=” sign and use proper mathematical notation. Notice that Excel color codes its parenthesis as you “open” and “close” parenthesis as we use them in the expression.

Notice that the expression is also shown in the formula box at the top of the workbook as we type. Be especially careful with entering fractions, to make sure you enclose the entire numerator and entire denominator in parenthesis. The resulting answer is $-892.56$.

(c) Calculate $\sqrt{356}$ in Excel.

**Solution:**
Square root functions can be entered in Excel by using the SQRT command. This is one of many “built-in” functions within Excel. You will notice that when you begin to type a command like =SQRT, Excel will display a menu of possible options to choose from.

In cell A4, try entering the expression: $\sqrt{356}$ by typing =sq and you will see the SQRT option displayed. Remember to ALWAYS begin with an “=” sign!

The resulting answer is 18.87.

Try it Now
1. Use Excel to find the numerical value of each of the following by completing the entire calculation in one cell, and have the Excel cells give the resulting answers as decimals rounded to 4 places.
   a. In cell A1 calculate $6(2.63)^3$
   b. In cell A2 calculate $\frac{1046}{25+4(1.03)^2}$
   c. In cell A3 calculate $76.2^{1.4} + \frac{1}{7}$
   d. In cell A4 calculate $8^{-21}+7-5.9$
Example 2: Cell Referencing in Excel (*Video Example Here)

In order to perform operations involving variables in Excel, we must use a process called cell referencing. Add a new sheet to your Excel workbook by clicking on the little + sign at the bottom of the sheet next to “Sheet 1.”

You can name any worksheet by simple clicking on the sheet1 label and typing a new name. Let’s call Sheet1 “Example 1” and let’s call Sheet2 “Example 2.”

Perform the following operations in this new Excel sheet. Evaluate each expression as directed.

(a) Calculate \( \frac{(2 \cdot (5.01)^3)}{9} \) in cell A2, and label it with the value “A” in cell A1.

**Solution:**

Store the expression \( \frac{(2 \cdot (5.01)^3)}{9} \) in cell A2. Call this value A, and label it with an “A” in cell A1.

This returns a value of 27.9478 in cell A2.

(b) Calculate \( 20 \sqrt{113} \) in cell B2, and label it with the value “B” in cell A2.

**Solution:**

Store the expression \( 20 \sqrt{113} \) in cell B2. Call this value B and label it with a “B” in cell B1.

This returns a value of 212.6029 in cell B2.

(c) In cell B3, calculate the value of 2A+5B in cell C2 using Excel cell referencing of cells B1 and B2. Label this new value as “C” in cell A3.

**Solution:**

In order to perform this last operation we must reference the cells that have the numerical values of A and B to find the value of C. We do so by clicking on the cells being referenced in the expression, or typing the names of those cells. Notice how Excel again color codes the cells for us for easy reference. Cell B3 now has the expression 2A + 5B. Excel will automatically pull the values within cells B1 and B2 in order to perform the calculation 2A+5B.

The resulting value of C=2A+5B is 1118.904.
Example 3: Input/Output Tables in Excel

Use Excel to create an input-output table from \( x = -10 \) to \( x = 10 \) in increments of one for the function \( f(x) = 7x^2 + 5x + 10 \).

**Solution:**

We can use our knowledge of cell references to produce this input/output tables in Excel. We can start by adding a new sheet in our Excel workbook, naming it Example 3.

Next, in cell A1, type “x”, and in cell B1 type "\( f(x) \)".

In column A we could fill in the \( x \)-values by simply entering the values -10, -9, -8, etc. However, this will not be feasible in the future when we have many more \( x \)-values. Luckily, Excel has a feature that can help us with this task: the Fill feature. To use the fill feature, first type the first number in our domain (the first \( x \)-value), in this case -10 in cell A2 and hit enter.

Then click back into cell A2 and click on Excel’s Fill feature which is found in the upper right corner of the Home menu in the editing tab, click on the little arrow to the right of “Fill” to see the dropdown menu. Then click on “Series” within the Fill menu.

Select the “Columns” button in order to direct Excel to automatically fill in the column in which your cursor currently sits.

Next type in your “Step value”, in this case our increment. is 1.

Lastly fill in your “Stop value”, in this case 10.

Then click “OK”.

You will notice that your column A is now automatically filled with your correct values starting at -10, incrementing in units of 1, and stopping at 10.
Next, type the expression for \( f(x) \) in cell B2, using a cell reference for cell A2 to pull the x-value into the calculation as shown below.

To evaluate the remaining cells in column B there are two techniques that work equally well.

Putting your cursor in the **bottom right** of the highlighted cell B2 box, you will notice that the cursor changes from               to             to             .

a) Once you see the bold black “plus” sign you can hold down the left mouse button, and drag the formula down the column to evaluate the function at each of the other input values.

**OR**

b) Once you see the bold “plus” sign you can double click the mouse and the formula will evaluate the function at each of the other input values.

If we wanted to produce the table with different input values, we could simply add that input value to the list in column A, or refill the table with different input values.

For example, if we would like to see the value of this function from -2 to 5 using increments of 0.5, then we can reproduce the table by editing the input values using the fill feature.

We could also manually type in any input values we wanted to produce the desired output. Say we wanted the following outputs of this function: \( f(12.2), f(22), f(19), f(0.23), f(11.3) \)
Try it Now
2. Use an Excel input/output table to evaluate the following functions at the given values. Round all answers to four decimal places.
\[ f(-2), \ f(-1), \ f(0), \ f(1), \ \text{and} \ f(2) \]
a. \( f(x) = 4 - 2x \)

b. \( f(x) = 8x^2 - 7x + 3 \)
c. \( f(x) = -x^3 + 2x \)

d. \( f(x) = 4 - \sqrt{7 + x} \)

Video Link: Installing Solver Add-In on a MAC

Video Link: Installing Solver Add-In in Windows

Example 4: Using Solver in Excel [Video Example Here]

Use Excel’s Solver tool to solve the equation \( f(x) = 4 - 2x = 0 \).

Solution:
Excel has a tool called **Solver** which will solve a given function for a desired input value. In other words, Solver will give you the input value to produce a given output value. We must be careful when using the Solver command when there is more than one input value that will give us the same output value.

We want to have Excel solve the following equation, \( f(x) = 4 - 2x = 0 \). We first produce an input/output table for this function using any desired input values. Random input values were chosen here.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>f(x) = 4 - 2x</td>
</tr>
<tr>
<td>-1</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>7</td>
<td>-10</td>
</tr>
</tbody>
</table>

We can see from the table that perhaps the function will have an OUTPUT of zero somewhere between \( x = 0 \) and \( x = 3 \) since the y-values change from positive to negative somewhere in between \( x=0 \) and \( x=3 \). Notice that the answer we seek here is the x-value that will produce an output of zero, and not finding \( f(0) \).

Clicking on the DATA tab at the very top of the Excel workbook, you will see Solver (once it is installed) all the way on the right, under the analysis tab.
Next, click back in Column B of your table in an output value cell that has an output value that is close to the value for which we are solving. In this case let’s click on the output of -2 since that is close to y=0 that we are trying to solve for.

Next click on Solver to produce the following.

1) Putting your cursor in the “Set Objective” area, click on the output cell that has the output of -2 (cell B4). This will tell Excel that we want it to use the formula in cell B4 to solve.

2) Next click on the “Value of” button and also type the desired output value in the box next to it, in this case we want a value of zero. This will tell Excel that we want it to solve for when the formula in cell B4 is equal to a value of 0.

3) Next put your cursor in the “By Changing Variable Cells” and click on the input cell that we want to change in order to solve this equation, in this case cell A4. This will tell Excel which x-value cell to change in order to solve the equation.

4) Next click the Solve button at the bottom. Excel will change the x-value in cell A4 in order to make cell B4 equal 0 (or very close to 0).

The following message will appear and we can simply click the “OK” button to get the solution.
Notice that the table values changed: B4 will have a value of zero or very close to zero, and A4 will have the corresponding x-value that produces a y-value of 0.

**Example 5: Solver in Excel with Multiple Solutions**

*Video Example Here*

Use Excel to find all solutions to the following equation. \( g(x) = -x^3 + 2x = 0 \).

**Solution:**

Let’s first produce an input/output table.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>g(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>-10</td>
<td>-10</td>
</tr>
</tbody>
</table>

Notice that the table produces one correct answer: when \( x = 0 \), \( g(x) = 0 \).

But there are other answers to this equation. Somewhere between \( x = -2 \) and \( x = -1 \) the function probably equals zero, and again somewhere between \( x = 1 \) and \( x = 2 \) the output looks to have been zero. We know this because there is a sign of the output value in between these x-values.
Here Excel’s Solver will need a little help to produce the correct input value. We need to give Excel some direction about which x-value we are seeking when we ask it to solve for the x-value that makes y=0. We can do that by selecting a cell for the “By Changing Variable Cells” that is closest to the x-value that we are seeking.

We begin as we did before, using Solver. Let’s find the zero that occurs between \( x = -2 \) and \( x = -1 \) first.

1) Set the objective as cell B6 by placing the cursor in the “Set Objective” box and clicking on cell B6.

2) Place the cursor in the “By Changing Variable Cells” and click on cell A6. This is the closes cell to the x-value we seek, and it will serve as a “seed” starting point for Excel to search for the x-value. Since it is closes to the x-value we seek, it will identify that solution.

   *NOTE: You must un-check the “Make Unconstrained Variables Non-Negative” when your x-value solution may be negative.

3) Click “Solve” and Excel will change cell A6 to the x-value closest to the value in A6 that solves the equation. In this case, that x-value is approximately -1.41421.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>g(x)</td>
</tr>
<tr>
<td>-5</td>
<td>115</td>
</tr>
<tr>
<td>-4</td>
<td>56</td>
</tr>
<tr>
<td>-3</td>
<td>21</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1.41421</td>
<td>9.34813E-08</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>-21</td>
</tr>
<tr>
<td>4</td>
<td>-56</td>
</tr>
<tr>
<td>5</td>
<td>-115</td>
</tr>
</tbody>
</table>

Notice the output is not quite zero but very close, giving the answer as scientific notation. This is the number 0.0000000934813. So, essentially, it is y=0.
We could format column A and/or column B to display the answers in standard form with 6 decimal places as we would like by again right clicking on the column, and formatting the output to a number to 6 decimal places.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>g(x)</td>
</tr>
<tr>
<td>-5</td>
<td>115.000000</td>
</tr>
<tr>
<td>-4</td>
<td>56.000000</td>
</tr>
<tr>
<td>-3</td>
<td>21.000000</td>
</tr>
<tr>
<td>-2</td>
<td>4.890000</td>
</tr>
<tr>
<td>-1.41421</td>
<td>0.000000</td>
</tr>
<tr>
<td>0</td>
<td>0.000000</td>
</tr>
<tr>
<td>1</td>
<td>1.000000</td>
</tr>
<tr>
<td>2</td>
<td>-4.000000</td>
</tr>
<tr>
<td>3</td>
<td>-21.000000</td>
</tr>
<tr>
<td>4</td>
<td>-56.000000</td>
</tr>
<tr>
<td>5</td>
<td>-115.000000</td>
</tr>
</tbody>
</table>

Repeating the process of using Solver, we could find the third value where the output of the function is equal to zero between the input values of x = 1 and x = 2.

Therefore, the three input values that satisfy the equation \( g(x) = -x^3 + 2x = 0 \) are \( x = 0 \), and \( x = -1.41421 \), and \( x = 1.41421 \).
Try it Now

3. Use an Excel to solve the following equations. Round the answers to 4 decimal places.

a. $k(x) = 3 + \sqrt{x + 3} = 10$

b. $h(x) = x^2 - 7.2x + 9.4 = 0$

c. $m(x) = \frac{x - 3}{x + 1} = 0.1$

d. $t(x) = 2^x = 9$
Example 6: Graphing from a Table of Values in Excel (*Video Example Here)

Using Excel, sketch a Scatterplot and Smooth-line graph of the following data table.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>74</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
</tr>
<tr>
<td>5</td>
<td>53</td>
</tr>
<tr>
<td>6</td>
<td>56</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>45</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>11</td>
<td>47</td>
</tr>
</tbody>
</table>

Solution:
Copy the table into Excel as it is shown. Next, highlight all the data from cell A1 through cell B11.

Click on the INSERT tab. In the Charts group, click on the arrow next to the picture of a scatter plot to produce a drop-down menu.

Click on the upper middle graph option to produce a scatterplot with data points.

Or, select the upper right graph option to produce a smooth line graph.
Try it Now

4. Using Excel, sketch a Scatterplot and **Smooth-line graph** of the following data table.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>62</td>
<td>8</td>
<td>7</td>
<td>38</td>
<td>86</td>
<td>73</td>
<td>70</td>
<td>39</td>
<td>75</td>
<td>34</td>
</tr>
</tbody>
</table>

Graphing from an algebraic function is done the same way, except you must select an appropriate domain and an input/output table first.

**Example 7: Graphing in Excel using a Formula** (*Video Example Here*)

Using Excel, produce an input/output table for the function \( f(x) = 8x^2 - 7x + 3 \). Then sketch a Smooth-line graph of the function. Use input values from -10 to 10.

**Solution:** We first create the input-output table. Then highlight it and click Insert, and click on smoothline graph.
Try it Now

5. Using Excel, produce an input/output table for the functions below. Then sketch a Smooth-line graph of the function. Use input values in an appropriate domain.

a. \( f(x) = (x - 3)(x - 7) \) from \( x=0 \) to \( x=10 \)

b. \( f(x) = 4 - \sqrt{x + 5} \) from \( x=0 \) to \( x=50 \)

---

**Important Topics of this Section**

- Perform calculations in Excel
- Use cell references to perform calculations in Excel
- Create input/output tables in Excel
- Create smooth-line graphs in Excel

---

**Try it Now Answers**

1.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6*2.63^3</td>
</tr>
<tr>
<td>3</td>
<td>1046/(25+4*1.03^2)</td>
</tr>
<tr>
<td>4</td>
<td>76.2^1.4+1/7</td>
</tr>
<tr>
<td>5</td>
<td>8^(-0.21+7-5.9)</td>
</tr>
</tbody>
</table>

1) ANSWERS:

a) 109.1487
b) 35.7685
c) 431.4007
d) 6.3643

2.
3.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
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<th>H</th>
</tr>
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<tr>
<td>1</td>
<td>x</td>
<td>k(x)=3+sqrt(x+3)</td>
<td>x</td>
<td>h(x)=x^2-7.2x+9.4</td>
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4.

<table>
<thead>
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<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>x</td>
<td>f(x)</td>
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<tr>
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<td>9</td>
<td>34</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) k(x)=10 when x is 46.

b) h(x)=0 when x is 1.7132 and also when x=5.4868

c) m(x)=0.1 when x=3.4444

d) t(x)=9 when x=3.1699.
5. Using Excel, produce an input/output table for the functions below. Then sketch a smooth-line graph of functions. Use input values in an appropriate domain.

a. \( f(x) = -(x-3)^2 + 7 \) for \( x \in [0, 10] \)

b. \( g(x) = -x^2 + 5 \) for \( x \in [0, 50] \)

Try it Now Answer

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
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<tbody>
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<td>0</td>
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<td>9</td>
<td>0</td>
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<tr>
<td>10</td>
<td>3</td>
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<table>
<thead>
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<tbody>
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<td>7</td>
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<td>8</td>
<td>-4.88237748</td>
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<tr>
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<tr>
<td>10</td>
<td>-5.21544407</td>
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<td>11</td>
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<tr>
<td>13</td>
<td>-6.74763146</td>
</tr>
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</table>

Try it Now Answer

<table>
<thead>
<tr>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
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<td>0.12701566</td>
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<td>9</td>
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<td>-5.74679440</td>
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<td>12</td>
<td>-6.24656201</td>
</tr>
<tr>
<td>13</td>
<td>-6.74763146</td>
</tr>
</tbody>
</table>
Section 1.2 Exercises

1. Create an Excel file that does all of the following things. Then save your file and submit it in the manner that your instructor requests. This will be an indication that you have access to Excel, and are able to understand the basic terminology of Excel.

   a) Right click on cell D7, select “Format Cells.” In the Formal Cells box, go to the “Fill” tab. Select the grey color in order to make cell D7 grey.

   b) Create a textbox in your Excel worksheet. Within your textbox, write “This is my textbox!”

   c) Send the completed file to your instructor.

2. Use Excel to create a smooth-line graph of the following:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>74</td>
<td>28</td>
<td>53</td>
<td>56</td>
<td>3</td>
<td>36</td>
<td>45</td>
<td>14</td>
<td>47</td>
<td></td>
</tr>
</tbody>
</table>

3. Use Excel to create a smooth-line graph of the following:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>62</td>
<td>8</td>
<td>7</td>
<td>38</td>
<td>86</td>
<td>73</td>
<td>70</td>
<td>39</td>
<td>75</td>
<td>34</td>
</tr>
</tbody>
</table>

4. Use Excel to create a smooth-line graph of each of the following:
   a) the function $f(x) = 5x^4 + x^2$ from $x = -5$ to $x = 5$.
   b) the function $g(x) = 15 + \sqrt{x} - 6$ from $x = 6$ to $x = 20$.
   c) the function $h(x) = 150(0.85)^x$ from $x = 0$ to $x = 10$.
   d) the function $g(x) = -17(x - 33)(x - 10)$ from $x = 0$ to $x = 50$

5. Use Solver in Excel to solve each of the following. Round to 4 places.
   a) $f(t) = 3^t + 5t$ solve for when $f(t) = 10$
   b) $g(x) = (x - 2)(x - 4)(x - 7)$ solve for when $g(x) = 3$. Note that there are 3 solutions!
   c) $f(q) = \frac{12 - q}{q - 5}$ solve for when $f(q) = 1$. Hint: Create your input-output table from $q = 6$ up to $q = 15$.
   d) $f(t) = 1500(0.75)^t$ solve for when $f(t) = 10000$
Section 1.3 Domain and Range

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create.

This table shows a relationship between circumference and height of a tree as it grows.

<table>
<thead>
<tr>
<th>Circumference, c</th>
<th>1.7</th>
<th>2.5</th>
<th>5.5</th>
<th>8.2</th>
<th>13.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height, h</td>
<td>24.5</td>
<td>31</td>
<td>45.2</td>
<td>54.6</td>
<td>92.1</td>
</tr>
</tbody>
</table>

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of $-3$ feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the domain and range of the function.

**Domain and Range**

**Domain:** The set of possible input values to a function

**Range:** The set of possible output values of a function

**Notation**

Often, especially in real-world circumstances, there are restrictions on the domain and the range which go beyond the restrictions within the mathematical formula. We will need to rely on our life experiences and the conditions of the situation given to determine the appropriate restrictions on the domain and range of a given problem as we study contextual problems. We will begin by finding a reasonable domain, then extract a reasonable range.
Interval Notation is often used to represent the domain and range of a function.

A **closed interval** is written in interval notation as \([a, b]\). This interval indicates all real numbers \(x\) for which \(a \leq x \leq b\). Closed intervals include their endpoints. The brackets indicate that \(x = a\) and \(x = b\) are included in the interval.

![Diagram of a closed interval](image)

An **open interval** is written in interval notation as \((a, b)\). This interval indicates all real numbers \(x\) for which \(a < x < b\). Open intervals DO NOT include their endpoints. The parenthesis indicate that \(x = a\) and \(x = b\) are not included in the interval.

![Diagram of an open interval](image)

**Half-open (or half closed) intervals** are represented by

\([a, b)\) which indicates all real numbers \(x\) for which \(a \leq x < b\) or

\((a, b]\) which indicates all real numbers \(x\) for which \(a < x \leq b\).

When using the concept of infinity in an interval, we use an **open interval**, and a parenthesis, as infinity is a concept of tending toward a larger and larger value, (or negative infinity as tending toward a smaller and smaller value) which does not have an endpoint.

![Diagram of an interval with infinity](image)
Remember when writing or reading interval notation:
Using a square bracket [ or ] means that value is included in the set.
Using a parenthesis ( or ) means that value is not included in the set.

Below are some examples of changing from inequality notation to interval notation.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Interval notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5 &lt; h \leq 10$</td>
<td>$(5, 10]$</td>
</tr>
<tr>
<td>$5 \leq h &lt; 10$</td>
<td>$[5, 10)$</td>
</tr>
<tr>
<td>$5 &lt; h &lt; 10$</td>
<td>$(5, 10)$</td>
</tr>
<tr>
<td>$h &lt; 10$</td>
<td>$(-\infty, 10)$</td>
</tr>
<tr>
<td>$h \geq 10$</td>
<td>$[10, \infty)$</td>
</tr>
<tr>
<td>All real numbers</td>
<td>$(-\infty, \infty)$</td>
</tr>
</tbody>
</table>

To combine two intervals together, using inequalities we can use the word “or.” In interval notation, we use the union symbol, $\cup$, to combine two unconnected intervals together.

**Example 1**
Covert each of the following from Inequality Notation to Interval Notation.

a) $0 < x \leq 10 \ or \ 23 \leq x < 30$

b) $x \leq -2 \ or \ x > 3$

**Solutions:**

a) $(0,10] \cup [23,30)$

b) $(-\infty,-2] \cup (3,\infty)$
Example 2

Using the table below, which gives the circumference and height of trees in feet, determine a reasonable domain and range of the function $h = f(c)$.

<table>
<thead>
<tr>
<th>Circumference, $c$</th>
<th>1.7</th>
<th>2.5</th>
<th>5.5</th>
<th>8.2</th>
<th>13.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height, $h$</td>
<td>24.5</td>
<td>31</td>
<td>45.2</td>
<td>54.6</td>
<td>92.1</td>
</tr>
</tbody>
</table>

Solution:
We could use the data together with our own experiences and reason to approximate the domain and range of the function $h = f(c)$.

For the domain, the table gives some example values for the input circumference $c$. So, we know that the domain must include the values 17, 2.5, 5.5, 8.2, and 13.7. It doesn’t make sense to have negative values or a value of 0, since the circumference of a tree must be positive. So, we know that $c > 0$. We could make an educated guess at a maximum reasonable value of the circumference, or we could look up that the maximum circumference of a tree measured is about 119 feet\(^1\). With this information, we would say a reasonable domain is $0 < c \leq 119$ feet, or in interval notation the domain is values of $c$ on the interval $(0, 119]$.

Example 3

A company has fixed initial costs of $1500, as well as fixed variable costs for producing a certain product. The total costs of production for a given set of items is shown in the table below. Determine the domain and range.

<table>
<thead>
<tr>
<th>Product costs</th>
<th>Number produced</th>
<th>Total Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1400</td>
<td>$2340</td>
</tr>
<tr>
<td></td>
<td>3000</td>
<td>$3300</td>
</tr>
<tr>
<td></td>
<td>7280</td>
<td>$5868</td>
</tr>
<tr>
<td></td>
<td>12,490</td>
<td>$8994</td>
</tr>
</tbody>
</table>

Solution:
Suppose we notate the number produced by $n$ and total cost by $C$, and set up a function named $C(n)$, where the number produced, $n$, is a function of total cost, $C$. $C = C(n)$.

Since acceptable number produced is 0 or higher, we can conclude that the domain is $n \geq 0$. This can also be written as $[0, \infty)$.

Since possible total costs are $1500 and higher, we can conclude that the range is $C \geq 1500$. This can also be written as $[1500, \infty)$.

Try it Now
1. The population of a small town in the year 1960 was 100 people. Since then the population has grown to 1400 people reported during the 2010 census. Choose descriptive variables for your input and output and use interval notation to write the domain and range.

Example 4

Describe the intervals of values shown on the line graph below using inequality and interval notations.

Solution:
To describe the values, \( x \), that lie in the intervals shown above we would say, “\( x \) is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5.”

As an inequality it is: \( 1 \leq x \leq 3 \) or \( x > 5 \)
In interval notation: \([1,3] \cup (5,\infty)\)

Try it Now
2. Given the following interval, write its meaning in words, and interval notation.
Domain and Range from Graphs

We can also talk about domain and range based on graphs. Since domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the graph (the horizontal values). Likewise, since range is the set of possible output values, the range of a graph we can see from the possible values along the vertical axis of the graph.

Be careful – if the graph continues beyond the window on which we can see the graph, the domain and range might be larger than the values we can see.

Example 5 (* Video Example Here)

Determine the domain and range of the graph below.

![Graph of Alaska Crude Oil Production](http://commons.wikimedia.org/wiki/File:Alaska_Crude_Oil_Production.PNG)

Data from US Energy Information Administration

Solution:

In the graph above, the input quantity along the horizontal axis appears to be “year”, which we could notate with the variable $y$. The output is “thousands of barrels of oil per day”, which we might notate with the variable $b$, for barrels. The graph would likely continue to the left and right beyond what is shown, but based on the portion of the graph that is shown to us, we can determine the domain is $1975 \leq y \leq 2008$, and the range is approximately $180 \leq b \leq 2010$.

In interval notation, the domain would be [1975, 2008] and the range would be about [180, 2010]. For the range, we have to approximate the smallest and largest outputs since they don’t fall exactly on the grid lines.

---

Remember that, as in the previous example, $x$ and $y$ are not always the input and output variables. Using descriptive variables is an important tool to remembering the context of the problem.

Try it Now
3. Given the graph below write the domain and range in interval notation

![Graph of World Population Increase](source: U.S. Bureau of the Census, International Data Base via Gapminder)
Domain and Range of each Toolkit functions (*Video Link Here)

We will now return to our set of toolkit functions to note the domain and range of each.

**Constant Function:** $f(x) = c$

The domain here is not restricted; $x$ can be anything. When this is the case we say the domain is all real numbers. The outputs are limited to the constant value of the function.

Domain: $(-\infty, \infty)$
Range: $[c]$  

*Since there is only one output value, we list it by itself in square brackets.*

**Identity Function:** $f(x) = x$

Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

**Quadratic Function:** $f(x) = x^2$

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

*Multiplying a negative or positive number by itself can only yield a positive output.*
**Cubic Function:**  \( f(x) = x^3 \)
- **Domain:** \((-\infty, \infty)\)
- **Range:** \((-\infty, \infty)\)

**Reciprocal Function:**  \( f(x) = \frac{1}{x} \)
- **Domain:** \((-\infty, 0) \cup (0, \infty)\)
- **Range:** \((-\infty, 0) \cup (0, \infty)\)

*We cannot divide by 0 so we must exclude 0 from the domain.*

*One divide by any value can never be 0, so the range will not include 0.*

**Reciprocal squared:**  \( f(x) = \frac{1}{x^2} \)
- **Domain:** \((-\infty, 0) \cup (0, \infty)\)
- **Range:** \((0, \infty)\)

*We cannot divide by 0 so we must exclude 0 from the domain.*
Cube Root: $f(x) = \sqrt[3]{x}$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Square Root: $f(x) = \sqrt{x}$, commonly just written as, $f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$
When dealing with the set of real numbers we cannot take the square root of a negative number so the domain is limited to 0 or greater.

Absolute Value Function: $f(x) = |x|
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Since absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.
It is sometimes difficult to determine the domain values of a function when given the function as a mathematical expression in function notation. There are two situations that we will need to be aware of when determining a real domain for a function. Both are concerned with our previous knowledge about real numbers.

- First, recall that we cannot take the square root of a negative number in the real number system. So, when we encounter functions that have the square root, we can only use input values in which the radicand (the value under the square root symbol) is greater than or equal to zero.
- Second, recall that fractional expressions are only defined in the real number system when the denominator expression is not equal to zero (we cannot divide by zero in the Real Number system).

<table>
<thead>
<tr>
<th>Input values allowable for square root and fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inputs to a square root:</td>
</tr>
<tr>
<td>The function $f(x) = \sqrt{\text{expression}}$ will only produce real-valued outputs if $(\text{expression}) \geq 0$.</td>
</tr>
<tr>
<td>Inputs to fractional functions:</td>
</tr>
<tr>
<td>The function $f(x) = \frac{A}{B}$ will only produce real-valued outputs if $B \neq 0$.</td>
</tr>
</tbody>
</table>

**Example 6**

Find the domain of each function:

a) $f(x) = 2\sqrt{x + 4}$

b) $g(x) = \frac{3}{6 - 3x}$

**Solutions:**

a) Since we cannot take the square root of a negative number, we need the inside of the square root to be non-negative.

$x + 4 \geq 0$

**Subtract 4 on both sides.**

$x \geq -4$.

In interval notation, the domain of $f(x)$ is $[-4, \infty)$.

b) We cannot divide by zero, so we need the denominator to be non-zero.

$6 - 3x \neq 0$

**Add 3x on both sides.**

$6 \neq 3x$

**Divide both sides by 3.**

$2 \neq x$

Then $x \neq 2$, so we must exclude 2 from the domain.

In interval notation, the domain of $g(x)$ is $(-\infty, 2) \cup (2, \infty)$. 
**Important Topics of this Section**

- Definition of domain
- Definition of range
- Inequalities
- Interval notation
- Domain and Range from graphs
- Domain and Range of toolkit functions

---

**Try it Now Answers**

1. Domain: \( y = \text{years} \ [1960, 2010] \); Range, \( p = \text{population} \), [100, 1400]

2. a. Values that are less than or equal to -2, or values that are greater than or equal to -1 and less than 3
   
   b. \( x \leq -2 \text{ or } -1 \leq x < 3 \)

   c. \( (-\infty, -2] \cup [-1, 3) \)

3. Domain; \( y = \text{years} \ [1952, 2002] \); Range, \( p = \text{population in millions} \), [40,88]
Section 1.3 Exercises

Write the domain and range of the function using interval notation.

1. 

2. 

Write the domain and range of each graph as an inequality.

3. 

4. 

Suppose that you are holding your toy submarine under the water. You release it and it begins to ascend. The graph models the depth of the submarine as a function of time, stopping once the sub surfaces. What is the domain and range of the function in the graph?

5. 

6.
Find the domain of each function

7. $f(x) = 3\sqrt{x-2}$
8. $f(x) = 5\sqrt{x+3}$
9. $f(x) = 3 - \sqrt{6-2x}$
10. $f(x) = 5 - \sqrt{10-2x}$
11. $f(x) = \frac{9}{x-6}$
12. $f(x) = \frac{6}{x-8}$
13. $f(x) = \frac{3x+1}{4x+2}$
14. $f(x) = \frac{5x+3}{4x-1}$
15. $f(x) = \frac{\sqrt{x+4}}{x-4}$
16. $f(x) = \frac{\sqrt{x+5}}{x-6}$

17. The following graph shows music cassette tape sales, $S(t)$ in millions of cassettes, over time, $t$ in years.

Find the following:

a. What is the input variable and what does it represent, with units?
b. What is the output variable and what does it represent with units?
c. Estimate the domain of this function, using interval notation.
d. Estimate the range of this function, using interval notation.
e. Estimate $S(1990)$. What is the practical meaning of this value, in context, using a complete sentence.
f. Estimate a value of $t$, such that $S(t) = 250$. What is the practical meaning of this value, in context, using a complete sentence.
g. In the interval given in the graph, what estimate the maximum value of $S(t)$, and estimate when it occurs. What is the practical meaning of this value, in context, using a complete sentence.
18. Below is a graph of the reported measles cases around the world, $M(t)$, in millions, where $t$, is the year from 1990 to 2019.

Find the following:

a. What is the input variable and what does it represent, with units?
b. What is the output variable and what does it represent with units?
c. Estimate the domain of this function, using interval notation.
d. Estimate the range of this function, using interval notation.
e. Estimate $M(2000)$. What is the practical meaning of this value, in context, using a complete sentence.
f. Estimate a value of $t$, such that $M(t) = 0.5$. What is the practical meaning of this value, in context, using a complete sentence.
g. In the interval given in the graph, estimate the minimum value of $M(t)$, and estimate when it occurs. What is the practical meaning of this value, in context, using a complete sentence.
19. The following graph shows the number of motorcycle fatalities, \( F(y) \) in the U.S. in year, \( y \), from 1994 to 2018. (Source: NHTSA National Highway Traffic Safety Association). Despite providing less than 1% of miles driven, motorcycles made up 15% of traffic deaths in 2012.

Find the following:

a. What is the input variable and what does it represent, with units?
b. What is the output variable and what does it represent with units?
c. Estimate the domain of this function, using interval notation.
d. Estimate the range of this function, using interval notation.
e. Estimate \( F(2004) \). What is the practical meaning of this value, in context, using a complete sentence.
f. Estimate a value of \( y \), such that \( F(y) = 4500 \). What is the practical meaning of this value, in context, using a complete sentence.
g. In the interval given in the graph, estimate the maximum and minimum value of \( F(y) \), and estimate when each occurs. What is the practical meaning of these value, in context, using a complete sentence.
Section 1.4 Rates of Change

Since functions represent how an output quantity varies with an input quantity, it is natural to ask about the rate at which the values of the function are changing.

For example, the function \( C(t) \) below gives the average cost, in dollars, of a gallon of gasoline \( t \) years after 2000.

<table>
<thead>
<tr>
<th>( t )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C(t) )</td>
<td>1.47</td>
<td>1.69</td>
<td>1.94</td>
<td>2.30</td>
<td>2.51</td>
<td>2.64</td>
<td>3.01</td>
<td>2.14</td>
</tr>
</tbody>
</table>

If we were interested in how the gas prices had changed between 2002 and 2009, we could compute that the cost per gallon had increased from $1.47 to $2.14, an increase of $0.67. While this is interesting, it might be more useful to look at how much the price changed per year. You are probably noticing that the price didn’t change the same amount each year, so we would be finding the average rate of change over a specified amount of time.

The gas price increased by $0.67 from 2002 to 2009, over 7 years, for an average of

\[
\frac{0.67}{7\text{ years}} \approx 0.096 \text{ dollars per year.}
\]

On average, the price of gas increased by about 9.6 cents each year.

Rate of Change

A rate of change describes how the output quantity changes in relation to the input quantity. The units on a rate of change are “output units per input units.”

Some other examples of rates of change would be quantities like:

- A population of rats increases by 40 rats per week
- A barista earns $9 per hour (dollars per hour)
- A farmer plants 60,000 onions per acre
- A car can drive 27 miles per gallon
- A population of grey whales decreases by 8 whales per year
- The amount of money in your college account decreases by $4,000 per quarter

Average Rate of Change

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

\[
\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}
\]
Example 1
Using the cost-of-gas function from earlier, find the average rate of change between 2007 and 2009. Interpret the meaning of the value in context.

Solution:
From the table, in 2007 the cost of gas was $2.64. In 2009 the cost was $2.14.

The input (years) has changed by 2. The output has changed by $2.14 − $2.64 = −0.50.

The average rate of change is then \(-\frac{0.50}{2 \text{ years}}\) = −0.25 dollars per year

Interpretation: Between 2007 and 2009 the cost of gas was decreasing, on average, by 25 cents per year.

Try it Now
1. Using the same cost-of-gas function, find the average rate of change between 2003 and 2008. Interpret the meaning of the value in context.

Notice that in the last example the change of output was negative since the output value of the function had decreased. Correspondingly, the average rate of change is negative.

Example 2
Given the function \(g(t)\) shown here, find the average rate of change on the interval \([0, 3]\). Interpret the value in relation to the graph.

Solution:
At \(t = 0\), the graph shows \(g(0) = 1\)
At \(t = 3\), the graph shows \(g(3) = 4\)

The output has changed by 3 while the input has changed by 3, giving an average rate of change of:
\[
\frac{4 - 1}{3 - 0} = \frac{3}{3} = 1
\]

Interpretation: On average, from \(t=0\) to \(t=3\), the graph is increasing at a rate of 1 unit increase in the \(g(t)\)-value (output) for every 1 unit of increase in the \(t\)-value (input).
Example 3 (* Video Example Here)

On a road trip, after picking up your friend who lives 10 miles away, you decide to record your distance from home over time. Find your average speed over the first 6 hours. Interpret the value in the context of the problem.

<table>
<thead>
<tr>
<th>$t$ (hours)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(t)$ (miles)</td>
<td>10</td>
<td>55</td>
<td>90</td>
<td>153</td>
<td>214</td>
<td>240</td>
<td>292</td>
<td>300</td>
</tr>
</tbody>
</table>

Solution:
Here, your average speed is the average rate of change.
You traveled 282 miles in 6 hours, for an average speed of \[
\frac{292 - 10}{6 - 0} = \frac{282}{6} = 47 \text{ miles per hour}.
\]

We can more formally state the average rate of change calculation using function notation.

**Average Rate of Change using Function Notation**

Given a function $f(x)$, the average rate of change on the interval $[a, b]$ is

\[
\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{f(b) - f(a)}{b - a}
\]

Example 4 By Hand Calculation

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval [2, 4]. Interpret what the value tells us about the graph of $f(x)$.

Solution:
We can start by computing the function values at each endpoint of the interval by hand as follows:

$f(2) = 2^2 - \frac{1}{2} = 4 - 0.5 = 3.5$

$f(4) = 4^2 - \frac{1}{4} = 16 - 0.25 = 15.75$

Now computing the average rate of change

$\text{Average rate of change} = \frac{f(4) - f(2)}{4 - 2} = \frac{15.75 - 3.5}{4 - 2} = \frac{12.25}{2} = 6.125$

Interpretation of the graph: This tells us that from $x=2$ to $x=4$, the graph of $f(x)$ increases, on average, by 6.125 units every time the x-value increases by 1 unit.

Note: These average rate of change calculations can be very cumbersome when we do them by hand. Instead, we will now use Excel to do the quick calculation for us so that we can focus more of our time and attention on interpreting the meaning of the resulting value.

Video Link: Find the Average Rate of Change Given a Function on [2,t]
Example 4 Using Excel for the Calculation

Compute the average rate of change of \( f(x) = x^2 - \frac{1}{x} \) on the interval \([2, 4]\) using Excel.

**Solution:**

In cell B2, the function formula is written in order to calculate \( f(2) \) where it will pull \( x=2 \) from cell A2.

In cell B3, the function formula is written in order to calculate \( f(4) \) where it will pull \( x=4 \) from cell A3.

In cell C3, the function formula is written in order to calculate

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(4) - f(2)}{4 - 2} = \frac{(cell \ B3 - cell \ B2)}{(cell \ A3 - cell \ A2)}
\]

The resulting values in Excel are as follows:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x )</td>
<td>( f(x) = x^2 - \frac{1}{x} )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3.5</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>15.75</td>
</tr>
</tbody>
</table>

The slope of the line that intersects \( f(x) \) at \( x=2 \) and \( x=4 \) is 6.125.

From \( x=2 \) to \( x=4 \), \( f(x) \) is, on average, increasing at a rate of “up 6.125 every time \( x \) increases by 1.”
Example 5 Using Excel to Calculate Average Rate of Change

The magnetic force $F$, measured in Newtons, between two magnets is related to the distance between the magnets $d$, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the magnets is increased from 2 cm to 6 cm.

Solution:

We are computing the average rate of change of $F(d) = \frac{2}{d^2}$ on the interval [2, 6].

In cell B2, the function formula is written in order to calculate $F(2)$ where it will pull $d=2$ from cell A2.

In cell B3, the function formula is written in order to calculate $F(6)$ where it will pull $d=6$ from cell A3.

In cell C3, the function formula is written in order to calculate

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{F(4) - F(2)}{4 - 2} = \frac{(cell\ B3 - cell\ B2)}{(cell\ A3 - cell\ A2)}$$

The resulting values in Excel are as follows (decimal format on the left, and fraction format on the right):

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>-0.111111111</td>
</tr>
</tbody>
</table>

Average rate of change $= \frac{-1}{9} \approx -0.11$ Newtons per centimeter. This tells us that from a distance of 2 cm apart up to a distance of 6 cm apart, the magnetic force between the magnets decreases, on average, by $\frac{1}{9}$ Newtons for each centimeter increase in the distance.

The slope of the line that intersects $F(d)$ at $d=2$ and $d=6$ is $-0.11$.

From $d=2$ to $d=6$, $F(d)$ is, on average, decreasing at a rate of "down 0.11 every time $d$ increases by 1."
Example 6 Using Excel for the Calculation

Find the average rate of change of \( g(t) = t^2 + 3t + 1 \) on the interval \([-5, -1]\). Interpret what this value tells us about the graph of \( g(t) \).

Solution:

In cell B2, the function formula is written in order to calculate \( g(-5) \) where it will pull \( t = -5 \) from cell A2.

In cell B3, the function formula is written in order to calculate \( g(-1) \) where it will pull \( t = -1 \) from cell A3.

In cell C3, the function formula is written in order to calculate

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{g(-1) - g(-5)}{-1 - (-5)} = \frac{\text{cell B3} - \text{cell B2}}{\text{cell A3} - \text{cell A2}}
\]

The resulting values in Excel are as follows:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>t</td>
<td>g(t)=t^2+3t+1</td>
</tr>
<tr>
<td>2</td>
<td>-5</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>-3</td>
</tr>
</tbody>
</table>

This tells us that from \( t = -5 \) up to \( t = -1 \), the graph of \( g(t) \) is, on average, decreasing by 3 units every time \( t \) increases by 1 unit.

The slope of the line that intersects \( g(t) \) at \( t = -5 \) and \( t = -1 \) is \(-3\).

From \( t = -5 \) to \( t = -1 \), \( g(t) \) is, on average, decreasing at a rate of "down 3 every time \( t \) increases by 1."

Try it Now

2. Use Excel to find the average rate of change of \( f(x) = x^3 + 2 \) on the interval \([5, 11]\). Interpret what the value tells us about the graph of \( f(x) \).
Section 1.4 Exercises

1. The table below gives the annual sales (in millions of dollars) of a product. Use Excel to find the average rate of change of annual sales in the given time period. Interpret, in context, what the value found tells us, using correct units.
   a) Between 2001 and 2002?  
   b) Between 2001 and 2004?

<table>
<thead>
<tr>
<th>year</th>
<th>sales</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998</td>
<td>201</td>
</tr>
<tr>
<td>1999</td>
<td>219</td>
</tr>
<tr>
<td>2000</td>
<td>233</td>
</tr>
<tr>
<td>2001</td>
<td>243</td>
</tr>
<tr>
<td>2002</td>
<td>249</td>
</tr>
<tr>
<td>2003</td>
<td>251</td>
</tr>
<tr>
<td>2004</td>
<td>249</td>
</tr>
<tr>
<td>2005</td>
<td>243</td>
</tr>
<tr>
<td>2006</td>
<td>233</td>
</tr>
</tbody>
</table>

2. The table below gives the population of a town, in thousands. Use Excel to find the average rate of change of population in the given time period. Interpret, in context, what the value found tells us, using correct units.
   a) Between 2002 and 2004?  
   b) Between 2002 and 2006?

<table>
<thead>
<tr>
<th>year</th>
<th>population</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>87</td>
</tr>
<tr>
<td>2001</td>
<td>84</td>
</tr>
<tr>
<td>2002</td>
<td>83</td>
</tr>
<tr>
<td>2003</td>
<td>80</td>
</tr>
<tr>
<td>2004</td>
<td>77</td>
</tr>
<tr>
<td>2005</td>
<td>76</td>
</tr>
<tr>
<td>2006</td>
<td>75</td>
</tr>
<tr>
<td>2007</td>
<td>78</td>
</tr>
<tr>
<td>2008</td>
<td>81</td>
</tr>
</tbody>
</table>

3. Based on the graph shown, use Excel to estimate the average rate of change from \( x = 1 \) to \( x = 4 \). Interpret what the value found tells us about the graph.

4. Based on the graph shown, use Excel estimate the average rate of change from \( x = 2 \) to \( x = 5 \). Interpret what the value found tells us about the graph.

5. Use Excel to find the average rate of change of each function on the interval specified.
   a) \( f(x) = x^2 \) on \([1, 5]\)
   b) \( g(x) = x^3 \) on \([-4, 2]\)
   c) \( h(x) = 3x^3 - 1 \) on \([-3, 3]\)
   d) \( q(x) = 5 - 2x^2 \) on \([-2, 4]\)

6. Algebraically find the average rate of change of each function on the interval specified. Your answers will be expressions involving a variable \( b \) or \( h \).
   a) \( f(x) = 4x^2 - 7 \) on \([1, b]\)
   b) \( g(x) = 2x^2 - 9 \) on \([4, b]\)
   c) \( h(x) = 3x + 4 \) on \([2, 2 + h]\)
   d) \( k(x) = 4x - 2 \) on \([3, 3 + h]\)
   e) \( f(x) = 2x^2 + 1 \) on \([x, x + h]\)
   f) \( g(x) = 3x^2 - 2 \) on \([x, x + h]\)
For each function graphed, do the following:
(a) Estimate the domain interval(s) on which the function is increasing. Give the answer in interval notation.
and
(b) Estimate the domain interval(s) on which the function is decreasing. Give the answer in interval notation.
For each table below, select whether the table represents a function that is increasing or decreasing.

<table>
<thead>
<tr>
<th></th>
<th>f(x)</th>
<th></th>
<th>g(x)</th>
<th></th>
<th>h(x)</th>
<th></th>
<th>k(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>90</td>
<td>1</td>
<td>300</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
<td>80</td>
<td>2</td>
<td>290</td>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>3</td>
<td>75</td>
<td>3</td>
<td>270</td>
<td>3</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>4</td>
<td>72</td>
<td>4</td>
<td>240</td>
<td>4</td>
<td>32</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>5</td>
<td>70</td>
<td>5</td>
<td>200</td>
<td>5</td>
<td>35</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>f(x)</th>
<th></th>
<th>g(x)</th>
<th></th>
<th>h(x)</th>
<th></th>
<th>k(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-10</td>
<td>1</td>
<td>-200</td>
<td>1</td>
<td>-100</td>
<td>1</td>
<td>-50</td>
</tr>
<tr>
<td>2</td>
<td>-25</td>
<td>2</td>
<td>-190</td>
<td>2</td>
<td>-50</td>
<td>2</td>
<td>-100</td>
</tr>
<tr>
<td>3</td>
<td>-37</td>
<td>3</td>
<td>-160</td>
<td>3</td>
<td>-25</td>
<td>3</td>
<td>-200</td>
</tr>
<tr>
<td>4</td>
<td>-47</td>
<td>4</td>
<td>-100</td>
<td>4</td>
<td>-10</td>
<td>4</td>
<td>-400</td>
</tr>
<tr>
<td>5</td>
<td>-54</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td></td>
<td>5</td>
<td>-900</td>
</tr>
</tbody>
</table>

For each function graphed, do the following:

(a) Estimate the domain interval(s) on which the function is increasing. Give the answer in interval notation.

and

(b) Estimate the domain interval(s) on which the function is decreasing. Give the answer in interval notation.
Section 1.5 Graphical Behavior of Functions

As part of exploring how functions change, it is important to explore where the function is increasing and decreasing, and where the function is concave up and concave down.

<table>
<thead>
<tr>
<th>Increasing/Decreasing</th>
<th>(*Video Link Increasing/ Decreasing Behavior)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A function is <strong>increasing</strong> on an interval if the function values increase as the inputs increase.</td>
<td></td>
</tr>
<tr>
<td>• When traveling from left-to-right on the graph, if the function goes up on the interval, then it is increasing on that interval.</td>
<td></td>
</tr>
<tr>
<td>• More formally, a function is increasing if ( f(b) &gt; f(a) ) for any two input values ( a ) and ( b ) in the interval with ( b &gt; a ).</td>
<td></td>
</tr>
<tr>
<td>• The average rate of change of an increasing function is <strong>positive</strong>.</td>
<td></td>
</tr>
</tbody>
</table>

A function is **decreasing** on an interval if the function values decrease as the inputs increase.

| • When traveling from left-to-right on the graph, if the function goes down on the interval, then it is decreasing on that interval. |
| • More formally, a function is decreasing if \( f(b) < f(a) \) for any two input values \( a \) and \( b \) in the interval with \( b > a \). |
| • The average rate of change of a decreasing function is **negative**. |

**Example 1**

Given the function \( p(t) \) graphed here, on what intervals does the function appear to be increasing?

**Solution:**
The function appears to be increasing from \( t = 1 \) to \( t = 3 \), and from \( t = 4 \) on.

In interval notation, we would say the function appears to be increasing on the interval \((1, 3)\) and the interval \((4, \infty)\).

Notice that we used open intervals (intervals that don’t include the endpoints) since the function is neither increasing nor decreasing at \( t = 1, 3, \) or \( 4 \).
**Local Extrema**

- A point where a function changes from increasing to decreasing is called a **local maximum**.
- A point where a function changes from decreasing to increasing is called a **local minimum**.
- Together, local maxima and minima are called the **local extrema**, or local extreme values, of the function.

**Example 2**

The table below gives cost of gasoline, \( C(t) \), where \( t \) is the number of years that have passed since they started recording. Estimate any local extrema using the table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C(t) )</td>
<td>1.47</td>
<td>1.69</td>
<td>1.94</td>
<td>2.30</td>
<td>2.51</td>
<td>2.64</td>
<td>3.01</td>
<td>2.14</td>
</tr>
</tbody>
</table>

**Solution:**

It appears that the cost of gas increased from \( t = 2 \) to \( t = 8 \). It appears the cost of gas decreased from \( t = 8 \) to \( t = 9 \), so the function appears to be decreasing on the interval \((8, 9)\). Since the function appears to change from increasing to decreasing at \( t = 8 \), there is local maximum at \( t = 8 \). So, there was a peak in gas prices 8 years after they started recording prices, and that peak price was $3.01 per gallon.

**Example 3**

Use the given Excel graph and table to estimate the local extrema of the function \( f(x) = x^3 - 55.9x^2 + 1021.8x - 5601.17 \). Then determine the interval(s) on which the function is increasing.

**Solution:**

The picture above shows the input-output table and graph from Excel. Choosing an appropriate domain (input values) is key to be able to see the features of the function on a graph. From the graph of the function, it appears there is a local maximum at approximately \( x = 16 \), where the function has an output of \( y = 533.23 \), and a local minimum at approximately \( x = 22 \), where the function has an output of \( y = 470.83 \).

Based on these estimates, we approximate that the function is increasing on the intervals \((-\infty, 16)\) and \((22, \infty)\). And we approximate that the function is decreasing on the interval \((16, 22)\).
It should be noted that Excel has a feature that allows us to find the local maximum and local minimum points to a large degree of accuracy. We will explore that feature in Excel in the next section.

**Try it Now**

3. Use the given Excel graph of the function \( f(x) = x^3 - 6x^2 - 15x + 20 \) to estimate the local extrema of the function. Also estimate the interval(s) on which the function is increasing, and the interval(s) on which the function is decreasing.

![Excel graph of the function](image)

**Concavity**

Concavity has to do with the “bend” of the curve. Graphically, concave down functions bend downwards like a frown, and concave up functions bend upwards like a smile.

<table>
<thead>
<tr>
<th>Concave Down</th>
<th>Increasing</th>
<th>Decreasing</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><img src="image" alt="Graph of concave down function" /></td>
<td><img src="image" alt="Graph of increasing function" /></td>
</tr>
<tr>
<td>Concave Up</td>
<td><img src="image" alt="Graph of increasing function" /></td>
<td><img src="image" alt="Graph of decreasing function" /></td>
</tr>
</tbody>
</table>
The total sales, in thousands of dollars, for two companies over 4 weeks are shown.

As you can see, the sales for each company are increasing, but they are increasing in very different ways. To describe the difference in behavior, we can investigate how the average rate of change varies over different intervals. Using tables of values in Excel, we get the following:

<table>
<thead>
<tr>
<th>Company A</th>
<th>Company B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week</td>
<td>Sales</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>7.1</td>
</tr>
<tr>
<td>3</td>
<td>8.7</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
</tbody>
</table>

From the tables, we can see that the rate of change for company A is decreasing since the average rates of change are 5, then 2.1, then 1.6, then 1.3. We see that the rate of change for company B is increasing since the average rates of change are 0.5, then 1.5, then 2.5, then 3.5.

**Relationship between the rate of change and the concavity:**
When the rate of change is getting smaller, as with Company A, we say the function is **concave down**. When the rate of change is getting larger, as with Company B, we say the function is **concave up**.
Concavity

A function is **concave up** if the rate of change is increasing.
A function is **concave down** if the rate of change is decreasing.
A point where a function changes from concave up to concave down or vice versa is called an **inflection point**.

**Example 4**
An object is thrown from the top of a building. The object’s height in feet above ground after \( t \) seconds is given by the function \( h(t) = 144 - 16t^2 \) for \( 0 \leq t \leq 3 \). Describe the concavity of the graph. Interpret the meaning of the concavity in context.

**Solution:**
Sketching a graph of the function, we can see that the function is decreasing. We can calculate some rates of change to explore the behavior of the average rates of change:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( h(t) )</th>
<th>rate of change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>144</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>128</td>
<td>-16</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>-48</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-80</td>
</tr>
</tbody>
</table>

Notice that the rates of change are *becoming more negative*, so the rates of change are *decreasing*. This means the function, \( h(t) \), is concave down.

Interpreting the concavity in context: The rate of change is interpreted as the *average speed* of the object as it falls towards the ground. The units of the average rate of change is *feet per second*. We see in the table that the object had an average speed of 16 feet per second from \( t = 0 \) seconds until \( t = 1 \) second. The object was then falling at an average speed of 48 feet per second from \( t = 1 \) second up to \( t = 2 \) seconds. The fact that the curve is concave down (the fact that the average rate of change is decreasing) in this case tells us that the object is falling towards the ground at a *faster and faster speed each second*.

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Example 5
The value, \( V \), of a car (in dollars) after \( t \) years is given in the table below. Is the value increasing or decreasing? Is the function concave up or concave down? Interpret the contextual meaning of the concavity.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V(t) )</td>
<td>28000</td>
<td>24342</td>
<td>21162</td>
<td>18397</td>
<td>15994</td>
</tr>
</tbody>
</table>

Solution:
Since the values of \( V(t) \) are getting smaller, we can determine that the car’s value is decreasing each year. We can compute average rates of change to determine concavity.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V(t) )</td>
<td>28000</td>
<td>24342</td>
<td>21162</td>
<td>18397</td>
<td>15994</td>
</tr>
<tr>
<td>Rate of change</td>
<td>-1829</td>
<td>-1590</td>
<td>-1382.5</td>
<td>-1201.5</td>
<td></td>
</tr>
</tbody>
</table>

Since these average rate of change values are becoming less negative, the average rates of change are increasing, so the function \( V(t) \) is concave up.

Interpreting the concavity in context: The average rate of change is the average increase or decrease in the value of the car each year. From year 0 to year 2, the value of the car decreased by $1,829 per year on average. From year 2 to year 4, the value of the car decreased by $1,590 per year on average. The fact that the curve is concave up (the fact that the average rate of change is increasing) in this case tells us that the car's value is decreasing, but the is decreasing by a smaller and smaller amount each year.

Try it Now
4. Is the function described in the table below concave up or concave down? Explain.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>10000</td>
<td>9000</td>
<td>7000</td>
<td>4000</td>
<td>0</td>
</tr>
</tbody>
</table>
Example 6 (* Video Example Here)

Give a left-to-right description of the behavior of the graph pictured. Your description should give clear descriptions of the behavior of the graph on each interval, as well as indicating the location of any local extrema and inflection points.

Solution:
On the interval \((-\infty, -2)\), the graph is decreasing concave up.
There is a local minimum at \(x = -2\).
On the interval \((-2, -1)\), the graph is increasing concave up.
There is an inflection point at \(x = -1\).
On the interval \((-1, 0)\), the graph is increasing concave down.
There is a local maximum at \(x = 0\).
On the interval \((0, 2)\), the graph is decreasing concave down.
There is an inflection point at \(x = 2\).
On the interval \((2, 3.5)\) the graph is decreasing concave up.
There is a local minimum at \(x = 3.5\).
On the interval \((3.5, \infty)\) the graph is increasing concave up.

Example 7

Use the left-to-right description of the behavior of the function to sketch a graph of the function.

- The function is increasing concave down on the interval \((-\infty, 8)\).
- There is a local maximum at \(x = 8\).
- The function is decreasing concave down on the interval \((8, 11)\).
- There is an inflection point at \(x = 11\).
- The function is decreasing concave up on the interval \((11, 14)\).
- There is a local minimum at \(x = 14\).
- The function is increasing concave up on the interval \((14, \infty)\).

Solution: Putting all the pieces together, we get a graph that looks like the following:
Try it Now

5. Use the graph of \( f(x) \) given below to give a complete left-to-right description of the function including increasing/decreasing, and concave up/down, and extrema and inflection points.

![Graph of \( f(x) \)]

6. Sketch a graph of a function that has a domain of \( x > 0 \) and has a range of all real numbers, and the function is increasing concave down on its entire domain.
Graphical Behaviors of the Toolkit Functions
We will now return to our toolkit functions and discuss their graphical behavior.

<table>
<thead>
<tr>
<th>Function</th>
<th>Increasing/Decreasing</th>
<th>Concavity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Function</td>
<td>Neither increasing nor decreasing</td>
<td>Neither concave up nor down</td>
</tr>
<tr>
<td>( f(x) = c )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Identity Function</td>
<td>Increasing</td>
<td>Neither concave up nor down</td>
</tr>
<tr>
<td>( f(x) = x )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| Quadratic Function| Increasing on \((0, \infty)\) Decreasing on \((-\infty, 0)\)
                    | Minimum at \(x = 0\)           | Concave up \((-\infty, \infty)\)
| \( f(x) = x^2 \)|                       |                                |
| Cubic Function   | Increasing            | Concave down on \((-\infty,0)\)
                    | Concave up on \((0, \infty)\)
                    | Inflection point at \((0,0)\) |
| \( f(x) = x^3 \)|                       |                                |
| Reciprocal       | Decreasing \((-\infty,0) \cup (0, \infty)\) | Concave down on \((-\infty,0)\)
                    |                                      | Concave up on \((0, \infty)\) |
| \( f(x) = \frac{1}{x} \)|                       |                                |
| Reciprocal squared| Increasing on \((-\infty,0)\) Decreasing on \((0, \infty)\) |
| \( f(x) = \frac{1}{x^2} \)|                       | Concave up on \((-\infty,0) \cup (0, \infty)\) |
| Cube Root        | Increasing            | Concave down on \((0, \infty)\)
                    | Concave up on \((-\infty,0)\)
                    | Inflection point at \((0,0)\) |
| \( f(x) = \sqrt{x} \)|                       |                                |
| Square Root      | Increasing on \((0, \infty)\) | Concave down on \((0, \infty)\) |
| \( f(x) = \sqrt[3]{x} \)|                       |                                |
| Absolute Value   | Increasing on \((0, \infty)\) Decreasing on \((-\infty,0)\) | Neither concave up or down |
| \( f(x) = |x| \) |                       |                                |
### Important Topics of This Section

Rate of Change  
Average Rate of Change  
Calculating Average Rate of Change using Function Notation  
Increasing/Decreasing  
Local Maxima and Minima (Extrema)  
Inflection points  
Concavity

---

**Try it Now Answers**

1. \[
\frac{3.01 - 1.69}{5\text{ years}} = \frac{1.32}{5\text{ years}} = 0.264 \text{ dollars per year. Interpretation: From 2003 to 2008, the cost of gas increased, on average, by 26 cents per year.}
\]

2. Average rate of change is 201. From x=5 up to x=11, the function, on average increases by 201 units every time x increases by 1 unit.

3. Based on the graph, the local maximum appears to occur at approximately (-1, 28), and the local minimum occurs at approximately (5,-80). We estimate that the function is increasing on the interval \((-\infty, -1)\) and also on the interval \((5, \infty)\). We estimate that the function is decreasing on the interval \((-1,5)\).

4. Calculating the rates of change, we see the rates of change become *more* negative, so the rates of change are *decreasing*. This function is concave down.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g(x))</td>
<td>10000</td>
<td>9000</td>
<td>7000</td>
<td>4000</td>
<td>0</td>
</tr>
<tr>
<td>Rate of change</td>
<td>-1000</td>
<td>-2000</td>
<td>-3000</td>
<td>-4000</td>
<td></td>
</tr>
</tbody>
</table>

5. The function is decreasing concave up on the interval \((-\infty, -1)\), there is a local minimum at \(x = -1\).  
The function is increasing concave up on the interval \((-1,1)\).  
There is an inflection point at \(x = 1\).  
The function is increasing concave down on the interval \((1,\infty)\).
There is a local maximum at $x = 5$. The function is decreasing concave down on the interval $(5, \infty)$. 

6.
**Section 1.5 Exercises**

1. The table below gives the annual sales (in millions of dollars) of a product. What was the average rate of change of annual sales…
   a) Between 2001 and 2002? Write a sentence to interpret the value in context.
   b) Between 2001 and 2004? Write a sentence to interpret the value in context.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>sales</td>
<td>201</td>
<td>219</td>
<td>233</td>
<td>243</td>
<td>249</td>
<td>251</td>
<td>249</td>
<td>243</td>
<td>233</td>
</tr>
</tbody>
</table>

2. The table below gives the population of a town, in thousands. What was the average rate of change of population…
   a) Between 2002 and 2004? Write a sentence to interpret the value in context.
   b) Between 2002 and 2006? Write a sentence to interpret the value in context.

<table>
<thead>
<tr>
<th>year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>population</td>
<td>87</td>
<td>84</td>
<td>83</td>
<td>80</td>
<td>77</td>
<td>76</td>
<td>75</td>
<td>78</td>
<td>81</td>
</tr>
</tbody>
</table>

3. Based on the graph shown on the right, estimate the average rate of change from \( x = 1 \) to \( x = 4 \).

4. Based on the graph shown on the right, estimate the average rate of change from \( x = 2 \) to \( x = 5 \).

For problems #5-10, use Excel to find the average rate of change of each function on the interval specified.

5. \( f(x) = x^3 \) on \([1, 5]\)
6. \( q(x) = x^3 \) on \([-4, 2]\)

7. \( g(x) = 3x^3 - 1 \) on \([-3, 3]\)
8. \( h(x) = 5 - 2x^2 \) on \([-2, 4]\)

9. \( k(t) = 6t^2 + \frac{4}{t^2} \) on \([-1, 3]\)
10. \( p(t) = \frac{t^2 - 4t + 1}{t^2 + 3} \) on \([-3, 1]\)
For problems #11-14, give a complete left-to-right description of the function behavior including increasing/decreasing, concave up/down, local extrema, and inflection points.

11. 

12. 

13. 

14. 

For problems #15-22, select whether the table represents a function that is increasing or decreasing, and whether the function is concave up or concave down.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>15. x</th>
<th>g(x)</th>
<th>16. x</th>
<th>h(x)</th>
<th>17. x</th>
<th>k(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1 90</td>
<td>1 90</td>
<td>1 300</td>
<td>1 0</td>
<td>1 0</td>
<td>1 0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2 80</td>
<td>2 80</td>
<td>2 290</td>
<td>2 15</td>
<td>2 15</td>
<td>2 15</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>3 75</td>
<td>3 75</td>
<td>3 270</td>
<td>3 25</td>
<td>3 25</td>
<td>3 25</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>4 72</td>
<td>4 72</td>
<td>4 240</td>
<td>4 32</td>
<td>4 32</td>
<td>4 32</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>5 70</td>
<td>5 70</td>
<td>5 200</td>
<td>5 35</td>
<td>5 35</td>
<td>5 35</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>19. x</th>
<th>g(x)</th>
<th>20. x</th>
<th>h(x)</th>
<th>21. x</th>
<th>k(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-10</td>
<td>1 -200</td>
<td>1 -200</td>
<td>1 -100</td>
<td>1 -50</td>
<td>1 -50</td>
<td>1 -50</td>
</tr>
<tr>
<td>2</td>
<td>-25</td>
<td>2 -190</td>
<td>2 -190</td>
<td>2 -50</td>
<td>2 -100</td>
<td>2 -100</td>
<td>2 -100</td>
</tr>
<tr>
<td>3</td>
<td>-37</td>
<td>3 -160</td>
<td>3 -160</td>
<td>3 -25</td>
<td>3 -200</td>
<td>3 -200</td>
<td>3 -200</td>
</tr>
<tr>
<td>4</td>
<td>-47</td>
<td>4 -100</td>
<td>4 -100</td>
<td>4 -10</td>
<td>4 -400</td>
<td>4 -400</td>
<td>4 -400</td>
</tr>
<tr>
<td>5</td>
<td>-54</td>
<td>5 0</td>
<td>5 0</td>
<td>5 0</td>
<td>5 900</td>
<td>5 900</td>
<td>5 900</td>
</tr>
</tbody>
</table>

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For problems #23-26, give a complete left-to-right description of the function behavior including increasing/decreasing, concave up/down, local extrema, and inflection points.

23. 

\[ f(x) = x^4 - 4x^3 + 5 \]

24. 

\[ g(t) = t\sqrt{t} + 3 \]

25. 

\[ m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4 \]

26. 

\[ n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2 \]

For problems #27-32, graph the function in Excel, and then use the graph to give a complete left-to-right description of the function behavior including increasing/decreasing, concave up/down, local extrema, and inflection points.

27. \[ f(x) = x^4 - 4x^3 + 5 \]

28. \[ h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1 \]

29. \[ g(t) = t\sqrt{t} + 3 \]

30. \[ k(t) = 3t^{2/3} - t \]

31. \[ m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4 \]

32. \[ n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2 \]

For problems #33-36, sketch a graph of a function that satisfies all aspects of the description.

33. 

- The graph is decreasing concave down on the interval \((-\infty, 1)\).
- The graph has an inflection point at \(x=1\).
- The graph is decreasing concave up on the interval \((1, 6)\).
- The graph has a local minimum at \(x=6\).
- The graph is increasing concave up on the interval \((6, 10)\).
- The graph has an inflection point at \(x=10\).
- The graph is increasing concave down on the interval \((10, \infty)\).

34. 

- The graph is increasing concave down on the interval \((-\infty, 0)\).
- The graph is increasing concave up on the interval \((0, \infty)\).

35. The graph is decreasing concave down on the interval \((-\infty, \infty)\).

36. The graph is increasing concave down on the interval \((-\infty, \infty)\).
Section 1.6 Local Maximum and Minimum on Excel

To find local extrema points mathematically by hand, we would need to use Calculus. Since Calculus is beyond the scope of this course, we will instead use Microsoft Excel to identify local extrema points. This section will explain how to use Microsoft Excel to find local maximum and local minimum points.

Local Extrema in Microsoft Excel

To find local extrema in Excel, follow the steps below.

1. Create an input-output table that includes the extrema point(s) within the span of the table. Note: while not necessary, it may be helpful to also have Excel create the graph from your table so that you can also use the graph to estimate the location of the extrema point(s).

2. Open Solver, found in the Data tab.

3. Select either “Max” or “Min” to match the local extrema you are seeking.

4. In the “Set Objective” box, clear out anything that is currently in the box. Then click on the cell in the worksheet containing the output value (the y-value) closest to the local max or min that we seek. Note: That y-value cell that is entered in the “Set Objective” box must have a formula of a function in it, and it must be referencing another cell from which it is pulling the input-value.

5. In the “By Changing Variable Cells” box, clear out anything that is currently in the box. Then click on the cell in the worksheet containing the input-value (the x-value) that matches up with the cell from step 4.

6. Click Solve
Example 1 (*Video Example Here*)

Use Excel to find any local extrema points of the function $f(x) = -x^3 + 13x^2 - 44x + 82$. Round the values to four decimal places.

Solution:
We first create an input-output table, as well as a graph in Excel.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$f(x) = -x^3 + 13x^2 - 44x + 82$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
</tr>
</tbody>
</table>

It appears that we have successfully selected a span of x-values that shows us all local minimum and local maximum points of this function since this is a cubic function which could only have at most one local max and one local min like we see in this graph. There appears to be a local minimum near $x = 2$, and there appears to be a local maximum near $x = 6$.

Next, we open Solver, found in the Data tab, and use it to find the local minimum point.

- The “Set Objective” box is referencing cell B4 which has the cell containing the y-value that is closest to the local minimum y-value.
- The “By Changing Variable Cells” box is referencing cell A4 that has the cell containing the x-value that is closest to the local minimum x-value.

When we click “Solve,” Excel will change the values in cells A4 and B4 to the values of the local minimum. The local minimum occurs at approximately $(2.3057, 37.4028)$. The values in row 4 are changed to the local minimum coordinate values.
Next, we open Solver, found in the Data tab, and use it to find the local maximum point.

- The “Set Objective” box is referencing cell B8 which has the cell containing the y-value that is closest to the local maximum y-value.

- The “By Changing Variable Cells” box is referencing cell A8 that has the cell containing the x-value that is closest to the local maximum x-value.

When we click “Solve,” Excel will change the values in cells A8 and B8 to the values of the local maximum. The local maximum occurs at approximately $(6.3609, 70.7453)$.

Try it Now

1. Use Excel to find the local extrema points of the function $f(x) = x^3 - 12x^2 + 41x + 8$. Round the values to four decimal places.
Example 2

The function $Q(p) = -0.23p + 362$ gives the number of items sold when the company sells the items for $p$ dollars per item. Find a formula for the revenue they would earn, $R(p)$ if they sold the items at $p$ dollars per item. Then use Excel to find the price they should charge in order to have the maximum revenue, and also identify what that maximum revenue would be, and the number of items they would be selling at that price.

Solution:

Revenue is always equal to (price per item)\(\cdot\) (number of items sold). We have been told in the problem that (price per item) = $p$. We have also been told that (number of items sold) = $Q(p)$.

Therefore, the function $R(p)$ is as follows:

$R(p) = (\text{price per item}) \cdot (\text{number of items sold})$

$R(p) = p \cdot Q(p)$

$R(p) = p \cdot (-0.23p + 362)$

$R(p) = -0.23p^2 + 362p$

We create an input-output table and a graph in Excel. To determine an appropriate span of input-values to use in the table, we observe that the input-values represent “price per item” and so it must be $0$ per item or higher. We don’t know how high we should let the values of $p$ get, and so we take a guess when we first make the table, and adjust as needed until we create a table and graph that clearly show the local maximum of the parabola.

We see that the local maximum occurs at approximately $p=800$. We use the solver Maximum feature, with the “Set Objective” cell set to $B10$, and the “By Changing Variable in Cells” set to $A10$. We select the “Solve” button, and Excel will change the row 10 values to display the coordinates of the local maximum. The local maximum is shown to be (786.96, 142439.13).

This tells us that the company should charge $786.96 per item, because that will generate the highest possible revenue of $142,439.13$. Note, that they would be selling $Q(786.96)=181$ items at that price.
Try it Now

2. The function \( P(q) = -0.61q + 1500 \) gives the price per item when the company sells a total of \( q \) items. Find a formula for the revenue they would earn, \( R(q) \) if they sell a total of \( q \) items. Then use Excel to find the number of items they would sell in order to maximize revenue, and identify the revenue they would get, and also price they would be charging per item.

---

Example 3

The function \( Q(p) = 10,004e^{-0.35p} \) gives the number of items sold when the company sells the items for \( p \) dollars per item. When the company produces the items, it costs them \$1,300 in initial fixed costs, and then an additional \$4.55 per item produced. Assume that they sell exactly the same number of items that they produce.

(a) Find a formula for the revenue they would earn, \( R(p) \) if they sold the items at \( p \) dollars per item.

(b) Find a formula for the total cost they would incur, \( C(p) \) if they sold the items at \( p \) dollars per item.

(c) Find a formula for the total profit they would make, \( F(p) \) if they sold the items at \( p \) dollars per item. Remember that we are assuming that they produce exactly the same number of items that they sell.

(d) Use Excel to find the price they should charge in order to have the maximum revenue, and also identify what that maximum revenue would be, and the number of items they would be selling at that price.

(e) Use Excel to find the price they should charge in order to have the maximum profit, and also identify what that maximum profit would be, and the number of items they would be selling at that price.

Solution:

(a) The formula for revenue is \( R(p) = \text{(price per item)} \cdot \text{(number items sold)} \) and so \( R(p) = p \cdot 10,004e^{-0.35p} \).

(b) The formula for total cost is \( C(p) = \text{(initial cost)} + \text{(cost per item)} \cdot \text{(number sold)} \) and so \( C(p) = 1,300 + 4.55 \cdot 10,004e^{-0.35p} \).

(c) The formula for profit is \( F(p) = \text{(revenue)} - \text{(total cost)} = R(p) - C(p) \) and so \( F(p) = p \cdot 10,004e^{-0.35p} - (1,300 + 4.55 \cdot 10,004e^{-0.35p}) \)

Simplified this is \( F(p) = p \cdot 10,004e^{-0.35p} - 1,300 - 4.55 \cdot 10,004e^{-0.35p} \)
(d) and (e) In Excel, the table below shows the values for all the functions discussed in this problem on a range of prices for $0 per item up to $10.00 per item. Solver has already been used to identify the local maximum point on R(p), and the local maximum point on F(p).

<table>
<thead>
<tr>
<th>p</th>
<th>Q(p) = Quantity items</th>
<th>R(p) = Revenue</th>
<th>C(p)=Cost</th>
<th>F(p)=R(p)-C(p)=Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.00</td>
<td>10,004</td>
<td>$0.00</td>
<td>46818.20</td>
<td>-$46,818.20</td>
</tr>
<tr>
<td>$1.00</td>
<td>7,050</td>
<td>$7,049.70</td>
<td>33376.13</td>
<td>-$26,326.43</td>
</tr>
<tr>
<td>$2.00</td>
<td>4,968</td>
<td>$9,935.68</td>
<td>23903.67</td>
<td>-$13,967.99</td>
</tr>
<tr>
<td>$2.86</td>
<td>3,680</td>
<td>$10,515.05</td>
<td>18045.21</td>
<td>-$7,530.16 Max Revenue</td>
</tr>
<tr>
<td>$4.00</td>
<td>2,467</td>
<td>$8,867.82</td>
<td>12524.65</td>
<td>-$2,656.83</td>
</tr>
<tr>
<td>$5.00</td>
<td>1,738</td>
<td>$8,692.17</td>
<td>9209.88</td>
<td>-$517.70</td>
</tr>
<tr>
<td>$6.00</td>
<td>1,225</td>
<td>$7,350.32</td>
<td>6874.00</td>
<td>$476.33</td>
</tr>
<tr>
<td>$7.41</td>
<td>749</td>
<td>$5,545.18</td>
<td>4706.25</td>
<td>$838.93 Max Profit</td>
</tr>
<tr>
<td>$8.00</td>
<td>608</td>
<td>$4,866.75</td>
<td>4067.96</td>
<td>$798.79</td>
</tr>
<tr>
<td>$9.00</td>
<td>429</td>
<td>$3,858.23</td>
<td>3250.55</td>
<td>$607.68</td>
</tr>
<tr>
<td>$10.00</td>
<td>302</td>
<td>$3,020.95</td>
<td>2674.53</td>
<td>$346.42</td>
</tr>
</tbody>
</table>

With the help of solver, we see that charging a price of $2.86 per item will create the highest revenue of $10,515.05. At that time, they would be selling about 3680 items.

With solver, we see that charging a price of $7.41 per item will create the highest profit of $838.93. At that time, they are selling about 749 items.

Try it Now Answers

1. Local maximum at (2.4725, 51.1285) and local minimum at (5.5275, 36.8716).

2. The local maximum is (1229.51, 922131.15). This tells us that they will sell about 1229 or 1230 items, which will result in a maximum revenue of $922,131.15. At that time, the price per item would be P(1229.51)=750 per item.
Section 1.6 Exercises

1. Use Excel to find the local maximum of \( f(x) = 7xe^{-0.4x} \). Use a window of \( 0 \leq x \leq 6 \).

2. Use Excel to find the local minimum of \( f(x) = \frac{20}{x} + 8x \). Use a window of \( 0.1 \leq x \leq 5 \). Your graph should look like the one pictured below:

![Graph](image)

3. Use Excel to create a table and graph that clearly shows one local maximum and one local minimum of the function \( f(x) = -0.85x^3 + 1.40x^2 + 3.75x \). Then use Excel to find the local maximum and local minimum.

4. Use Excel to create a table and graph that clearly shows one local maximum and one local minimum of the function \( f(x) = x^3 - 472.3x^2 + 62,663.5x - 2,517,930 \). Then use Excel to find the local maximum and local minimum.

5. Suppose the function \( Q(p) = 4500e^{-0.24p} \) gives the number of items sold when the company prices the items at \( p \) dollars per item. Find a formula for the total revenue earned, \( R(p) \), when charging \( p \) dollars per item. Then use Excel to identify the price they should charge in order to maximize revenue. Also identify the revenue they would be earning at the price, as well as the number of items they would be selling.

6. An aluminum company wants to make an aluminum can that holds 10.5 ounces of soup. Note that 10.5 ounces is the same as 18.95 cubic inches. The volume of a cylinder is given by the formula \( V = \pi r^2 h \), and the surface area of a cylinder is given by the formula \( S = 2\pi r^2 + 2\pi rh \).

   a) Re-write the volume equation so that the variable \( V \) is replaced by \( V = 18.95 \), and then the equation is solved for the variable \( h \).
b) Use the equation from part a) to re-write the surface area formula as a function \( S(r) \). The re-written surface area formula should now have only one input variable, \( r \).

c) Use Excel to find the radius of the can that will minimize the amount of aluminum used to create the can. Also identify the amount of aluminum they would be using to make the can, and the height of the can that they create. Be sure to use the correct units when you give your answers!

7. An aluminum company wants to make an aluminum can using 25 square inches of aluminum. The volume of a cylinder is given by the formula \( V = \pi r^2 h \), and the surface area of a cylinder is given by the formula \( S = 2\pi r^2 + 2\pi rh \).

a) Re-write the surface area equation so that the variable \( S \) is replaced by \( S = 25 \), and then the equation is solved for the variable \( h \).

b) Use the equation from part a) to re-write the volume formula as a function \( V(r) \). The re-written volume formula should now have only one input variable, \( r \).

c) Use Excel to find the radius of the can that will maximize the volume of the can. Also identify the volume of the can, and the height of the can that they create. Be sure to use the correct units when you give your answers!

8. The function \( P(q) = 85,000e^{-0.48q} \) gives the price per item when the company produces and sells \( q \) items. When the company produces the items, it costs them $850 in initial fixed costs, and then an additional $1.15 per item produced. Assume that they sell exactly the same number of items that they produce.

a) Find a formula for the revenue they would earn, \( R(q) \) if they produced and sold \( q \) items.

b) Find a formula for the total cost they would incur, \( C(q) \) if they produced and sold \( q \) items.

c) Find a formula for the total profit they would make, \( F(q) \) if they produced and sold \( q \) items. Remember that we are assuming that they produce exactly the same number of items that they sell.

d) Use Excel to find the number of items they would produce and sell in order to have the maximum revenue, and also identify what that maximum revenue would be, and the price they would charge per item.

e) Use Excel to find the number of items they would produce and sell in order to have the maximum profit, and also identify what that maximum profit would be, and the price they would charge per item.
Section 2.1 Introduction to Linear Functions

Linear Function

A linear function is a function whose graph produces a straight, non-vertical line. Linear functions have a constant, fixed, rate of change called the slope of the line.

\[ f(x) = mx + b \]

is the equation of a linear function with slope \( m \) and y-intercept \((0,b)\).

\( f(x) = mx + b \) is called the slope-intercept form of a linear function because you can see the slope and the y-intercept in the equation itself. All linear functions can be written in the slope-intercept form, and each line has a unique slope-intercept form.

\[ f(x) = m(x - x_1) + y_1 \]

is the equation of a linear function with slope \( m \) that passes through \((x_1, y_1)\).

\( f(x) = m(x - x_1) + y_1 \) is called the point-slope form of a linear function because the equation displays a generic point on the line, and the equation also displays the slope of the line. The equation of linear functions can be written in point-slope form in an infinite number of ways because any point on the line can be used in the equation.

The standard form of a linear function is \( Ax + By = C \) where \( A \) and \( B \) and \( C \) are real numbers.

Note that all linear functions can be re-written in slope-intercept form.

Example 1

For each equation, identify the form of the equation, and identify the information we know about the line in its current form.

a) \( f(t) = 4t - 9 \)

b) \( P(q) = 1.45(q - 750) + 8200 \)

c) \( A(w) = -30(w + 17) - 20 \)

Solution:

a) The function \( f(t) \) is a linear function written in slope-intercept form. The slope of the line is 4 and the y-intercept of the line is \((0,-9)\).

b) The function \( P(q) \) is a linear function written in point-slope form. The slope of the line is 1.45 and the line passes through the point \((750,8200)\).

c) The function \( A(w) \) is a linear function written in point-slope form. The slope of the line is \(-30\) and the line passes through the point \((-17,-20)\).
Try it Now

1. For each equation, identify the form of the equation, and identify the information we know about the line in its current form.
   a) \( M(k) = -7k + 12 \)
   b) \( L(r) = 4.26(r + 15) + 45 \)
   c) \( W(d) = -\frac{2}{5}(d - 800) - 1600 \)

Example 2 (* Video Example Here)

Re-write the equation \( 5x + 3y = 60 \) in slope-intercept form. Then identify the slope and y-intercept of the line.

Solution:
To write the equation in slope-intercept form we need to solve the equation for \( y \) so that it has the form \( y = mx + b \).

\[
5x + 3y = 60 \\
3y = 60 - 5x \\
y = \frac{60}{3} - \frac{5x}{3} \\
y = 20 - \frac{5}{3}x
\]

\( y = 20 - \frac{5}{3}x \) \( \Rightarrow \) \( y = b + mx \).

So, the slope-intercept form of the line is \( y = 20 - \frac{5}{3}x \).

The slope of the line is \(-\frac{5}{3}\).

The y-intercept of the line is (0,20).

Try it Now

2. For each equation, re-write the equation in slope-intercept form and then identify the value of the slope and identify the y-intercept as a point.
   a) \( y = 7x \)
   b) \( 4y = 2x - 5 \)
   c) \( y = 8 \)
   d) \( x = 7 - y \)
   e) \( y = \frac{2}{3}(x - 15) + 7 \)
   f) \( 5x + 7y = 28 \)
Example 3 (* Video Example Here)
The pressure on a diver, $P$, in pounds per square inch (PSI), depends upon their depth below the water surface, $d$, in feet. The relationship between the pressure on the diver and the depth is given by the function $P(d) = 14.696 + 0.434d$. Identify the slope and y-intercept, and interpret the components of this function in context.

Solution:
The equation can be written as $P(d) = 0.434d + 14.696$ which is slope-intercept form. The slope is $m = 0.434$, and the y-intercept is (0,14.696).

**Interpretation:** The slope is 0.434.
The units of the slope are \( \frac{\text{output}}{\text{input}} = \frac{\text{pressure}}{\text{depth}} = \frac{\text{PSI}}{\text{ft}} = \text{PSI per foot} \). This tells us that the pressure on the diver increases by 0.434 PSI for each foot their depth increases.

The y-intercept is (0,14.696). This tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 PSI.

**Slope of a Line**

$m$ is the **constant rate of change** of a linear function, which is called the **slope** of the line. The sign of the slope of a line determines if the line is increasing or decreasing:

- The line is **increasing** if $m > 0$. A positive slope means the linear function is increasing.
- The line is **decreasing** if $m < 0$. A negative slope means the linear function is decreasing.
- The cases when the slope is 0 or undefined are discussed at the end of this section.

Given any two points, \((x_1, f(x_1))\) and \((x_2, f(x_2))\) on a linear function, the slope is the rate of change between the two points. Since the rate of change is **constant** on a linear function, any two points on the linear function will produce the same value when calculating the slope:

\[
slope = m = \frac{\text{change in output}}{\text{change in input}} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]
Example 4 (* Video Example Here)

Match each equation with one of the lines in the graph below. Explain how you know which is which.

\[
\begin{align*}
  f(x) &= 2x + 3 \\
  g(x) &= 2x - 3 \\
  h(x) &= -2x + 3 \\
  j(x) &= \frac{1}{2}x + 3
\end{align*}
\]

Solution:

Only one graph has a vertical intercept of \(-3\). So, we can immediately match that graph with \(g(x)\) since \(g(x) = 2x - 3\) has a \(y\)-intercept of \((0, -3)\). \(Green = g(x)\).

For the three graphs with a vertical intercept at 3, only one is decreasing, which would indicate a negative slope. So, we can match that decreasing line with \(h(x) = -2x + 3\) since \(h(x)\) has a slope of \(-2\). \(Purple = h(x)\).

Of the other two lines, the steeper line would have a larger slope. So, we can match the steeper line with equation \(f(x) = 2x + 3\), since it has a slope of \(\frac{1}{2}\). \(Blue = f(x)\).

We can match less-steep line with the equation \(j(x) = \frac{1}{2}x + 3\) since it has a slope of \(\frac{1}{2}\). \(Red = j(x)\).

Example 5

A company finds that the number of items that they sell, \(Q(p)\), decreases a fixed number of items for every dollar that they increase the price of the items. They find that they sell 350,260 items when the price of the items is $18.42 each. They also find that, at a price of $25.42 per item, they are able to sell 350,246 items. Find the slope of \(Q(p)\) and interpret that value in context.

Solution:

\[
\text{slope} = m = \frac{Q(p_2) - Q(p_1)}{p_2 - p_1} = \frac{350,260 - 350,260}{25.42 - 18.42} = \frac{-14}{7} = -2
\]

The units of the slope in this problem are

\[
\frac{\text{units of output}}{\text{units of input}} = \frac{\text{number of items sold}}{\text{dollar price of items}} = \text{items sold per dollar price}
\]

The slope here is negative. To travel on this line, we could travel “over to the right 1 unit, and down 2 units” to land back on the line.

Interpretation: The contextual meaning of this is that \(\text{the company will sell 2 fewer items every time they raised the price by 1 dollar}\).
**Example 6 (*) Video Example Here)**

Suppose the population of a city increased at a constant rate each year from 2002 to 2006. Suppose that the population increased from 23,400 to 27,800 during that time period. Find the rate of change of the population during this time span, and interpret the meaning of that value in context.

**Solution:**

We use the formula to find the rate of change, which is the slope:

\[
slope = m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{27800 - 23400}{2006 - 2002} = \frac{4400}{4} = 1100
\]

The rate of change, which is also called the slope, is \( m = 1100 \).

To interpret the value in context, we first add units to the slope. The units of slope are as follows:

\[
\frac{\text{units of output}}{\text{units of input}} = \frac{\text{people}}{\text{year}} = \text{people per year}
\]

**Interpretation:** So, the slope is “1100 people per year.” This tells us that the population of the city, between 2002 and 2006, increased at a constant rate of 1100 people per year.

Note that you can check your answer by noting that (1100 people per year) *(4 years passing) = 4400 people added to the population in that time period. This checks out since the population did increase from 23,400 to 27,800, which is 4400 people total in that time period.

**Try it Now**

3. The function \( D(m) \) gives the distance that a person is from home in miles, where \( m \) is the number of minutes that have passed since the person began to travel. If we know that \( D(0) = 240 \), and we know that \( D(45) = 206.25 \), then find the slope and interpret the meaning of that value in context.
Finding Horizontal Intercepts

The **horizontal intercept** of the function is where the graph crosses the horizontal axis. The output of the function will equal zero at every horizontal intercept. If a function has a horizontal intercept, you can always find it by solving \( f(x) = 0 \).

Example 7

Find the horizontal intercept of \( f(x) = -3 + \frac{1}{2}x \).

Solution:
Setting the function equal to zero to find what input will put us on the horizontal axis,

\[
0 = -3 + \frac{1}{2}x
\]

\[
3 = \frac{1}{2}x
\]

\[
x = 6
\]

The graph crosses the horizontal axis at (6,0).

Try it Now

4. The balance in your college payment account, \( C \), is a function of the number of quarters, \( q \), you attend. Interpret the components of the function \( C(a) = 20000 - 4000q \) in context. How many quarters of college can you pay for until this account is empty?

Break-Even Point

The **break-even point** of a profit function, \( P(x) \), is the x-intercept (horizontal intercept) of the profit function. The break-even point is the point where the profit is exactly 0.
Important Topics of this Section

Identify features of a linear function from a graph
Given the equation of a linear function, identify features of the line shown in the given equation.
Re-write a linear equation in slope-intercept form
Identify if a linear equation is in slope-intercept form, or point-slope form, or standard form
Find the vertical intercept
Find the horizontal intercept
Finding the slope/rate of change, $m$
Interpret values in a linear function in context

Try it Now Answers

1.

a) The function $M(k)$ is a linear function written in slope-intercept form. The slope of the line is $-7$ and the $y$-intercept of the line is $(0,12)$.

b) The function $L(r)$ is a linear function written in point-slope form. The slope of the line is $4.26$ and the line passes through the point $(-15,45)$.

c) The function $W(d)$ is a linear function written in point-slope form. The slope of the line is $-2/5$ and the line passes through the point $(800,-1600)$.

2.

a) $y = 7x + 0$

b) $y = \frac{1}{2} x - \frac{5}{4}$

c) $y = 0x + 8$

d) $y = -1x + 7$

e) $y = \frac{2}{3}x - 3$

f) $y = -\frac{5}{7}x + 4$

3. $slope = \frac{206.25 - 240}{45 - 0} = \frac{-33.75}{45} = -0.75$. In context, this tells us that every minute, the person’s distance from home decreases by 0.75 miles. In other words, the person is travelling towards home at a speed of 0.75 miles per minute. That is the same as 45 miles per hour.

4. Solving $C(a) = 0$ gives $a = 5$. The horizontal intercept is $(5,0)$. You can pay for 5 quarters before the money in this account is gone.
Section 2.1 Exercises

1. For each equation, identify the form of the equation, and identify the information we know about the line in its current form.
   a) \( Q(p) = -3.12(q - 150) + 14,700 \)
   b) \( D(t) = 65t + 1400 \)
   c) \( g(x) = 7(x + 1) + 48 \)

2. For each equation, identify the form of the equation, and identify the information we know about the line in its current form.
   a) \( f(x) = 40(x + 7) - 85 \)
   b) \( m(b) = -6b + 9 \)
   c) \( k(r) = -0.2(r - 9) + 20 \)

3. Match each linear equation with its graph.
   a) \( f(x) = -x - 1 \)
   b) \( f(x) = -2x - 1 \)
   c) \( f(x) = -\frac{1}{2}x - 1 \)
   d) \( f(x) = 2 \)
   e) \( f(x) = 2 + x \)
   f) \( f(x) = 3x + 2 \)

4. Match each linear equation with its graph.
   a) \( f(x) = \frac{1}{2}x + 3 \)
   b) \( f(x) = 2x + 3 \)
   c) \( f(x) = 2x - 3 \)
   d) \( f(x) = -3x + 3 \)
5. For each equation, re-write the equation in slope-intercept form and then identify the value of the slope and identify the y-intercept as a point.
   a) \( y = \frac{2}{3}(x - 15) + 7 \)
   b) \( y = -3 \)
   c) \( y = 10x \)
   d) \( x = 13 - 17y \)
   e) \( 10x - 5y = 60 \)
   f) \( \frac{2}{3}y = 6 - 8x \)

6. For each equation, re-write the equation in slope-intercept form and then identify the value of the slope and identify the y-intercept as a point.
   a) \( -8y = 5x + \frac{7}{16} \)
   b) \( x = -y \)
   c) \( y = -0.005(x + 1800) - 600 \)
   d) \( y = 1900 \)
   e) \( -15x - 12y = 288 \)
   f) \( y = 150x \)

7. A phone company charges for service according to the formula: \( C(n) = 26 + 0.04n \), where \( n \) is the number of minutes talked, and \( C(n) \) is the monthly charge, in dollars.
   a) Identify the slope and interpret in context.
   b) Identify the vertical intercept and interpret in context.
   c) Identify the horizontal intercept and interpret in context.

8. Terry is skiing down a steep hill. Terry's elevation, \( E(t) \), in feet after \( t \) seconds is given by \( E(t) = 3000 - 70t \).
   a) Identify the slope and interpret in context.
   b) Identify the vertical intercept and interpret in context.
   c) Identify the horizontal intercept and interpret in context.

9. A phone company charges for service according to the formula: \( C(n) = 24 + 0.1n \), where \( n \) is the number of minutes talked, and \( C(n) \) is the monthly charge, in dollars.
   a) Identify the slope and interpret in context.
   b) Identify the vertical intercept and interpret in context.
   c) Identify the horizontal intercept and interpret in context.

10. Maria is climbing a mountain. Maria's elevation, \( E(t) \), in feet after \( t \) minutes is given by \( E(t) = 1200 + 40t \).
    a) Identify the slope and interpret in context.
    b) Identify the vertical intercept and interpret in context.
    c) Identify the horizontal intercept and interpret in context.

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11. Sonya is walking home from a friend’s house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. Find the slope and interpret in context.

12. A gym membership with two personal training sessions costs $125, while gym membership with five personal training sessions costs $260. Find the slope and interpret in context.

13. A city's population in the year 1960 was 287,500. In 1989 the population was 275,900. Find the slope and interpret in context.

14. A city's population in the year 1958 was 2,113,000. In 1991 the population was 2,099,800. Find the slope and interpret in context.

15. Find the break-even point of the profit function \( P(q) = 3.55q - 12,400 \) where \( q \) is the number of items sold, and \( P(q) \) is the profit earned in dollars. Interpret the meaning of the break-even point in context.

16. Find the break-even point of the profit function \( P(q) = 4.8(1 - 250) + 7500 \) where \( q \) is the number of items sold, and \( P(q) \) is the profit earned in dollars. Interpret the meaning of the break-even point in context.
Section 2.2 Finding Equations of Linear Functions

As you hop into a taxi in Las Vegas, the meter will immediately read $3.50; this is the initial charge made when the taximeter is activated. After that initial fee, the taximeter will add $2.76 for each mile the taxi drives\(^1\). In this scenario, the total taxi fare depends upon the number of miles ridden in the taxi, and we can ask whether it is possible to model this type of scenario with a function.

Using descriptive variables, we choose \(m\) for miles and \(C\) for Cost (in dollars) as a function of miles. So, the total cost of a taxi ride is given by the function \(C(m)\).

We know for certain that \(C(0) = 3.50\) since the $3.50 initial charge is assessed regardless of how many miles are driven.

Since $2.67 is added for each mile driven, then \(C(1) = 3.50 + 2.67 = 6.17\) .
This means that the total cost of a 1-mile taxi ride would be $6.17.

If we then drove a second mile, another $2.67 would be added to the cost.
\(C(2) = 3.50 + 2.67 + 2.67 = 3.50 + 2.67(2) = 8.84\)
This means that the total cost of a 2-mile taxi ride would be $8.84.

If we drove a third mile, another $2.67 would be added to the cost:
\(C(3) = 3.50 + 2.67 + 2.67 + 2.67 = 3.50 + 2.67(3) = 11.51\)

From this we might observe the pattern, and conclude that if \(m\) miles are driven, \(C(m) = 3.50 + 2.67m\) because we start with a $3.50 initial fee and then add $2.67 per mile times the number of miles driven.

It is good to verify that the units make sense in this equation. The $3.50 drop charge is measured in dollars; the $2.67 charge is measured in dollars per mile.

\[ C(m) \text{ total dollars} = (3.50 \text{ dollars}) + \left(\frac{2.67 \text{ dollars}}{1 \text{ mile}}\right) \times (m \text{ miles}) \]

When dollars per mile is multiplied by miles, the result is dollars. So, the units of the left side of the equation match the units of the right side of the equation.

Notice this equation \(C(m) = 3.50 + 2.67m\) consisted of two terms. The first term is the fixed $3.50 charge which does not change based on the value of the input. The second is the $2.67 dollars per mile value, which is a rate of change, multiplied by the input value (miles).

\(^1\) Nevada Taxicab Authority, retrieved May 8, 2017. There is also a waiting fee assessed when the taxi is waiting at red lights, but we’ll ignore that in this discussion.
Looking at this same taxi ride problem in table format, we can also see the total cost changes by $2.67 for every 1-mile increase.

<table>
<thead>
<tr>
<th>m miles driven</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>C(m) total cost in dollars</td>
<td>3.50</td>
<td>6.17</td>
<td>8.84</td>
<td>11.51</td>
</tr>
</tbody>
</table>

It is important to note that in this equation, the rate of change is constant. Over any interval, the rate of change is the same, constant value: 2.67 dollars per mile.

Graphing this equation, \( C(m) = 3.50 + 2.67m \) we see the shape is a line, which is how these functions get their name: linear functions.

When the number of miles is zero, then the cost is $3.50, which is the point \((0, 3.50)\) on the graph. This is the vertical-intercept.

The graph is increasing in a straight line from left to right because for each mile added, the cost goes up by $2.67. This is a constant rate of change, which we call the slope. Every time the input-value increases by 1, the output value increases by 2.67.

In the previous example, we saw the total cost of the taxi ride represented in the following 4 ways:

- modeled in words
- given as an equation
- given as a table
- given in graphical form.

Whenever possible, ensure that you can link these four representations together to continually build your skills. It is important to note that you will not always be able to find all 4 representations for a problem and so being able to work with all 4 forms is very important.
Finding the equation of a Linear Function

To find the equation of a line when the y-intercept is given, it is easiest to use the slope-intercept equation \( f(x) = mx + b \) where \( m \) is the slope of the line, and \((0,b)\) is the y-intercept of the line.

To find the equation of a line when we only know a generic point, \((x_1, y_1)\) that is on the line, then it is easiest to use the point-slope equation \( f(x) = m(x - x_1) + y_1 \) where \( m \) is the slope of the line, and \((x_1, y_1)\) is any point on the line.

The standard form equation, \( Ax + By = C \) is useful when we are given rates related to x and rates related to y and a total value that has units equivalent to the units of \( Ax + By \).

Example 1

Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, \( N \), in his iTunes collection as a function of the number of months, \( m \). How many songs will he own in a year?

Solution:
The initial value, which is the y-intercept value, is 200, since he currently owns 200 songs, so \( N(0) = 200 \).

The number of songs increases by 15 songs per month, so the slope is 15 songs per month.

With this information, we can write the formula: \( N(m) = 200 + 15m \).

With this formula we can predict how many songs he will have in 1 year (12 months): \( N(12) = 200 + 15(12) = 200 + 180 = 380 \). Marcus will have 380 songs in 12 months.

Try it Now
1. Suppose a company has fixed costs of \$45,000. Suppose the company spends \$3 for each item produce, and also that they charge \$7 per item when they sell the items. If we assume that the company sells exactly the same number of items that they produce, then find a formula for the total profit earned, \( P(q) \), when producing and selling \( q \) items.
   a) Identify the vertical intercept of the function and interpret the meaning of that point in context.
   b) Identify the slope of the function and interpret the meaning of the slope in context.
   c) Identify the equation of \( P(q) \).
Example 2
Write an equation for the linear function graphed to the right.

Solution:
Looking at the graph, we observe that it passes through the points (0, 7) and (4, 4).

So, we know the y-intercept of the equation, and we could use the equation \( f(x) = mx + b \) where \( b = 7 \).

We can use the two points to find the slope of the line:
\[
m = \frac{4 - 7}{4 - 0} = \frac{-3}{4}
\]

This allows us to write the equation:
\[
f(x) = b + mx = (7) + \left(\frac{-3}{4}\right)x
\]
And so, the equation of the line is \( f(x) = 7 - \frac{3}{4}x \)

Example 3
Johnson Drilling finds that they were able to drain oil out of a well at a steady pace of 350 gallons per hour, and the well ran dry after 1379 hours of continuous drilling.

a) Find an equation for the function \( W(t) \) that gives the number of gallons of oil in the well after \( t \) hours of drilling.
b) Find the amount of oil that was in the well when they began drilling.

Solution:
a) The problem gives the slope as \(-350\) since oil is draining out of the well, that is the rate of decrease of oil in the well. The problem also states that when \( t = 1379 \), then \( W(t) = 0 \). That means that the horizontal intercept is \((1379,0)\). We can use the point-slope form of a line to write the equation:
\[
W(t) = -350(t - 1379) + 0
\]
We can distribute the -350 and simplify in order to write the equation in slope-intercept form as
\[
W(t) = -350t + 482,650
\]
b) The y-intercept of \( W(t) \) is the point \((0, 482650)\). That point tells us that when the number of hours of drilling was 0, then the well had 482,650 gallons of oil in it. So, the well started with 482,650 gallons of oil.

Try it Now
2. A company finds that the number of Sprockets they sell decreases by 13 items every time they raise the price of Sprockets by $1.00. When pricing the Sprockets at $168 each, they are able to sell 510,300 Sprockets. Find an equation for \( S(p) \) that gives the number of Sprockets sold if they are priced at \( p \) dollars each.
Example 4: If \( f(x) \) is a linear function, \( f(3) = -2 \), and \( f(8) = 1 \), find an equation for the function.

Solution:

We first find the slope of the line first: \( \text{slope} = m = \frac{1 - (-2)}{8 - 3} = \frac{3}{5} \).

In this case, we do not know the y-intercept, and so we will use the point-slope form of the line where \( m = \frac{3}{5} \). We could use either \((3, -2)\) or \((8,1)\) as the point in the equation. Both will result in the same equation. The point-slope form of a line is \( f(x) = m(x - x_1) + y_1 \).

- If we use the point \((3, -2)\) to write the equation, then we would have \( f(x) = \frac{3}{5}(x - 3) - 2 \). This can be re-written as \( f(x) = \frac{3}{5}x - \frac{9}{5} - 2 \) which is \( f(x) = \frac{3}{5}x - \frac{19}{5} \).

- If we use the point \((8,1)\) to write the equation, then we would have \( f(x) = \frac{3}{5}(x - 8) + 1 \). This can be re-written as \( f(x) = \frac{3}{5}x - \frac{24}{5} + 1 \) which is \( f(x) = \frac{3}{5}x - \frac{19}{5} \).

So, it doesn’t matter which point is used when writing the equation of the line. Either point results in the same equation. The equation of this function could be written in any of the following forms:

\[ f(x) = \frac{3}{5}(x - 3) - 2 \quad \text{or} \quad f(x) = \frac{3}{5}(x - 8) + 1 \quad \text{or} \quad f(x) = \frac{3}{5}x - \frac{19}{5}. \]

We see that the function \( f(x) \) has a slope of \( 3/5 \). The first equation shows us that the line passes through \((3, -2)\). The second equation show us that the line passes through \((8,1)\). The third equation show us that the line passes through \((0, -19/5)\). The equations all look different, and give us different information, but they are all describing the same linear function.
Example 5

Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so Ilya’s weekly income, \( I \), depends on the number of new policies, \( n \), he sells during the week. Last week he sold 3 new policies, and earned $760 for the week. The week before, he sold 5 new policies, and earned $920.

a) Find an equation for \( I(n) \) in slope-intercept form.

b) Interpret the meaning of the components of the equation.

Solution:

a) The given information gives us two points: \((3, 760)\) and \((5, 920)\). We start by finding the slope.

\[
m = \frac{920 - 760}{5 - 3} = \frac{160}{2} = 80
\]

We can use either point to find the equation. So we could use \((3, 760)\) or we could use \((5, 920)\). It doesn’t matter which point is used since they will both result in the same slope-intercept equation in the end.

We will use the point \((3, 760)\) to find the equation:

\[
I(n) = 80(n - 3) + 760 = 80n - 240 + 760 = 80n + 520 = 520 + 80n
\]

b) The point \((0, 520)\) is the \(y\)-intercept of the function. This means that **Ilya’s income when no new policies are sold is $520**. This is a base salary that he earns before any policies are sold.

Keeping track of units can help us interpret the slope. Income increased by $160 when the number of policies increased by 2, so the rate of change is $80 per policy; **Ilya earns a commission of $80 for each policy sold during the week**.

Try it Now

3. A function \( A(w) \) gives the pounds of apples in a warehouse waiting to be sent to the applesauce production room, where \( w \) is the number of weeks since the apples arrived in the warehouse. At 3 weeks there are 12,000 pounds of apples. At 7 weeks there are 3,000 pounds of apples. The apples are moved out of the warehouse at a constant rate each week. Find an equation for \( A(w) \) in slope-intercept form. Then interpret the components of the equation in context.
Example 6

Given the table below write a linear equation that represents the table values

<table>
<thead>
<tr>
<th>$w$, number of weeks</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(w)$, number of rats</td>
<td>1000</td>
<td>1080</td>
<td>1160</td>
<td>1240</td>
</tr>
</tbody>
</table>

Solution:
We can see from the table that the initial value of rats is 1000 so the y-intercept is (0,1000).

Rather than solving for $m$, we can notice from the table that the population goes up by 80 for every 2 weeks that pass. This rate is consistent from week 0, to week 2, 4, and 6. The rate of change is 80 rats per 2 weeks. This can be simplified to 40 rats per week and we can write $P(w) = b + mw$ as $P(w) = 1000 + 40w$

If you didn’t notice this from the table you could still solve for the slope using any two points from the table. For example, using (2, 1080) and (6, 1240),

$$m = \frac{1240 - 1080}{6 - 2} = \frac{160}{4} = 40 \text{ rats per week}$$

Example 7

Acme Rubber Factory has 25,000 ounces of rubber that they will use to produce only rubber duckies and rubber frogs. It takes 1.8 ounces of rubber to produce each rubber duckie, and it takes 2 ounces of rubber to produce each frog.

(a) Write an equation that relates the number of duckies, $d$, they can produce and the number of frogs they can make, $f$, if they use all of the 25,000 ounces of rubber.

(b) Re-write the equation in slope-intercept form where we treat $f$ as the dependent variable.

(c) Identify the vertical intercept, and interpret the meaning of that point in context.

(d) Identify the number of ducks they would create if they produced 200 frogs.

(e) Identify the slope of the line and interpret the meaning in context.

Solutions:
(a) We can use the units to help us write an equation in standard form:

$$\left( \frac{1.8 \text{ ounces rubber}}{1 \text{ duck produced}} \right) * (d \text{ ducks produced}) + \left( \frac{2 \text{ ounces rubber}}{1 \text{ frog produced}} \right) * (f \text{ frogs produced}) = 25,000 \text{ ounces rubber}$$

$$d (\text{ ducks produced}) + f (\text{ frogs produced}) = 25,000$$

(b) Re-write the equation in slope-intercept form where we treat $f$ as the dependent variable:

$$f = \left( \frac{2}{1} \right) d - \left( \frac{25,000}{1} \right)$$

(c) Identify the vertical intercept, and interpret the meaning of that point in context:

$$f = 0 \implies d = \frac{25,000}{2} = 12,500$$

(d) Identify the number of ducks they would create if they produced 200 frogs:

$$d = \frac{25,000 - 2(200)}{1.8} = 13,222.22$$

(e) Identify the slope of the line and interpret the meaning in context:
So the equation can be written as $1.8d + 2f = 25,000$.

(b) To re-write the equation in slope-intercept form, we solve for $f$ since we were asked to treat the $f$ as the dependent variable.

1. Subtract $1.8d$ on both sides.

2. $2f = 25,000 - 1.8d$

3. Divide by 2 on both sides.

4. $f = \frac{25,000}{2} - \frac{1.8}{2}d$

So, the equation in slope-intercept form is $f = 12,500 - 0.9d$.

(c) The vertical intercept is (0, 12500). The 0 represents the $d$ value, and the 12500 represents the $f$ value. So, if they produce 0 duckies, then they can produce 12,500 frogs using the 25000 ounces of rubber.

(d) If $f = 200$, then

1. $200 = 12,500 - 0.9d$

2. Subtract 12,500 on both sides.

3. $-12,300 = -0.9d$

4. Divide both sides by $-0.9$.

5. $d = 13,666.7$

So, if they produce 200 frogs, then they can produce 13,666 duckies (and they’d have a bit of rubber left over).

(e) The slope of $f = 12,500 - 0.9d$ is $-0.9$. The units of the slope are frogs produced per duckies produced.

So the slope is $-0.9$ frogs produced per duckie produced.

**Interpretation:** Every time they produce another duckie, 0.9 fewer frogs are produced.

---

### Example 8

(*Video Example Here*)

We pour together Mixture A and Mixture B in order to create a new mixture. Mixture A contains 2.5% alcohol, and Mixture B contains 4.1% alcohol. When mixed together, the new mixture contains 150 liters of alcohol.

(a) Write an equation that relates the amount of Mixture A, which we will call $A$, and the amount of Mixture B, which we will call $B$.

(b) Re-write the equation in slope-intercept form where we treat $B$ as the dependent variable.

(c) Identify the vertical intercept, and interpret the meaning of that point in context.

(d) Identify the amount of Mixture $A$ they would use if they used 25 Liters of Mixture $B$.

(e) Identify the slope of the line and interpret the meaning in context.
Solutions:

(a) We can use the units to help us write an equation in standard form:
\[
\left(\text{0.025 liters alcohol \over 1 liters of A}\right) \times (A \text{ liters of A}) + \left(\text{0.041 liters alcohol \over 1 liter of B}\right) \times (B \text{ liters of B}) = 150 \text{ total liters}
\]
So the equation can be written as \(0.025A + 0.041B = 150\).

(b) To re-write the equation in slope-intercept form, we solve for \(B\) since we were asked to treat the \(B\) as the dependent variable.
\[
0.025A + 0.041B = 150 \quad \text{Subtract 0.025A on both sides}
\]
\[
0.041B = 150 - 0.025A \quad \text{Divide by 0.041 on both sides.}
\]
\[
B = \frac{150}{0.041} - \frac{0.025}{0.041}A \quad \text{Simplify each term on the right.}
\]
\[
B \approx 3,658.5 - 0.61A
\]
Notice that we are using a wavy equal sign since we have rounded the two constants in the equation. So the equation in slope-intercept form is \(B \approx 3,658.5 - 0.61A\).

(c) The vertical intercept is \((0, 3658.5)\). The 0 is the \(A\), and the 3658.5 is the \(B\). So, if we use 0 liters of Mixture \(A\), then we will use 3658.5 liters of mixture \(B\) in border to have 150 total liters of alcohol in the final mixture.

(d) If \(B = 25\), then
\[
25 \approx 3,658.5 - 0.61A \quad \text{Subtract 3658.5 on both sides.}
\]
\[
-3,633.5 \approx -0.61A \quad \text{Divide both sides by } -0.61.
\]
\[
A \approx 5956.6
\]
So, if we use 25 liters of Mixture \(A\), then we need to use 5956.6 liters of Mixture \(B\) in order to get 150 liters of alcohol in the final mixture.

(e) The slope of \(B \approx 3,658.5 - 0.61A\) is \(-0.61\). The units of the slope are
\[
\frac{\text{units of } B}{\text{units of } A} = \frac{\text{liters of Mixture } B}{\text{liters of Mixture } A} = \text{liters of Mixture } B \text{ per liter of Mixture } A
\]
So the slope is \(-0.61 \text{ liters of Mixture } B \text{ per liter of Mixture } A\).

Interpretation: Every time we use 1 more liter of Mixture \(A\), we have to use 0.61 fewer liters of Mixture \(B\).
Horizontal and Vertical Lines (*Video Link Here)

**Horizontal lines:**
- The slope of horizontal line is \( \frac{\text{change in } y}{\text{change in } x} = \frac{0}{1} = 0 \). To travel on the line, we travel to the right 1, and then up/down 0.
- The equation of a horizontal line can be written as \( f(x) = 0x + b \) which simplifies to \( f(x) = b \).
- Horizontal lines pass through all the points in the plane that have a y-value equal to b. For example, the horizontal line passes through (0,b) and (1,b) and (2,b) and (-1,b) and (-2,b). The y-value is always equal to \( y=b \), and so it makes sense to describe the horizontal line with the equation \( y=b \) since it is all points in the plane that have a y-value of y=b.

**Vertical lines:**
- The slope of a vertical line is \( \frac{\text{change in } y}{\text{change in } x} = \frac{1}{0} = \text{undefined} \). To travel on the line, we travel up 1, and then left/right 0. Division by 0 is undefined.
- The equation of a vertical line can be written as \( x = a \).
- Vertical lines pass through all the points in the plane that have an x-value equal to a. For example, the vertical line passes through \( (a,0) \) and \( (a,1) \) and \( (a,2) \) and \( (a,-1) \) and \( (a,-2) \). The x-value is always equal to \( x = a \), and so it makes sense to describe the vertical line with the equation \( x=a \) since it is all points in the plane that have an x-value of \( x = a \).
- Vertical lines are not functions since they fail the vertical line test.

**Example 9**

Write an equation for the horizontal line graphed to the right.

**Solution:**
This line would have equation \( f(x) = 2 \). Notice that between any two points, the change in the outputs is 0. In the slope equation, the numerator will be 0, resulting in a slope of 0.

Using a slope of 0 in the \( f(x) = mx + b \) equation, we would have \( f(x) = 0x + 2 \), which simplifies to \( f(x) = 2 \).
Example 10

Write an equation for the vertical line graphed to the right.

Solution:
This line would have equation \( x = 2 \)

In the case of a vertical line, like the one graphed to the right, notice that between any two points, the change in the inputs is zero. In the slope equation, the denominator will be zero, and so, the slope of a vertical line is undefined. The vertical line is not a function.

This line passes through points such as (2,0) and (2,1) and (2,2).
Every point on the line has an \( x \)-value equal to \( x = 2 \).
For this reason, we describe the line as all points in the plane where \( x = 2 \).

Parallel and Perpendicular Lines (*Video Link Here)

When two lines are graphed together, the lines will be parallel if they are increasing at the same rate. Parallel lines have the same slope. In this case, the graphs will never cross (unless they’re the same line called colinear lines).

**Parallel Lines**

Two lines are parallel if the slopes are equal (or, if both lines are vertical).

In other words, given two linear equations \( f(x) = b + m_1x \) and \( g(x) = b + m_2x \), the lines will be parallel if \( m_1 = m_2 \).

If two lines are not parallel, one other interesting possibility is that the lines are perpendicular, which means the lines form a right angle (90 degree angle – a square corner) where they meet. In this case, the slopes when multiplied together will equal -1. Solving for one slope leads us to the definition:

**Perpendicular Lines**

Given two linear equations \( f(x) = b + m_1x \) and \( g(x) = b + m_2x \)

The lines will be perpendicular if \( m_1m_2 = -1 \), and so \( m_2 = \frac{-1}{m_1} \)

We often say the slope of a perpendicular line is the “negative reciprocal” of the other line’s slope.
Example 1
(* Video Example Here)
Find a line parallel to \( f(x) = 6 + 3x \) that passes through the point (3, 0).

Solution:
We know the line we’re looking for will have the same slope as the given line, \( m = 3 \). Using this and the given point, we can solve for the new line’s vertical intercept:

\[
g(x) = b + 3x \quad \text{then at (3, 0),} \\
0 = b + 3(3) \\
b = -9
\]

The line we’re looking for is \( g(x) = -9 + 3x \).

Example 12
Find the slope of a line perpendicular to a line with:
a) a slope of 2.
b) a slope of -4.
c) a slope of \( \frac{2}{3} \).

Solutions:
a) If the original line had slope 2, the perpendicular line’s slope would be \( -\frac{1}{2} \).
b) If the original line had slope -4, the perpendicular line’s slope would be \( \frac{1}{4} \).
c) If the original line had slope \( \frac{2}{3} \), the perpendicular line’s slope would be \( -\frac{3}{2} \).

Example 13 (*Video Example Here)
Find the equation of a line perpendicular to \( f(x) = 6 + 3x \) and passing through the point (3, 0).

Solution:
The original line has slope \( m = 3 \). The perpendicular line will have slope \( m = -\frac{1}{3} \). Using this and the given point, we can find the equation for the line.

\[
g(x) = b - \frac{1}{3}x \quad \text{then at (3, 0),} \\
0 = b - \frac{1}{3}(3) \\
b = 1
\]

The line we’re looking for is \( g(x) = 1 - \frac{1}{3}x \).
Try it Now
4. Given the line \( h(t) = -4 + 2t \), find an equation for the line passing through \((0, 0)\) that is…
   a) parallel to \( h(t) \). Call the function \( f(x) \).
   b) perpendicular to \( h(t) \). Call the function \( g(x) \).

---

**Important Topics of this Section**
- Find equation of line given slope and y-intercept
- Find equation of line given slope and generic point on the line
- Find equation of line given two points
- Horizontal and vertical lines
- Parallel and perpendicular lines

**Try it Now Answers**
1. 
   a) The vertical intercept is \((0, -45000)\). This tells us that the profit is -$45,000 when producing and selling no items. These are the fixed costs.
   b) The slope is 4, which tells us that the profit increases by $4 for every additional item produced and sold.
   c) \( P(q) = -45,000 + 4q \)

2. \( S(p) = -13(p - 168) \)

3. The slope of the line is \(-\frac{9,000 - 12,000}{7 - 3} = \frac{-3,000}{4} = -750\). The equation can be written as either \( A(w) = -750(w - 3) + 12,000 \) or as \( A(w) = -750(w - 7) + 9,000 \). Both of these equations can be simplified to slope-intercept form as \( A(w) = -750w + 14,250 \). The slope tells us that the apples in the warehouse are decreasing at a constant rate of 750 pounds per week. The y-intercept is \((0, 14250)\) which tells us that there was 14,250 pounds of apples when they were first delivered to the warehouse.

4. 
   a) The slope of the line is the same as the slope of \( h(t) \). So, the slope of the line is 2. The equation of the line parallel to \( h(t) \) would be \( f(x) = 0 + 2x \). This simplifies to \( f(x) = 2x \).
   b) The slope of the line is \(-\frac{1}{2}\) since it is the negative reciprocal of the slope of \( h(t) \). The equation of the line perpendicular to \( h(t) \) would be \( g(x) = 0 - \frac{1}{2}x \). This simplifies to \( g(x) = -\frac{1}{2}x \).
**Section 2.2 Exercises**

1. Find an equation of the line with slope $-2/3$ that passes through the point (7, 18).

2. Find an equation of the line with slope $-8/9$ that passes through the point (-15, 250).

3. Find an equation of the line that passes through (-1, 4) and (-5, 2).

4. Find an equation of the line that passes through (-2, -8) and (4, -6).

5. Find an equation of the line in the graph.

6. Find an equation of the line in the graph.

7. Which of the following tables could represent a linear function? For each that could be linear, find a linear equation models the data. For each table that is not linear, explain how you know it is not linear.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>$x$</th>
<th>$h(x)$</th>
<th>$x$</th>
<th>$f(x)$</th>
<th>$x$</th>
<th>$k(x)$</th>
</tr>
</thead>
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<td>15</td>
<td>70</td>
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<td>73</td>
</tr>
</tbody>
</table>
8. Which of the following tables could represent a linear function? For each that could be linear, find a linear equation models the data. For each table that is not linear, explain how you know it is not linear.

<table>
<thead>
<tr>
<th>x</th>
<th>g(x)</th>
<th>x</th>
<th>h(x)</th>
<th>x</th>
<th>f(x)</th>
<th>x</th>
<th>k(x)</th>
</tr>
</thead>
<tbody>
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<td>-4</td>
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<td>53</td>
<td>8</td>
<td>56</td>
<td>8</td>
<td>231</td>
</tr>
</tbody>
</table>

9. A car rental company offers two plans for renting a car.
   Plan A: 30 dollars per day and 18 cents per mile
   Plan B: 50 dollars per day with free unlimited mileage
   a) Find an equation A(m) that gives the total cost of Plan A if we drive m miles.
   b) Find an equation B(m) that gives the total cost of Plan B if we drive m miles.

10. You’re comparing two cell phone companies.
    Company A: $20/month for unlimited talk and text, and $10/GB for data.
    Company B: $65/month for unlimited talk, text, and data.
    a) Find an equation A(d) that gives the total cost of using Company A if we use d GB of data.
    b) Find an equation B(d) that gives the total cost of using Company B if we use d GB of data.

11. A town's population has been decreasing the same amount each year. In 2003, the population was 45,000, and the population has been decreasing by 1700 people each year. Write an equation, \( P(t) \), for the population \( t \) years after 2003.

12. A town's population has been declining at a constant rate each year. In 2005, the population was 69,000, and the population has been growing by 2500 people each year. Write an equation, \( P(t) \), for the population \( t \) years after 2005.

13. Sonya is currently 10 miles from home, and is walking towards home at 2 miles per hour. Write an equation for her distance from home, \( D(t) \), \( t \) hours from now.

14. A boat is 100 miles away from the marina, sailing directly towards the marina at 10 miles per hour. Write an equation for the distance of the boat from the marina, \( D(t) \), after \( t \) hours.

15. Jackson’s finds that their profit increases at a steady rate of $3.85 for every additional trinket they produce and sell. When producing and selling 1400 trinkets, they have a profit of $12,500. Write an equation for the total profit, \( P(t) \), when producing and selling \( t \) trinkets.
16. A company finds that the number of sproogles they sell decreases by 13 items every time they raise the price of the items by 1 additional dollar. When charging $53.75 per item, they sell a total of 320,000 items. Write an equation for the total number of items sold, \( Q(p) \), when charging \( p \) dollars per item.

17. A car is driving towards home at 60 mph. After 2.4 hours of driving, the car is 320 miles away from home.
   a) Write an equation that gives the car’s distance from home, \( D(t) \), after \( t \) hours of driving.
   b) How far was the car from home to start?
   c) After how many hours of driving will the car reach home?

18. A regional shipping facility receives the holiday season’s supply of holiday lights for the region. They sell and ship out lights at a steady pace of 215 lights per week, and after 3 weeks they have 270 lights remaining in stock.
   a) Write an equation that gives the number of lights remaining in the shipping facility, \( L(w) \), where \( w \) is the number of weeks after the supply arrived at the shipping facility.
   b) How many lights initially arrived at the shipping facility?
   c) How long will it take until the shipping facility runs out of lights?

19. A garbage company collects tires and stores them in a big pile outside of town. They add 24 tires per week to the pile, since the company caps their tire collection at 24 each week. After 68 weeks, they pile has 1380 tires in it.
   a) Write an equation that gives the number of tires in the pile, \( T(w) \), where \( w \) is the number of weeks after the pile was started.
   b) How many tires were in the pile initially?
   c) The town counsel passes an ordinance about the pile of tires stating that the company has to limit the tire pile to 2000 tires or less. When will the tire pile reach this cap?

20. A clothing business finds there is a linear relationship between the number of shirts, \( n \), it can sell and the price, \( p \), it can charge per shirt. In particular, historical data shows that 1000 shirts can be sold at a price of $30, while 3000 shirts can be sold at a price of $22. Find a linear equation in the form \( p = mn + b \) that gives the price \( p \) they can charge for \( n \) shirts.

21. A farmer finds there is a linear relationship between the number of bean stalks, \( n \), she plants and the yield, \( y \), each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationship in the form \( y = mn + b \) that gives the yield when \( n \) stalks are planted.

22. A company produces plastic toy airplanes. When producing the toys, they have total costs of $19,550 when producing 3000 toy airplanes. When producing 7400 toy airplanes, they have total costs of $21,090.
   a) Write an equation that gives the total cost, \( C(a) \), when producing \( a \) toy airplanes.
   b) What are the companies fixed costs?
   c) Interpret the contextual meaning of the slope.
23. The ticket booth at an event allows people to enter the venue at a fixed rate per hour. 45 minutes after opening the doors, there are 540 people inside. There were 240 people inside 20 minutes after opening the doors.
   a) Write an equation for \( P(m) \), which gives the number of people inside \( m \) minutes after opening the doors.
   b) The fire-codes allow a maximum of 600 people in the venue. How many minutes will pass until they reach the max capacity?

24. Mixture A contains 8% alcohol, and Mixture B contains 14% alcohol. We pour the two mixtures together in order to create a new mixture that contains 13 gallons of alcohol.
   a) Write an equation that relates \( A \), the gallons of mixture A, and \( B \), the gallons of mixture B, that would be necessary to produce the new mixture.
   b) If we used 0 gallons of mixture A, then how many gallons of mixture B would be necessary to create the new mixture?
   c) If we used 0 gallons of mixture B, then how many gallons of mixture A would be necessary to create the new mixture?
   d) Solve the equation from part (a) for the variable \( B \) so that it is written in slope-intercept form.
   e) Write a sentence that interprets the contextual meaning of the slope of the equation from part (d).

25. Mixture X contains 35% alcohol, and Mixture Y contains 18% alcohol. We pour the two mixtures together in order to create a new mixture that contains 27 gallons of alcohol.
   a) Write an equation that relates \( X \), the gallons of mixture X, and \( Y \), the gallons of mixture Y, that would be necessary to produce the new mixture.
   b) If we used 0 gallons of mixture X, then how many gallons of mixture Y would be necessary to create the new mixture?
   c) If we used 0 gallons of mixture Y, then how many gallons of mixture X would be necessary to create the new mixture?
   d) Solve the equation from part (a) for the variable \( Y \) so that it is written in slope-intercept form.
   e) Write a sentence that interprets the contextual meaning of the slope of the equation from part (d).

26. A person invests some money in an account that gives 5% interest per year, and invests some other money in a different account that gives 3% interest per year. At the end of the year, the person has earned a total of $350 of interest from the accounts all together.
   a) Write an equation that relates the amount of money invested in each account. Be sure to describe the meaning of each variable that you use in the equation.
   b) Re-write your equation in slope-intercept form.
   c) Identify the vertical intercept of your equation and write a sentence that interprets the contextual, real-world meaning of the vertical intercept.
   d) Identify the horizontal intercept of your equation and write a sentence that interprets the contextual, real-world meaning of the horizontal intercept.
27. A person invests some money in an account that gives 1.8% interest per year, and invests some other money in a different account that gives 2.4% interest per year. At the end of the year, the person has earned a total of $725 of interest from the accounts all together.

a) Write an equation that relates the amount of money invested in each account. Be sure to describe the meaning of each variable that you use in the equation.

b) Re-write your equation in slope-intercept form.

c) Identify the vertical intercept of your equation and write a sentence that interprets the contextual, real-world meaning of the vertical intercept.

d) Identify the horizontal intercept of your equation and write a sentence that interprets the contextual, real-world meaning of the horizontal intercept.

28. A company produces toy boats, and also produces toy cars. The toy boats require 1.2 ounces of plastic per boat and the toy cars require 0.8 ounces of plastic per car. They have a total supply of 450 ounces of plastic to produce boats and cars.

a) Write an equation that relates the number of cars and the number of boats they are able to produce with the amount of plastic they have available. Be sure to describe the meaning of each variable that you use in the equation.

b) Re-write your equation in slope-intercept form.

c) Identify the vertical intercept of your equation and write a sentence that interprets the contextual, real-world meaning of the vertical intercept.

d) Identify the horizontal intercept of your equation and write a sentence that interprets the contextual, real-world meaning of the horizontal intercept.

29. A company produces toy boats, and also produces toy cars. The toy boats cost $0.45 per boat to produce and the toy cars cost $0.22 per car to produce. They are directed to only spend $900 on the production of these two toys.

a) Write an equation that relates the number of cars and the number of boats they are able to produce with the amount of money they have available. Be sure to describe the meaning of each variable that you use in the equation.

b) Re-write your equation in slope-intercept form.

c) Identify the vertical intercept of your equation and write a sentence that interprets the contextual, real-world meaning of the vertical intercept.

d) Identify the horizontal intercept of your equation and write a sentence that interprets the contextual, real-world meaning of the horizontal intercept.
30. While speaking on the phone to a friend in Oslo, Norway, you learned that the current temperature there was -23 Celsius (-23°C). After the phone conversation, you wanted to convert this temperature to Fahrenheit degrees, °F, but you could not find a reference with the correct formulas. You then remembered that the relationship between °F and °C is linear. [UW]
   
a) Using this and the knowledge that 32°F = 0 °C and 212 °F = 100 °C, find an equation that computes Celsius temperature in terms of Fahrenheit; i.e. an equation of the form C = “an expression involving only the variable F.”
   
b) Likewise, find an equation that computes Fahrenheit temperature in terms of Celsius temperature; i.e. an equation of the form F = “an expression involving only the variable C.”
   
c) How cold was it in Oslo in °F?
   
d) 31. Write the equation of the line shown

32. a) Write an equation for a line parallel to \( f(x) = -5x - 3 \) and passing through the point (2,-12)
   
b) Write an equation for a line parallel to \( g(x) = 3x - 1 \) and passing through the point (4,9)
   
c) Write an equation for a line perpendicular to \( h(t) = -2t + 4 \) and passing through the point (-4,-1)
   
d) Write an equation for a line perpendicular to \( p(t) = 3t + 4 \) and passing through the point (3,1)
Section 2.3 Intersections of Lines on Excel

In this section we will learn how to use the Solver feature in Excel to find the intersection of two lines. This will give us a technological method for solving for when two functions are equal.

Example 1

Use Excel to find the intersection of the lines $f(x) = 2x - 10$ and $g(x) = -3x + 10$.

Solution:
In Excel we first create an input-output table and graph for $f(x)$ and $g(x)$ using a domain that allows us to see the point of intersection. We see in the table and graph below that we have identified a range of $x$-values that allows us to see the intersection point.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
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</thead>
<tbody>
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<td>$g(x) = -3x + 10$</td>
<td></td>
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</tr>
</tbody>
</table>

We can conclude that the function $f(x)$ intersects the function $g(x)$ at the point $(4, -2)$. We can see this in the table because at the $x$-value $x = 4$, we see that the two functions have the same $y$-value of $-2$. So, both functions have $(4, -2)$ on their graphs.

Example 1 above was a simple example because the table we initially created in Excel showed the exact intersection point. Usually, when finding the intersection of two functions, the point of intersection will not be obvious, or easy to see in the table or graph.
Find Intersection Point in Microsoft Excel

To find an intersection point of two linear functions, \( f(x) \) and \( g(x) \) in Excel, follow the steps below.

1. Create an input-output table that includes both \( f(x) \) and \( g(x) \) using a window that includes the intersection point.

2. Add a column for \( f(x) - g(x) \). Note that the point where \( f(x) - g(x) = 0 \) is the point where \( f(x) = g(x) \), which is the intersection point.

3. Open Solver, found in the Data tab.

4. In the “Set Objective” box, clear out anything that is currently in the box. Then click on the cell in the worksheet that contains the output value closes to the solution you seek (closest to where the intersection y-value exists). The cell you click on needs to contain the Excel formula.

5. Select “Value of.”

6. In the “By Changing Variable Cells” box, clear out anything that is currently in the box. Then click on the cell in the worksheet containing the input-value (the x-value) that matches up with the cell from step 5.

7. Click Solve.
Example 2

Use Excel to find the intersection of the lines \( f(x) = -0.35x + 31.4375 \) and \( g(x) = 6.16x - 237.1 \).

Solution:
In Excel we first create an input-output table and graph for \( f(x) \) and \( g(x) \) using a domain that allows us to see the point of intersection.

```
<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>g(x)</th>
<th>f(x) - g(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>31.4375</td>
<td>-237.1</td>
<td>268.5375</td>
</tr>
<tr>
<td>5</td>
<td>29.6875</td>
<td>-206.3</td>
<td>235.9875</td>
</tr>
<tr>
<td>10</td>
<td>27.9375</td>
<td>-175.5</td>
<td>203.4375</td>
</tr>
<tr>
<td>15</td>
<td>26.1875</td>
<td>-144.7</td>
<td>170.8875</td>
</tr>
<tr>
<td>20</td>
<td>24.4375</td>
<td>-113.9</td>
<td>138.3375</td>
</tr>
<tr>
<td>25</td>
<td>22.6875</td>
<td>-83.1</td>
<td>105.7875</td>
</tr>
<tr>
<td>30</td>
<td>20.9375</td>
<td>-52.3</td>
<td>73.2375</td>
</tr>
<tr>
<td>35</td>
<td>19.1875</td>
<td>-21.5</td>
<td>40.6875</td>
</tr>
<tr>
<td>40</td>
<td>17.4375</td>
<td>9.3</td>
<td>8.1375</td>
</tr>
<tr>
<td>45</td>
<td>15.6875</td>
<td>40.1</td>
<td>-24.4125</td>
</tr>
<tr>
<td>50</td>
<td>13.9375</td>
<td>70.9</td>
<td>-56.9625</td>
</tr>
<tr>
<td>55</td>
<td>12.1875</td>
<td>101.7</td>
<td>-89.5125</td>
</tr>
<tr>
<td>60</td>
<td>10.4375</td>
<td>132.5</td>
<td>-122.0625</td>
</tr>
<tr>
<td>65</td>
<td>8.6875</td>
<td>163.3</td>
<td>-154.6125</td>
</tr>
<tr>
<td>70</td>
<td>6.9375</td>
<td>194.1</td>
<td>-187.1625</td>
</tr>
<tr>
<td>75</td>
<td>5.1875</td>
<td>224.9</td>
<td>-197.125</td>
</tr>
<tr>
<td>80</td>
<td>3.4375</td>
<td>255.7</td>
<td>-252.2625</td>
</tr>
<tr>
<td>85</td>
<td>1.6875</td>
<td>286.5</td>
<td>-284.8125</td>
</tr>
</tbody>
</table>
```

We can see that the point of intersection occurs a bit after \( x = 40 \).

We now add another column that calculates \( f(x) - g(x) \).

In cell C2 we write the formula =B2 - C2, and then we drag it down.

Notice that \( f(x) - g(x) \) is closest to 0 near \( x=40 \).

This makes sense because the solution to \( f(x) = g(x) \) would be the same as the solution to \( f(x) - g(x) = 0 \).

Now we will use Solver to find the solution.

(Continued on next page)
We now need to use Solver to let Excel identify the exact point where \( f(x) - g(x) = 0 \). This will tell us where \( f(x) = g(x) \).

In the Set Objective Cell, we click on the cell that is closest to where \( f(x) - g(x) = 0 \). That is cell D10.

We select “Value Of”.

In the “By Changing Variable Cells” we click on the x-value cell corresponding to cell D10 above. That is cell A10.

We click Solve.

Excel gives a solution of (41.25, 17).

When \( x = 41.25 \) we see that \( f(x) - g(x) = 0 \).

This means that when \( x = 41.25 \) we have \( f(x) = g(x) \).

We see that both \( f(x) \) and \( g(x) \) have a y-value of \( y=17 \) at that point.

So the functions \( f(x) \) and \( g(x) \) intersect at (41.25, 17).
Example 3

A company produces Snarkles, and sells them on demand (so they only produce Snarkles as they are ordered). The company has initial costs of $28,400 and also has to spend $6.72 for each Snarkle produced. The company then sells the Snarkles for $11.25 each.

(a) Find a formula for $C(q)$ that gives the total cost of producing $q$ Snarkles.

(b) Find a formula for $R(q)$ that gives the total revenue from selling $q$ Snarkles.

(c) Find, and simplify, the formula for $P(q)$ that gives the total profit when producing and selling $q$ Snarkles.

(d) Use Excel to find the point of intersection of the functions $C(q)$ and $R(q)$. Interpret the meaning in context.

(e) Use Excel to find the point when the function $P(q)=0$. Interpret the meaning in context.

Solutions:

(a) Total cost is $(Total \ Cost) = (initial \ costs) + (cost \ per \ item) \times (number \ items \ sold)$.
So, the total cost function is $C(q) = 28,400 + 6.72q$.

(b) Revenue is $(Revenue) = (price \ per \ item) \times (number \ items \ sold)$.
So, the revenue function is $R(q) = 11.25q$.

(c) Profit is $(Profit) = (revenue) - (cost)$.
So, the total profit is $P(q) = R(q) - C(q)$.
This means $P(q) = (11.25q) - (28,400 + 6.72q)$
This simplifies to $P(q) = -28,400 + 4.53q$.

(d) In Excel, we see that $R(q)$ intersects $C(q)$ near $q=6000$ items.

<table>
<thead>
<tr>
<th>q items</th>
<th>R(q) revenue</th>
<th>C(q) cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>$11,250.00</td>
<td>$35,120.00</td>
</tr>
<tr>
<td>2000</td>
<td>$22,500.00</td>
<td>$41,840.00</td>
</tr>
<tr>
<td>3000</td>
<td>$33,750.00</td>
<td>$48,560.00</td>
</tr>
<tr>
<td>4000</td>
<td>$45,000.00</td>
<td>$55,280.00</td>
</tr>
<tr>
<td>5000</td>
<td>$56,250.00</td>
<td>$62,000.00</td>
</tr>
<tr>
<td>6000</td>
<td>$67,500.00</td>
<td>$68,720.00</td>
</tr>
<tr>
<td>7000</td>
<td>$78,750.00</td>
<td>$75,440.00</td>
</tr>
<tr>
<td>8000</td>
<td>$90,000.00</td>
<td>$82,160.00</td>
</tr>
<tr>
<td>9000</td>
<td>$101,250.00</td>
<td>$88,880.00</td>
</tr>
<tr>
<td>10000</td>
<td>$112,500.00</td>
<td>$95,600.00</td>
</tr>
</tbody>
</table>
In order to find the point of intersect in Excel, we first add another column in Excel which calculates $R(q) - C(q)$.

We then use Solver to find where $R(q) - C(q) = 0$.

We see that the intersection of $R(q)$ and $C(q)$ occurs at the point (6269.3, 70529.80). This tells us that the revenue earned and the total cost incurred are both $70,529.80 when the company produces and sells 6269.3 items. So, revenue will equal cost at approximately 6269 or 6270 items. Revenue will be slightly below cost at 6269 items, and revenue will be slightly above cost at 6270 items.
(e) To find where profit is equal to 0, we need to find where \( P(q) = 0 \). Notice that we have already
solved this problem in part (d). To find where profit is 0 is the same thing as finding where revenue is
equal to cost.

\[
P(q) = 0 \quad \text{and we know} \quad P(q) = R(q) - C(q)
\]

So \( P(q) = 0 \) is the same as \( R(q) - C(q) = 0 \)

Since we solved this problem in part (d), we can conclude that the profit is equal to $0 when they sell
approximately 6269.3 items. Profit will be slightly negative when producing and selling 6269 items,
and profit will be slightly positive when producing and selling 6270 items.

**Find Intersection Point by Hand**

To find the intersection point of \( f(x) \) and \( g(x) \), we set the two functions equal to one another by solving
\( f(x) = g(x) \). Then we solve the equation in order to find the input that satisfies the equation.

**Example 4** *(Video Example Here)*

Find the intersection of the lines \( h(t) = 3t - 4 \) and \( j(t) = 5 - t \).

**Solution:**

Setting \( h(t) = j(t) \),

\[
3t - 4 = 5 - t
\]

\[
4t = 9
\]

\[
t = \frac{9}{4} = 2.25
\]

This tells us the lines intersect when the input is \( t = 2.25 \).

We can then find the output value of the intersection point by
evaluating either function at this input \( j(2.25) = 5 - 2.25 = 2.75 \).

These lines intersect at the point \( (2.25, 2.75) \).
Looking at the graph, this result seems reasonable.
Section 2.3 Exercises

1) Use Excel to find the intersection of each of the following:
   a) $f(x) = 3(x - 7) + 12$ and $g(x) = -7x + 8$
   b) $2x - 5y = 15$ and $y = -3x + 2$
   c) $f(x) = 1.28x - 50$ and $g(x) = 0.82x$

2) A car rental company offers two plans for renting a car.
   Plan A: 30 dollars per day and 18 cents per mile
   Plan B: 50 dollars per day with free unlimited mileage
   a) Find an equation $A(m)$ that gives the total cost of Plan A if we drive $m$ miles.
   b) Find an equation $B(m)$ that gives the total cost of Plan B if we drive $m$ miles.
   c) Use Excel to find the intersection point of $A(m)$ and $B(m)$. Interpret the meaning of the point in context.
   d) Describe the circumstances when you should choose Plan A if you rent a car at this company.

3) You’re comparing two cell phone companies.
   Company A: $20/month for unlimited talk and text, and $10/GB for data.
   Company B: $65/month for unlimited talk, text, and data.
   a) Find an equation $A(d)$ that gives the total cost of using Company A if we use $d$ GB of data.
   b) Find an equation $B(d)$ that gives the total cost of using Company B if we use $d$ GB of data.
   c) Use Excel to find the intersection point of $A(d)$ and $B(d)$. Interpret the meaning of the point in context.
   d) Describe the circumstances when you should choose Company B if you are choosing between these two companies.

4) We mix Solution A, which contains 3% alcohol, with Solution B, which contains 7% alcohol. When the two solutions are mixed together, the new solution contains a total of 350 liters of alcohol.
   a) Write an equation that relates $A$, the total amount of Solution A that is used in the mixture, to $B$, the total amount of Solution B that is used in the mixture.
   b) Put the equation in slope-intercept form where $B$ is the dependent variable.
   c) Graph the function in Excel using a span of values from $A=0$ to $A=10,000$.
   d) Use Excel to find the amount of Solution B that will be used if they use 6200 liters of Solution A.
   e) Use Excel to find the amount of Solution A that will be used if they use 2000 liters of Solution B.
5) We mix Solution x, which contains 15% alcohol, with Solution y, which contains 4% alcohol. When the two solutions are mixed together, the new solution contains 25 gallons of alcohol. We want to create a total of 500 gallons of solution.
   a) Write an equation that relates x, the total amount of Solution x that is used in the mixture, to y, the total amount of Solution y that is used in the mixture.
   b) Write a second equation that relates x and y from the given equation.
   c) Graph the two functions together in Excel using a span of values from x=0 to x=200. Let y be the dependent variable.
   d) Use Excel to find the point of intersection. Then write a complete sentence to clearly explain the contextual meaning of that point.

6) A company pays production line workers who work during regular hours $38 per hour, and they pay them $57 per hour when they work outside of the regular work day. The company has to do some maintenance on some machines during regular business hours, and so they want to look at the employee costs if they move some employee hours outside the regular work day in order to get the maintenance done. They need employees to keep the total budget for employee hours at $336,000 or less during this maintenance.
   a) Write an equation that relates x, the total number of regular work day hours, to y, the total number of non-regular work day hours.
   b) Put the equation in slope-intercept form where y is the dependent variable.
   c) Graph the function in Excel using a span of values from x=0 to x=9000.
   d) Use Excel to find the amount of non-regular work hours employees would work if they worked 5248 regular work hours.
   e) Use Excel to find the amount of regular work hours that employees would work if they worked 3295 non-regular work hours.
   f) Identify the vertical intercept and write a sentence to interpret the meaning in context.
   g) Identify the horizontal intercept and write a sentence to interpret the meaning in context.
   h) Now suppose that the company has the additional restriction that they want to have double the amount of regular working hours than they do non-regular working hours. Write a second equation that shows this relationship between x and y.
   i) Find the intersection of the equation from part (h) and the equation from part (a). Then write a complete sentence to clearly explain the contextual meaning of that point.

7) At Bobco, new employees are able to produce an average of 72 sprockets per hour, and senior employees are averaging 128 sprockets per hour. It would be great if they company could have all employees producing sprockets at a rate of 128 sprockets per hour, but they don’t have enough senior employees to do that and senior employees also cost more in hourly wages. The company needs to make 240,000 sprockets each week. They can do that with various mixes of new and senior employee hours.
   a) Write an equation that relates x, the total number of hours worked by new employees, to y, the total number of hours worked by senior employees.
   b) Put the equation in slope-intercept form where y is the dependent variable.
   c) Graph the function in Excel using a span of values from x=0 to x=3400.
d) Use Excel to find the amount of hours the senior employees would need to work if the new employees worked a total of 750 hours.

e) Use Excel to find the amount of hours the new employees would need to work if the senior employees worked 500 hours.

f) Identify the vertical intercept and write a sentence to interpret the meaning in context.

g) Identify the horizontal intercept and write a sentence to interpret the meaning in context.

h) Now suppose that the company has the additional restriction that they want to have twice as many hours from senior works as they do from new workers. Write a second equation that shows this relationship between x and y.

i) Find the intersection of the equation from part (h) and the equation from part (a). Then write a complete sentence to clearly explain the contextual meaning of that point.

8) Carlos invests some money in Bank A, and invests some more money in Bank B. Bank A pays interest at a 2.8% annual percentage rate, and Bank B pays interest at a 3.4% annual percentage rate. Carlos wants to have gained $4000 in total interest by the end of the year. He could put all the money in the bank with the higher interest rate, but that bank’s rate isn’t guaranteed. Carlos is just crossing his fingers that he’ll get that 3.4% rate for the entire year. So, he is splitting his money up between the two banks for now.

a) Write an equation that relates variable A and variable B to the information given in the problem. Be sure to clearly define the contextual meaning of the variables A and B in your equation!

b) Put the equation in slope-intercept form where B is the dependent variable.

c) Graph the function in Excel using a span of values from A=0 to A=150,000.

d) Use Excel to find the amount of money he would need to invest in Bank B if he invested $50,000 in Bank A.

e) Use Excel to find the amount of money he would need to invest in Bank A if he invested $50,000 in Bank B.

f) Identify the vertical intercept and write a sentence to interpret the meaning in context.

g) Identify the horizontal intercept and write a sentence to interpret the meaning in context.

h) Now suppose we also tell you that Carlos is investing a total of $120,000 between the two banks. Write a second equation that relates A and B.

i) Find the intersection of the equation from part (h) and the equation from part (a). Then write a complete sentence to clearly explain the contextual meaning of that point.

9) A company produces and sells Sproogles on demand (so they produce exactly the number that they sell). The company has initial costs of $12,800 and it also costs the company an additional $12.84 to produce each Sproogle. They sell each Sproogle for $18.75.

a) Find a formula for C(q) that gives the total cost of producing q Sproogles.

b) Find a formula for R(q) that gives the total revenue from selling q Sproogles.

c) Find, and simplify, the formula for P(q) that gives the total profit when producing and selling q Sproogles.

d) Use Excel to find the point of intersection of the functions C(q) and R(q). Interpret the meaning in context.

e) Use Excel to find the point when the function P(q)=0. Interpret the meaning in context.
Section 2.4: Systems of Two Linear Equations in Two Variables

In section 2.3 we learned how to use Excel to find the intersection of two lines. Doing so allowed us to solve interesting problems by finding a pair of values that satisfied two different equations. While we didn’t call it this at the time, we were solving a system of two linear equations in two variables. To start out, we’ll review an example of the type of problem we’ve solved before.

**System of Two Linear Equations in Two Variables**

A **system of two linear equations in two variables** is two linear equations, with the same variables, given simultaneously. We will refer to this, simply, as a system or a system of equations.

A **solution** to a system of two linear equations in two variables, if it exists, is a point(s) that solves both equations. A system of equations has three possibilities for solutions:

1. There is exactly one solution, which is the one point of intersection of the two lines.
2. There is no solution, since the two lines are parallel and never intersect.
3. There are an infinite number of solutions, which are all the points on the line(s) since the two linear equations are actually the same line.

**Example 1  Identify if a Point is a Solution to a System**

For each system of equation given below, identify if the given point is a solution to the system of equations or not.

a) System:
   
   \[
   \begin{align*}
   y &= 3x - 10 \\
   2x + 5y &= 10
   \end{align*}
   \]
   
   Point: (8,14)

b) System:
   
   \[
   \begin{align*}
   B &= 2A + 15 & \text{where } B \text{ is the dependent variable} \\
   B &= -5A + 1 & \text{where } B \text{ is the dependent variable}
   \end{align*}
   \]
   
   Point: (0,0)

c) System:
   
   \[
   \begin{align*}
   M + 9 &= Q - 12 & \text{where } M \text{ is the dependent variable} \\
   2M &= Q & \text{where } M \text{ is the dependent variable}
   \end{align*}
   \]
   
   Point: (42, 21)

**Solution:**

a) A point is a solution to a system of equations if the point is a solution to both equations (that the point is on the graph of both equations). So, we need to see if the point (8,14) is a solution to both equations.

For the first equation, if we let \(x=8\), we get \(y = 14\). So, the point (8,14) is a solution to the first equation.
For the second equation, if we let $x=8$, we get $y = -\frac{6}{5}$. So, the point $(8, -\frac{6}{5})$ is a solution to the second equation. The point $(8, 14)$ is not a solution to the second equation.

Since the point $(8, 14)$ is not a solution to both equations, we conclude that $(8, 14)$ is not a solution to the system of equations.

b) A point is a solution to a system of equations if the point is a solution to both equations (that the point is on the graph of both equations). So, we need to see if the point $(0,0)$ is a solution to both equations.

For the first equation, if we let $x=0$, we get $y = 15$. So, the point $(0,0)$ is not a solution to this equation. We could stop here and conclude that the point $(0,0)$ is not a solution to the system since it isn’t a solution to this equation.

For the second equation, if we let $x=0$, we get $y = 1$. So, the point $(0, 0)$ is also not a solution to the second equation.

Since the point $(0, 0)$ is not a solution to either equation, we conclude that $(0, 0)$ is not a solution to the system of equations.

c) A point is a solution to a system of equations if the point is a solution to both equations (that the point is on the graph of both equations). So, we need to see if the point $(42,21)$ is a solution to both equations.

For the first equation, if we let $Q=42$, we get $M = 21$. So, the point $(42,21)$ is a solution to the first equation.

For the second equation, if we let $Q=42$, we get $M = 21$. So, the point $(42, 21)$ is a solution to the second equation.

Since the point $(42, 21)$ is a solution to both equations, we conclude that $(42, 21)$ is a solution to the system of equations. This point is on both graphs. So, this point is an intersection point for these two linear equations.

Solving Systems of Two Linear Equations in Two Variables

There are three common methods for solving systems of linear equations with two variables. We covered one method back in section 2.3, and so we will focus on the other two methods here in this section.

1. Graphing and finding the point(s) of intersection (covered in section 2.3)
2. Substitution method.
3. Elimination method.
**Substitution Method to Solve a System of Two Linear Equations**

In the substitution method, we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable.
4. Write the solution as an ordered pair.
5. Check the solution in both equations.

---

**Example 2  Use Substitution to Solve a System**

Solve the system using the substitution method.

\[-x + y = -5\]
\[2x - 5y = 1\]

**Solution:**

We first pick one of the equations and solve for one of the variables. We will pick the first equation, and we will solve it for \(y\). NOTE: you can pick either equation, and you can pick either variable! It doesn’t matter!

\[-x + y = -5\]

Add \(x\) on both sides.

\[y = x - 5\]

Second, we substitute the expression \((x - 5)\) in the second equation wherever we see the variable \(y\) since \(y\) is equal to \((x - 5)\). Then we solve the resulting equation.

\[2x - 5y = 1\]

Substitute \((x-5)\) in for \(y\).

\[2x - 5(x - 5) = 1\]

Solve for \(x\). The first step is to distribution \(-5\).

\[2x - 5x + 25 = 1\]

Combine \(2x - 5x\) to get \(-3x\) on the left, and subtract 24 on both sides.

\[\text{Combine } 2x - 5x \text{ to get } -3x \text{ on the left, and subtract 24 on both sides.}\]

\[-3x = -24\]

Divide both sides by \(-3\).

\[x = \frac{24}{3} = 8\]

We now know that the point of intersection occurs when \(x = 8\). Next, we need to find the \(y\)-value of that point. To find the \(y\)-value, we let \(x=8\) in either equation since, at \(x=8\), the two equations will have the same \(y\)-value.

\[-x + y = -5\]

Let \(x=8\).

\[-(8) + y = -5\]

Add 8 on both sides.

\[y = 3\]

So the solution to the system if \((8,3)\).
Check:

\[-x + y = -5 \quad 2x - 5y = 1\]
\[-(8) + (3) = -5 \quad 2(8) - 5(3) = 1\]
\[-5 = -5 \quad \text{true!}\]

Try it Now

1. Use substitution to solve the system.

\[x = y + 3\]
\[4 = 3x - 2y\]

Example 3 Application of Systems of Equations

Julia has just retired, and has $600,000 in her retirement account that she needs to reallocate to produce income. She is looking at two investments: a very safe guaranteed annuity that will provide 3% interest, and a somewhat riskier bond fund that averages 7% interest. She would like to invest as little as possible in the riskier bond fund, but needs to produce $40,000 a year in interest to live on. How much should she invest in each account? Use the substitution method to solve.

Solution:

Notice there are two unknowns in this problem: the amount she should invest in the annuity and the amount she should invest in the bond fund. We can start by defining variables for the unknowns:

\[a: \text{The amount (in dollars) she invests in the annuity}\]
\[b: \text{The amount (in dollars) she invests in the bond fund.}\]

Our first equation comes from noting that together she is going to invest $600,000:

\[a + b = 600,000\]

Our second equation will come from the interest. She earns 3% on the annuity, so the interest earned in a year would be 0.03a. Likewise, the interest earned on the bond fund in a year would be 0.07b. Together, these need to total $40,000, giving the following equation:

\[0.03a + 0.07b = 40,000\]

Together, these two equations form our system:

\[a + b = 600,000\]
\[0.03a + 0.07b = 40,000\]

The first equation is an ideal candidate for the first step of substitution. We can easily solve the equation for \(a\) or \(b\): \(a = 600,000 - b\)

(continued on next page)
Then we can substitute this expression for $a$ in the second equation and solve.

\[ 0.03a + 0.07b = 40,000 \]

Substituting $(600,000 - b)$ for $a$.

\[ 0.03(600,000 - b) + 0.07b = 40,000 \]

Distribute the 18,000.

\[ 18,000 - 0.03b + 0.07b = 40,000 \]

Combine the $-0.03b + 0.07b$ on the left, subtract 18,000 on both sides.

\[ 0.04b = 22,000 \]

Divide both sides by 0.04.

\[ b = 550,000 \]

Now substitute this back into the equation $a = 600,000 - b$ to find $a$.

\[ a = 600,000 - b \]
\[ a = 600,000 - 550,000 \]
\[ a = 50,000 \]

The solution to the system of equation is $a = 50,000$ and $b = 550,000$.

In order to reach her goal, Julia will have to invest $550,000 in the bond fund, and $50,000 in the annuity.

---

**Elimination Method to Solve a System**

In the elimination method, we add two terms that have the same variable, but opposite coefficients, so that the sum is zero. This creates a new equation that has eliminated one of the variables.

1. Write one equation above the other, lining up corresponding like-terms vertically. If necessary, add/subtract term(s) on both sides of one equation so to that they line up.

2. If necessary, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation. You can skip this step if the two equations start with terms that are opposite one another (like $3x$ in one equation, and $-3x$ in the other equation).

3. Add the equations together to create a new third equation that has eliminated one of the variables (since the terms were opposite).

3. Solve the resulting new equation for the remaining variable.

4. Substitute that value into one of the original equations and solve for the second variable.

5. Check the solution by substituting the values into the other equation.
Example 4  Use Elimination to Solve a System

Solve the system using the elimination method.
\[ x + 2y = -1 \]
\[ -x + y = 3 \]

Solution:
Both equations are already written so that the like terms line up vertically. Notice that the coefficient of \( x \) in the second equation, \(-1\), is the opposite of the coefficient of \( x \) in the first equation, \(1\). That means that we don’t need to multiply by a constant since these terms will already cancel when we add them together.

Now we will line the equations up vertically, and we will add them together to create a new equation:

\[
\begin{align*}
0 + 3y &= 2 \\
x + 2y &= -1 \\
-x + y &= 3 \\
\end{align*}
\]

\[ x + (−x) = 0 \quad 2y + y = 3y \quad -1 + 3 = 2 \]

\[ 3y = 2 \]
\[ y = \frac{2}{3} \]

Divide both sides by 3.

Now we substitute \( y=2/3 \) back into one of the equations in order to solve for \( x \):
\[ -x + y = 3 \]
\[ -x + \frac{2}{3} = 3 \]

Substitute \( 2/3 \) in for \( y \).

Add \( x \) on both sides, and subtract 3 on both sides.
\[ 3 + 2/3 = x \]
\[ x = \frac{11}{3} \]

So the solution to the system of equations is the point \( \left( \frac{11}{3}, \frac{2}{3} \right) \).

Often, using the elimination method will require first multiplying one or both equations by a constant so the terms will eliminate. We will see this in the next example.
Example 5  Use Elimination to Solve a System  (*Video Example Here)

Solve the system using the elimination method.

$$3x + 5y = -11$$
$$x - 2y = 11$$

Solution:
Adding these equations as presented will not eliminate a variable. If we add 3x to x we would get 4x, and so the x’s would not cancel. If we add 5y to -2y we would get 3y. So, before adding the equations together, we will first multiply the left side and right side of one of the equations by a value that will result in coefficients that will help us eliminate.

Notice that multiplying the second equation by $-3$ on both sides will result in the following:

$$-3 \cdot (x - 2y) = -3 \cdot (11)$$

And so the second equation could be written as

$$-3x + 6y = -33$$

Notice that this is still an equivalent equation to the original equation $x - 2y = 11$.

Now we can re-write the system of equations as follows:

$$3x + 5y = -11$$
$$-3x + 6y = -33$$

Notice that, because we have re-written the second equation in this new equivalent form, we could now add the two equations together, and it would result in the elimination of the x-variable:

$$3x + 5y = -11$$
$$+(-3x + 6y = -33)$$
$$0 + 11y = -44$$

And so we can solve the equation for y:

$$11y = -44$$

$$y = \frac{-44}{11} = -4$$

And lastly, plug y=-4 into either original equation in order to solve for x:

$$x - 2y = 11$$

$$x - 2(-4) = 11$$

$$x + 8 = 11$$

$$x = 3$$

The solution to the system of equations is the point $(3, -4)$. 
Try it Now

2. Solve the system.
   \[8x - 7y = 5\]
   \[2x + y = -4\]

3. Solve the system.
   \[x = y + 3\]
   \[4 = 3x - 2y\]

Try it Now Answers

1. \((-2, -5)\)

2. \(\left(\frac{-23}{22}, \frac{-21}{11}\right)\)

3. \((-2, -5)\)
Section 2.4 Exercises

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

1) \(5x - y = 4\)  
   \(x + 6y = 2\)  
   point: \((4, 0)\)

2) \(-3x - 5y = 13\)  
   \(-x + 4y = 10\)  
   point: \((-6, 1)\)

3) \(3x + 7y = 1\)  
   \(2x + 4y = 0\)  
   point: \((2, 3)\)

4) \(-2x + 5y = 7\)  
   \(2x + 9y = 7\)  
   point: \((-1, 1)\)

For the following exercises, solve each system by substitution.

5) \(x + 3y = 5\)  
   \(2x + 3y = 4\)

6) \(3x - 2y = 18\)  
   \(5x + 10y = -10\)

7) \(4x + 2y = -10\)  
   \(3x + 9y = 0\)

8) \(2x + 4y = -3.8\)  
   \(9x - 5y = 1.3\)

For the following exercises, solve each system by elimination.

9) \(-2x + 5y = -42\)  
   \(7x + 2y = 30\)

10) \(6x - 5y = -34\)  
    \(2x + 6y = 4\)

11) \(5x - y = -2.6\)  
    \(-4x - 6y = 1.4\)

12) \(7x - 2y = 3\)  
    \(4x + 5y = 3.25\)

For the following exercises, solve for the desired quantity.

13) A stuffed animal business has a total cost of production \(C(x) = 12x + 30\) and a revenue function \(R(x) = 20x\). Find the break-even point.

14) A fast-food restaurant has a cost of production \(C(x) = 1.1x + 120\) and a revenue function \(R(x) = 5x\). When does the company start to turn a profit?

15) A cell phone factory has a cost of production \(C(x) = 150x + 10,000\) and a revenue function \(R(x) = 200x\). What is the break-even point?

16) A venue charges a total \(C(X) = 64x + 20,000\), where \(x\) is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

17) A guitar factory has a cost of production \(C(x) = 75x + 50,000\). The price of each guitar is $325. Write the revenue function. Determine the number of guitars the company needs to sell to break even? What is the cost and the revenue when they break even?

18) For the following exercises, use a system of linear equations with two variables and two
equations to solve.

19) A moving company charges a flat rate of $150, and an additional $5 for each box. If a taxi service would charge $20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

20) If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?
Section 2.5 Linear Regression

In the real world, rarely do things follow trends perfectly. When we expect the trend to behave linearly, or when inspection suggests the trend is behaving linearly, it is often desirable to find an equation to approximate the data. Finding an equation to approximate the data helps us understand the behavior of the data and allows us to use the linear model to make predictions about the data, inside and outside of the data range.

Example 1

The table below shows the number of cricket chirps in 15 seconds, and the air temperature, in degrees Fahrenheit. Plot this data, and determine whether the data appears to be linearly related.

<table>
<thead>
<tr>
<th>chirps</th>
<th>44</th>
<th>35</th>
<th>20.4</th>
<th>33</th>
<th>31</th>
<th>35</th>
<th>18.5</th>
<th>37</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp</td>
<td>80.5</td>
<td>70.5</td>
<td>57</td>
<td>66</td>
<td>68</td>
<td>72</td>
<td>52</td>
<td>73.5</td>
<td>53</td>
</tr>
</tbody>
</table>

Solution: Let’s plot this data in Excel. Produce an input/ output table with the given data. Next insert a graph of a scatter plot with no lines.

We can next add labels to the graph by double clicking on the graph, then clicking on “Add Chart Elements → Axis Titles”, then type in the desired labels in the horizontal and vertical axes.

Plotting this data, it appears there may be a trend, and that the trend appears roughly linear, though certainly not perfectly so.

The simplest way to find an equation to approximate this data is to try to “eyeball” a line that seems to fit the data pretty well, then find an equation for that line based on the slope and intercept.

You can see from the trend in the data that the number of chirps increases as the temperature increases. As you consider a function for this data you should know that you are looking at an increasing function or a function with a positive slope.

**Flashback**

1. a. What descriptive variables would you choose to represent Temperature & Chirps?
   b. Which variable is the independent variable and which is the dependent variable?
   c. Based on this data and the graph, what is a reasonable domain & range?
   d. Based on the data alone, is this function one-to-one, explain?

**Example 2**

Using the table of values from the previous example, find a linear function that fits the data by “eyeballing” a line that seems to fit.

**Solution:**

On a graph, we could try sketching in a line. Note the scale on the axes have been adjusted to start at zero to include the vertical axis and vertical intercept in the graph.

Using the starting and ending points of our “hand drawn” line, points (0, 30) and (50, 90), this graph has a slope of

\[ m = \frac{90 - 30}{50 - 0} = 1.2 \]

and a vertical intercept at 30, giving an equation of \( T(c) = 30 + 1.2c \)

where \( c \) is the number of chirps in 15 seconds, and \( T(c) \) is the temperature in degrees Fahrenheit.

This linear equation can then be used to approximate the solution to various questions we might ask about the trend. While the data does not perfectly fall on the linear equation, the equation is our best guess as to how the relationship will behave outside of the values we have data for. There is a difference, though, between making predictions inside the domain and range of values we have data for, and outside that domain and range.
Interpolation and Extrapolation

**Interpolation:** When we predict a value inside the domain and range of the data

**Extrapolation:** When we predict a value outside the domain and range of the data

For the Temperature as a function of chirps in our hand drawn model above,

- Interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44.
- Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

**Example 3**

a) Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

b) Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

**Solution:**

With our cricket data, our number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model: 

\[ T(30) = 30 + 1.2(30) = 66 \] 

\[ T = 40 + 1.2c \]

\[ 10 = 1.2c \]

\[ c \approx 8.33 \]

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

When our model no longer applies after some point, it is sometimes called **model breakdown**. We need to be especially careful of the possibility of model breakdown when extrapolating.

**Try it Now**

1. What temperature would you predict if you counted 20 chirps in 15 seconds?
Finding Best-Fit Models using Excel

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values\(^2\). This technique is called **least-square regression**, and can be computed by many graphing calculators, spreadsheet software like Excel or Google Docs, statistical software, and many web-based calculators\(^3\).

**Example 4.  (* Video Example Here*)**

Use Excel to find the equation of line of best fit using the cricket chirp data from above.

Solution: In Excel, right-click on one of that data points on the graph (you will notice the points change to an “x” shape, and all of the points are now selected. A drop-down box appears as shown below.

Click on Add Trendline….. and the “Formal Trendline” box appears.

The Linear option is the default setting and is selected by default. Notice the least-squares regression line is already drawn on the scatter plot of the data.

In order to show the equation of that line, click the box “Display Equation on chart”.

This will then display the linear equation on the graph of the data.

NOTICE, the variables will always be “x” and “y” in this linear equation so may need to be edited for the variables in the given problem.

---

\(^2\) Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.

\(^3\) For example, [http://www.shodor.org/unchem/math/l1s/leastsq.html](http://www.shodor.org/unchem/math/l1s/leastsq.html)
You can also edit the number of decimal places displayed in the equation. To do this, right click on the equation that is displayed on the graph.

Change the category to number.
Change to the desired number of decimal places (here we used 5 decimal places).
We usually “un-click” the comma separator option.

So using Excel, and using 3 decimal places, we obtain the equation: \( T(c) = 1.143c + 30.281 \)

Notice that this line is quite similar to the equation we “eyeballed”, but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

\[ T(30) = 1.143(30) + 30.281 = 64.571 \approx 64.6 \text{ degrees}. \]
Most calculators and computer software will also provide you with the correlation coefficient, a measure of how closely the line fits the data. Statistics courses go into the details of the correlation coefficient. For, now, the information on this page is the extent to which we will consider the correlation coefficient:

<table>
<thead>
<tr>
<th>Correlation Coefficient</th>
<th><em>Video Example Here</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>The correlation coefficient is a value, r, between $-1$ and $1$.</td>
<td></td>
</tr>
<tr>
<td>$r &gt; 0$ suggests a positive (increasing) relationship</td>
<td></td>
</tr>
<tr>
<td>$r &lt; 0$ suggests a negative (decreasing) relationship</td>
<td></td>
</tr>
<tr>
<td>The closer the value is to 0, the more scattered the data</td>
<td></td>
</tr>
<tr>
<td>The closer the value is to 1 or $-1$, the less scattered the data is</td>
<td></td>
</tr>
</tbody>
</table>

The correlation coefficient provides an easy way to get some idea of how close the data falls in relation to the best-fit line on average.

We should only compute the correlation coefficient for data that follows a linear pattern; if the data exhibits a non-linear pattern, the correlation coefficient is meaningless. To get a sense for the relationship between the value of $r$ and the graph of the data, here are some large data sets with their correlation coefficients:

**Examples of Correlation Coefficient Values**

<table>
<thead>
<tr>
<th>1.0</th>
<th>0.8</th>
<th>0.4</th>
<th>0.0</th>
<th>$-0.4$</th>
<th>$-0.8$</th>
<th>$-1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="http://en.wikipedia.org/wiki/File:Correlation_examples.png" alt="Image" /></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

Example 5

Gasoline consumption in the US has been increasing steadily. Consumption data from 1994 to 2004 is shown below.\(^5\) Determine if the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

<table>
<thead>
<tr>
<th>Year</th>
<th>'94</th>
<th>'95</th>
<th>'96</th>
<th>'97</th>
<th>'98</th>
<th>'99</th>
<th>'00</th>
<th>'01</th>
<th>'02</th>
<th>'03</th>
<th>'04</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption (billions of gallons)</td>
<td>113</td>
<td>116</td>
<td>118</td>
<td>119</td>
<td>123</td>
<td>125</td>
<td>126</td>
<td>128</td>
<td>131</td>
<td>133</td>
<td>136</td>
</tr>
</tbody>
</table>

Solution:
To make things simpler, a new input variable is introduced, \(t\), representing years since 1994.

Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend. The least-squares regression equation is:

\[
C(t) = 113.318 + 2.209t
\]

Using this to predict consumption in 2008 \((t = 14)\),

\[
C(14) = 113.318 + 2.209(14) = 144.244
\]

billions of gallons

The model predicts 144.244 billion gallons of gasoline will be consumed in 2008.

Try it Now
2. Use the model created by technology in example 5 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

Important Topics of this Section

Fitting linear models to data by hand
Fitting linear models to data using technology
Interpolation
Extrapolation
Correlation coefficient

Flashback Answers
1. a. T = Temperature,  C = Chirps (answers may vary)
   b. Independent (Chirps) , Dependent (Temperature)
   c. Reasonable Domain (18.5, 44) , Reasonable Range (52, 80.5) (answers may vary)
   d. NO, it is not one-to-one, there are two different output values for 35 chirps.

Try it Now Answers
1. 54 degrees Fahrenheit
2. 150.871 billion gallons; extrapolation
Section 2.5 Exercises

1. The following is data for the first and second quiz scores for 8 students in a class. Plot the points, then sketch a line that fits the data.

<table>
<thead>
<tr>
<th>First Quiz</th>
<th>11</th>
<th>20</th>
<th>24</th>
<th>25</th>
<th>33</th>
<th>42</th>
<th>46</th>
<th>49</th>
</tr>
</thead>
<tbody>
<tr>
<td>Second Quiz</td>
<td>10</td>
<td>16</td>
<td>23</td>
<td>28</td>
<td>30</td>
<td>39</td>
<td>40</td>
<td>49</td>
</tr>
</tbody>
</table>

2. Eight students were asked to estimate their score on a 10 point quiz. Their estimated and actual scores are given. Plot the points, then sketch a line that fits the data.

<table>
<thead>
<tr>
<th>Predicted</th>
<th>5</th>
<th>7</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>9</th>
<th>10</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>10</td>
<td>6</td>
</tr>
</tbody>
</table>

Based on each set of data given, calculate the regression line using your calculator or other technology tool, and determine the correlation coefficient.

3. x y
   5  4
   7 12
  10 17
  12 22
  15 24

4. x y
   8 23
  15 41
  26 53
  31 72
  56 103

5. x y
   3 21.9
   4 22.22
   5 22.74
   6 22.26
   7 20.78
   8 17.6
   9 16.52
  10 18.54
  11 15.76
  12 13.68
  13 14.1
  14 14.02
  15 11.94
  16 12.76
  17 11.28
  18 9.1

6. x y
   4 44.8
   5 43.1
   6 38.8
   7 39
   8 38
   9 32.7
  10 30.1
  11 29.3
  12 27
  13 25.8
  14 24.7
  15 22
  16 20.1
  17 19.8
  18 16.8
7. A regression was run to determine if there is a relationship between hours of TV watched per day \((x)\) and number of situps a person can do \((y)\). The results of the regression are given below. Use this to predict the number of situps a person who watches 11 hours of TV can do.

\[
y = ax + b \\
a = -1.341 \\
b = 32.234 \\
r^2 = 0.803 \\
r = -0.896
\]

8. A regression was run to determine if there is a relationship between the diameter of a tree \((x, \text{ in inches})\) and the tree’s age \((y, \text{ in years})\). The results of the regression are given below. Use this to predict the age of a tree with diameter 10 inches.

\[
y = ax + b \\
a = 6.301 \\
b = -1.044 \\
r^2 = 0.940 \\
r = 0.970
\]

Match each scatterplot shown below with one of the four specified correlations.

9. \(r = 0.95\)  
10. \(r = -0.89\)  
11. \(r = 0.26\)  
12. \(r = -0.39\)

![Scatterplots A, B, C, D]

13. The US census tracks the percentage of persons 25 years or older who are college graduates. That data for several years is given below. Determine if the trend appears linear. If so and the trend continues, in what year will the percentage exceed 35%?

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent Graduates</td>
<td>21.3</td>
<td>21.4</td>
<td>22.2</td>
<td>23.6</td>
<td>24.4</td>
<td>25.6</td>
<td>26.7</td>
<td>27.7</td>
<td>28</td>
<td>29.4</td>
</tr>
</tbody>
</table>

14. The US import of wine (in hectoliters) for several years is given below. Determine if the trend appears linear. If so and the trend continues, in what year will imports exceed 12,000 hectoliters?

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Imports</td>
<td>2665</td>
<td>2688</td>
<td>3565</td>
<td>4129</td>
<td>4584</td>
<td>5655</td>
<td>6549</td>
<td>7950</td>
<td>8487</td>
<td>9462</td>
</tr>
</tbody>
</table>
Section 2.6 Absolute Value Functions

So far in this chapter we have been studying the behavior of linear functions. The Absolute Value Function is a piecewise-defined function made up of two linear functions. The name, Absolute Value Function, should be familiar to you from Section 1.2. In its basic form \( f(x) = |x| \) it is one of our toolkit functions.

### Absolute Value Function

The absolute value function can be defined as

\[
f(x) = |x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
\end{cases}
\]

The absolute value function is commonly used to determine the distance between two numbers on the number line. Given two values \( a \) and \( b \), then \( |a - b| \) will give the distance, a positive quantity since distance is always positive, between these values, regardless of which value is larger.

**Example 1**

Describe all values, \( x \), within a distance of 4 from the number 5.

**Solution:**

We want the distance between \( x \) and 5 to be less than or equal to 4. The distance can be represented using the absolute value, giving the expression

\[ |x - 5| \leq 4 \]

**Example 2**

A 2010 poll reported 78% of Americans believe that people who are gay should be able to serve in the US military, with a reported margin of error of 3%\(^1\). The margin of error tells us how far off the actual value could be from the survey value\(^2\). Express the set of possible values using absolute values.

**Solution:**

Since we want the size of the difference between the actual percentage, \( p \), and the reported percentage to be less than 3%,

\[ |p - 78| \leq 3 \]

---

\(^1\) [http://www.pollingreport.com/civil.htm](http://www.pollingreport.com/civil.htm), retrieved August 4, 2010  
\(^2\) Technically, margin of error usually means that the surveyors are 95% confident that actual value falls within this range.
Try it Now

1. Students who score within 20 points of 80 will pass the test. Write this as a distance from 80 using the absolute value notation.

**Important Features**
The most significant feature of the absolute value graph is the **corner point** where the graph changes direction. This point is very helpful for determining the range of the function.

**Example 3**
Determine the domain and range of the absolute function \( f(x) = 2|x - 3| - 2 \).

**Solution:**
Let’s graph this function in Excel to help us determine the domain and range.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>x</td>
<td>f(x)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-10</td>
<td>2*ABS(A2-3)-2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-9</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-8</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-7</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-6</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-5</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-4</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>-1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>13</td>
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<td>14</td>
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<td>0</td>
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<tr>
<td>15</td>
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<td>-2</td>
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<td>16</td>
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<td>0</td>
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<td>23</td>
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</tbody>
</table>

In Excel, the ABS command will give us the absolute value.

We know that any \( x \)-value entered into this function will give us a real number output, so the domain of this function would be all real numbers, or using interval notation, \((-\infty, \infty)\).

Graphing this function in Excel, in the domain window \([-10, 10]\), yields the following graph.

We can conclude from the graph that the range (output) results in values greater than or equal to \(-2\), which can be written as \( f(x) \geq -2 \), or in interval notation, \([-2, \infty)\).

**NOTE:** In Excel, the program tries to “round” the curve to graph the function, but we know that this function is comprised of two linear functions that meet at the point \((3, -2)\). This means that there is actually a sharp point at the point \((3, -2)\), and not a rounded curve.
Example 4

Find the two linear equations that make up the absolute function \( f(x) = 2|x - 3| - 2 \).

Solution: We can graph the function in Excel in order to see that the “turn” in the graph occurs at the point \((3, -2)\).

![Graph of absolute value function](image)

To find the equation of the line that makes up the left side of the function, first find two points on the left side of the absolute value graph using either the table or the graph. We find that both (0,4) and (1,2) are on the graph.

\[
\begin{align*}
\text{Choose two points} & \quad (0,4) \text{ and } (1,2) \\
\text{Find the slope} & \quad m = \frac{2-4}{1-0} = \frac{-2}{1} = -2 \\
\text{Write the equation of the line using the slope and y-intercept.} & \quad y = -2x + 4
\end{align*}
\]

Next find two points on the right side of the absolute value graph using either the table or the graph.

\[
\begin{align*}
\text{Choose two points} & \quad (5, 2) \text{ and } (6, 4) \\
\text{Find the slope} & \quad m = \frac{4-2}{6-5} = \frac{2}{1} = 2 \\
\text{Write the equation of the line in point-slope form, then simplify.} & \quad y = 2(x - 5) + 2 = 2x - 8
\end{align*}
\]

So, the equation of the absolute value function could be written as

\[
y = \begin{cases} 
-2x + 4 & \text{for } x \leq 3 \\
2x - 8 & \text{for } x \geq 3
\end{cases}
\]

[Video Link - Graphing Absolute Value in Excel]
The graph of an absolute value function will have a vertical intercept when the input is zero. The graph may or may not have horizontal intercepts. It is possible for the absolute value function to have zero, one, or two horizontal intercepts.

Zero horizontal intercepts  
One horizontal intercepts  
Two horizontal intercepts

To find the horizontal intercepts, solve for the x-value that makes the output equal to 0.

**Solving Absolute Value Equations**

To solve an equation like \( 8 = |2x - 6| \), we can notice that the absolute value will be equal to eight if the quantity inside the absolute value were 8 or -8. This leads to two different equations we can solve independently:

\[
2x - 6 = 8 \quad \text{or} \quad 2x - 6 = -8
\]

\[
2x = 14 \quad 2x = -2
\]

\[
x = 7 \quad x = -1
\]

**Solutions to Absolute Value Equations**

An equation of the form \( |expression| = N \), where \( N \) represents a number with \( N \geq 0 \), then \( expression = N \) and also \( -(expression) = N \)

Solve both equations in order to identify all existing solutions to the absolute value equation.
Example 5 (*Video Example Here)

Find where the graph of \( f(x) = |4x + 1| - 7 \) reaches a height of 71.

Solution: In order to solve by hand, we need to solve the equation
\[
|4x + 1| - 7 = 71
\]
Isolate the absolute value on one side of the equation by adding 7 on both sides.

\[
|4x + 1| = 78
\]

Now we need to solve two separate equations: \( 4x + 1 = 78 \) and also \( 4x + 1 = -78 \).

\[
\begin{align*}
4x + 1 &= 78 \\
4x &= 77 \\
x &= \frac{77}{4} = 19.25
\end{align*}
\]

\[
\begin{align*}
4x + 1 &= -78 \\
4x &= -79 \\
x &= -\frac{79}{4} = -19.75
\end{align*}
\]

The graph reaches a height of 71, at \( x = 19.25 \) and also \( x = -19.75 \).

Using Excel: Now we could also solve this equation in Excel using Solver.

We first enter the function \( f(x) = |4x + 1| - 7 \) in Excel on a domain that allows us to see the function reach a height of 71. In the graph to the right, we see that \( f(x) \) reaches a height of 71 somewhere between \( x = -20 \) and \( x = -15 \). We see that \( f(x) \) reaches a height of 71 for a second time somewhere between \( x = 15 \) and \( x = 20 \). We now use solver to have Excel identify the x-values where \( f(x) \) reaches a height of 71.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>x</td>
</tr>
<tr>
<td>2</td>
<td>-30</td>
</tr>
<tr>
<td>3</td>
<td>-25</td>
</tr>
<tr>
<td>4</td>
<td>-19.75</td>
</tr>
<tr>
<td>5</td>
<td>-15</td>
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<tr>
<td>6</td>
<td>-10</td>
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<tr>
<td>7</td>
<td>-5</td>
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<td>8</td>
<td>0</td>
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<tr>
<td>16</td>
<td>40</td>
</tr>
<tr>
<td>17</td>
<td></td>
</tr>
</tbody>
</table>
Example 6

Solve $4|x - 2| + 2 = 1$.

Solution:

$4|x - 2| + 2 = 1$

*Isolate the absolute value on one side of the equation by first subtracting 2.*

$4|x - 2| = -1$

*Isolate the absolute value on one side of the equation by dividing by 4.*

$|x - 2| = -\frac{1}{4}$

At this point, we notice that this equation has no solutions because the absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value.

Using Excel, we could graph the function $f(x) = 4|x - 2| + 2$ and try to solve for when the function $f(x)$ reaches a height of 1.

![Excel graph](image-url)

We see from the Excel graph that this absolute value function decreases down to the point $(2, 2)$ and then turns to begin increasing. So, this function has a range of $y \geq 2$. So, since the graph never reaches a height of 1, we can conclude that $4|x - 2| + 2 = 1$ has no solutions.

Try it Now

2. Find the horizontal & vertical intercepts for the function $f(x) = -|x + 2| + 3$
**Important Topics of this Section**

- The properties of the absolute value function
- Solving absolute value equations
- Finding intercepts

**Try it Now Answers**

1. Using the variable \( p \), for passing, \( |p - 80| \leq 20 \).

2. The vertical intercept is at (0,1). \( f(x) = 0 \) when \( x = -5 \) and \( x = 1 \) so the horizontal intercepts are at \((-5,0) & (1,0)\).
Section 2.6 Exercises

Find the piece-wise equation for the given absolute value functions. This means that you should give the two linear equations as well as the domain on which each equation applies. Graph the equations in Excel to check your answer.

1. Graph the equations in Excel to check your answer.

Solve each the equation.

5. \(|5x - 2| = 11\)
6. \(|4x + 2| = 15\)
7. \(2|4 - x| = 7\)
8. \(3|5 - x| = 5\)
9. \(3|x + 1| - 4 = -2\)
10. \(5|x - 4| - 7 = 2\)

Find the horizontal and vertical intercepts of each function

11. \(f(x) = 2|x + 1| - 10\)
12. \(f(x) = 4|x - 3| + 4\)
13. \(f(x) = -3|x - 2| - 1\)
14. \(f(x) = -2|x + 1| + 6\)
Chapter 3: Polynomial and Rational Functions

Section 3.1 Power Functions with Negative and Fractional Exponents

Let’s begin with a brief review of fractional and negative exponent rules.

Integer and Fractional Exponents

In this chapter we are going to discuss functions and equations containing integer and fractional powers of variables. For example, we will discuss functions like \( f(x) = x^{-1} = \frac{1}{x} \) or \( f(x) = \sqrt{x} = x^{1/2} \) and many other functions where the exponent will be an integer or a fraction. This may involve converting expressions involving radicals to expressions involving rational exponents, or vice versa. Recall that if “\(a\)” is a real number and “\(n\)” is a positive integer, then \(a^n\) represents the base of \(a\) as a factor \(n\) times (i.e. \(2^4 = 2 \cdot 2 \cdot 2 \cdot 2\)).

In the expression, \(f(x) = a^n\), \(a\) is called the base and \(n\) is called the exponent.

We define an expression raised to a zero power and to a negative power as follows:

<table>
<thead>
<tr>
<th>Negative Exponents and Zero in the Exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (a^0 = 1)</td>
</tr>
<tr>
<td>2. (a^{-1} = \frac{1}{a})</td>
</tr>
<tr>
<td>3. (a^{-n} = \frac{1}{a^n})</td>
</tr>
<tr>
<td>4. ((\frac{a}{b})^{-n} = (\frac{b}{a})^n)</td>
</tr>
</tbody>
</table>

Example 1 [*Video Example Here]*

Simplify the following expressions by removing all zero and negative exponents.

a) \((2x)^0\) b) \(2x^0\) c) \((3x)^{-1}\) d) \(3x^{-1}\) e) \(\left(\frac{2}{3}\right)^{-2}\) f) \(5x^{-3}\)

**Solutions:**

a) \((2x)^0 = 1\) \hspace{1cm} Using rule #1 above.

b) \(2x^0 = 2 (1) = 2\) \hspace{1cm} Notice here the “2” is not being raised to a power of zero, only the \(x\).

c) \((3x)^{-1} = \frac{1}{3x}\) \hspace{1cm} Using rule #2 above. Notice here **both** the number and variable are raised to the power of -1.

d) \(3x^{-1} = \frac{3}{x}\) \hspace{1cm} Using rule #2 above. Notice here only the “\(x\)” is being raised to the power of -1, therefore the number 3 stays in the numerator.

e) \(\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}\) \hspace{1cm} Using rule #4 above, then squaring each number.

f) \(5x^{-3} = \frac{5}{x^3}\) \hspace{1cm} Using rule #3 above. Notice here only the “\(x\)” is being raised to the power of -3, therefore the number 5 stays in the numerator.
Exponential expressions with rational exponents are defined in terms of radicals. NOTE that for expressions for $a \geq 0$ and $b \geq 0$,

$$\sqrt{a} = b \text{ if and only if } a = b^2$$

and we define $\sqrt{a} = a^{1/2}$ for $a \geq 0$.

We define the connection between rational exponents and radicals as follows:

<table>
<thead>
<tr>
<th>Rational Exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\sqrt{x} = x^{1/2}$</td>
</tr>
<tr>
<td>2. $\sqrt[3]{x} = x^{1/3}$</td>
</tr>
</tbody>
</table>

If $a$ and $b$ are positive integers,

$$3. \sqrt[b]{x^a} = x^{a/b} \quad \text{OR} \quad x^{\left(\frac{\text{power}}{\text{root}}\right)} = \sqrt[\text{root}]{x^\text{power}} \quad \text{(for example: } \sqrt[2]{7^3} = \sqrt[2]{7^3})$$

**Example 2** *(Video Example Here)*

Rewrite the following expressions by writing the expression from exponential form to radical form, or from radical form to exponential form.

a) $\sqrt[3]{x^2}$  
b) $x^{3/5}$  
c) $(4x)^{2/3}$  
d) $\sqrt{(3x)^5}$  
e) $\sqrt[4]{(2x)^3}$  
f) $5x^{-1/3}$

**Solutions:**

a) $\sqrt[3]{x^2} = x^{2/3}$  
Using rule #3 above.

b) $x^{3/5} = \frac{3\sqrt[5]{x^3}}{5}$  
Using rule #3 above.

c) $(4x)^{2/3} = 3\sqrt[3]{(4x)^2} = 3\sqrt[3]{16x^2}$  
Using rule #3 above. Then simplify the “inside” of the radical.

d) $\sqrt{(3x)^5} = (3x)^{5/2}$  
Using rule #1 & 3 above.

e) $\sqrt[4]{(2x)^3} = (2x)^{3/4}$  
Using rule #3 above.

f) $5x^{-1/3} = \frac{5}{x^{1/3}} = \frac{5}{\sqrt[3]{x}}$  
Using our rules for negative exponents and rule #3 above.
Recall also the following properties of exponents,

**Exponent Rules: Product, Quotient, Power (**Video Link Here)**

For all integers \( m \) and \( n \),

1. \( x^m \cdot x^n = x^{m+n} \) (Product Property)

2. \( \frac{x^m}{x^n} = x^{m-n} \) (Quotient Property)

3. \( (x^m)^n = x^{m \cdot n} \) (Power Property)

4. \( (x \cdot y)^n = x^n \cdot y^n \) and \( \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \)

Now that we have reviewed some basic definition and properties of exponents, let’s look at functions that use these exponents.

If we wanted a function for the volume of a cube with each side having some length \( L \), you may recall volume of a rectangular box can be found by multiplying length by width by height, which are all equal for a cube, giving the formula:

\[
V(L) = L^3
\]

This is an example of a **power function**, functions that are some power of the variable.

**Power Function**

A **power function** is a function that can be represented in the form

\[
f(x) = x^p
\]

Where the base is a variable and the exponent, \( p \), is a rational number.
Example 3 (Video Link Here)

Which of our toolkit functions are power functions?

Solution:
The **constant function** \( f(x) = 1 \), and the **identity function** \( g(x) = x \) are technically (by definition) power functions, since they can be written as \( f(x) = x^0 \) and \( g(x) = x^1 \) respectively, but we think of these primarily as linear functions because their graphs produce straight lines.

The **basic quadratic function** \( f(x) = x^2 \) and the **cubic function** \( g(x) = x^3 \) are both considered power functions also (by definition) with whole number powers, but we think of these primary as special types of polynomial functions, which we will explore in the next section.

The **reciprocal function** \( f(x) = \frac{1}{x} \) and **reciprocal squared function** \( g(x) = \frac{1}{x^2} \) are both power functions with negative whole number powers since they can be written as \( f(x) = \frac{1}{x} = x^{-1} \) and \( g(x) = \frac{1}{x^2} = x^{-2} \).

The **square root function** \( f(x) = \sqrt{x} \) and **cube root function** \( g(x) = \sqrt[3]{x} \) are both power functions with fractional powers since they can be written as \( f(x) = \sqrt{x} = x^{1/2} \) and \( f(x) = \sqrt[3]{x} = x^{1/3} \).

These last two types of functions are what we will consider in this section.
Characteristics of Special Power Functions

**Square Root** \( f(x) = \sqrt{x} = x^{1/2} \)
Domain: \([0, \infty)\)

**Cube Root** \( f(x) = \sqrt[3]{x} = x^{1/3} \)
Domain: \((-\infty, \infty)\)

**Reciprocal function** \( f(x) = \frac{1}{x} = x^{-1} \)
Domain: \((-\infty, 0) \cup (0, \infty)\)

**Reciprocal Squared function** \( f(x) = \frac{1}{x^2} = x^{-2} \)
Domain: \((-\infty, 0) \cup (0, \infty)\)
Example 4

Enter the special power functions below in Excel, and create graphs of each function.

(a) \( f(x) = \sqrt{x} = x^{1/2} \)  
(b) \( f(x) = \sqrt[3]{x} = x^{1/3} \)  
(c) \( (x) = \frac{1}{x} = x^{-1} \)

Solution: When entering these functions into Excel it is important to note the domain values that can and cannot be used in these functions.

(a) As we can see when we enter \( f(x) = \sqrt{x} = x^{1/2} \) into an Excel spreadsheet, the output of the function shows up as “#NUM!” whenever the input value is negative, indicating that the output cannot be displayed in the column. Expanding the column will not correct this as the output is undefined in the negative domain, (that is we cannot take the square root of a negative number in the real number system!).

Notice when we enter this function into Excel that we enter parenthesis around the fractional exponent! This is vitally important! Always use ( ) around a fraction in Excel.

Excel has another way to indicate the square root of a number, it is the SQRT command. This may be used as an alternative for the square root of a number.

It is also important to know the domain of the Square Root function because Excel will attempt to graph the function using the input values you put into the spreadsheet (see below).

NOT correct graph of \( f(x) = \sqrt{x} \)  
Use ONLY positive input values!

CORRECT graph of \( f(x) = \sqrt{x} \)
If we delete the values that are undefined in our Excel chart, a correct graph can be displayed.

(b) When we enter the $f(x) = \sqrt[3]{x} = x^{1/3}$ into an Excel spreadsheet, we see that the output IS defined for negative input values.

Notice that we define the function using a fractional exponent and once again make sure to use parenthesis around the fraction.

The graph in Excel is accurate here because the Cube Root function can be graphed in a negative domain. Your graph in Excel will look similar to our toolkit function: $f(x) = \sqrt[3]{x} = x^{1/3}$.
(c) When we enter, \( f(x) = \frac{1}{x} = x^{-1} \), we can enter the function as a fraction or as a negative exponent, but notice Excel gives us the following table of values (see table on the left below).

The output of “#DIV/0!” in cell B7 is indicating that the function is being divided by zero when the input of the function is zero, and we know that a denominator of zero is not defined in the real number system! However, if we try to use Excel to graph this function, and include zero in the domain, Excel will produce an **INCORRECT graph** (see below)!

However, if you eliminate zero from your domain, (see the table at right) Excel will produce the **CORRECT GRAPH**! Of course, if we want more accuracy we would need to give more input values as \( x \) approaches zero from both the positive and negative side. The same will be true for all reciprocal functions, like \( f(x) = \frac{1}{x^2} = x^{-2} \), where zero is NOT in the domain of the function!
Long Run Behavior

The behavior of the graph of a function as the input takes on large negative values (as \( x \to -\infty \)) and large positive values (as \( x \to \infty \)) as is referred to as the long run (tail) behavior of the function.

- To say that “\( x \) approaches infinity”, written as \( x \to \infty \), we are describing a behavior of the output of the function, as \( x \) is getting large in the positive direction. We are describing the behavior of the function on the right tail.
- To say that “\( x \) approaches negative infinity”, written as \( x \to -\infty \), we are describing a behavior of the output of the function, as \( x \) is getting smaller in the negative direction. We are describing the behavior of the function on the left tail.

Example 5

Identify the long run (tail) behavior of each of the special power functions below.

(a) \( f(x) = \sqrt{x} \)  
(b) \( f(x) = \sqrt[3]{x} \)  
(c) \( f(x) = x^{-1} \)  
(d) \( f(x) = x^{-2} \)

Solution:

(a) The right tail end behavior of the function \( f(x) = \sqrt{x} \) is as follows: \( x \to \infty, f(x) \to \infty \)
This means that as the input values of the function increase without bound (tending to infinity), the output values also increase without bound (tending to infinity).

The function is not defined for x-values below 0, and so we would not consider the left tail end behavior.

(b) Looking at the function, \( f(x) = \sqrt[3]{x} \), we can describe both the left and right tail end behavior.

The left tail behavior of \( f(x) = \sqrt[3]{x} \) is as follows: \( x \to -\infty, f(x) \to -\infty \)
This means that as the input values of the function decrease without bound (tending to negative infinity, large negative input values), the output values also decrease without bound (tending to negative infinity, large negative output values).

The right tail behavior of \( f(x) = \sqrt[3]{x} \) is as follows: \( x \to +\infty, f(x) \to +\infty \)
This tells us as the input values of the function increase without bound (tending to infinity), the output values also increase without bound (tending to infinity).
(c) Looking at the function, \( f(x) = x^{-1} = \frac{1}{x} \), we can describe both the left and right tail end behavior.

The left tail behavior of \( f(x) = x^{-1} \) is as follows: \( x \to -\infty, f(x) \to 0 \)
This means that as the input values of the function decrease without bound (tending to negative infinity), the output values approach \( y = 0 \).

The right tail behavior of the function \( f(x) = x^{-1} \) is as follows: \( x \to +\infty, f(x) \to 0 \).
This means that as the input values of the function decrease without bound (tending to negative infinity), the output values approach \( y = 0 \).

(d) Looking at the function, \( f(x) = x^{-2} \), we can describe both the left and right tail end behavior.

The left tail behavior of \( f(x) = x^{-2} \) is as follows: \( x \to -\infty, f(x) \to 0 \)
This means that as the input values of the function decrease without bound (tending to negative infinity), the output values approach \( y = 0 \).

The right tail behavior of \( f(x) = x^{-2} \) is as follows: \( x \to +\infty, f(x) \to 0 \)
This means that as the input values of the function increase without bound (tending to positive infinity), the output values approach \( y = 0 \).

You may use words or symbols to describe the long run behavior of these functions.

**Try it Now**
1. Describe in words and symbols the long run behavior of, \( f(x) = -x^{-2} \), and \( g(x) = -\sqrt{x} \)

**Important Topics of this Section**
- Integer Exponents
- Fractional Exponents
- Power Functions
- Long run behavior
1. As \( x \) approaches positive and negative infinity, \( f(x) \) approaches zero because the function is the reflection of the toolkit function \( x^{-2} \) over the x-axis and the end behavior still approaches zero.
\[
\lim_{x \to \pm \infty} f(x) = 0
\]

As \( x \) approaches positive infinity, \( g(x) \) approaches negative infinity because the function is the reflection of the toolkit function \( x^{1/2} \) over the x-axis and the end behavior will tend toward negative infinity.
\[
\lim_{x \to +\infty} g(x) = -\infty
\]
Section 3.1 Exercises

Convert the following expressions by writing the expression with only positive exponents.

1) \(7x^{-5}\)  
2) \(\frac{1}{10x^{-2}}\)  
3) \(\frac{4x^{-3}}{x^2}\)  
4) \((2x)^{-4}\)  
5) \(\frac{14}{x^{-6}}\)  
6) \(\left(\frac{x}{4}\right)^{-2}\)  
7) \((7x^2)^{-1}\)  
8) \(5(x^4)^{-2}\)

Convert the following expressions by writing the expression from exponential form to radical form, or from radical form to exponential form.

9) \(\sqrt[3]{x^5}\)  
10) \(x^{7/4}\)  
11) \((3x)^{4}\)  
12) \(\sqrt[3]{(9x)^3}\)  
13) \(\frac{5}{(2x)^{3}}\)  
14) \(12x^{\frac{1}{3}}\)

Use Excel to help find the symbolic long run behavior of each function as \(x \to \infty\) and \(x \to -\infty\). Make sure you graph over a correct domain!!! Describe in words the long run behavior of each.

15) \(f(x) = \frac{5}{x^2}\)  
16) \(g(x) = 5\sqrt{x}\)  
17) \(h(x) = 4x^{-1}\)  
18) \(j(x) = x^{2/3}\)  
19) \(k(x) = 4x^{3/2}\)  
20) \(m(x) = -7x^{-2}\)  
21) \(n(x) = \frac{6}{x^{\frac{1}{2}}}\)  
22) \(p(x) = -9x^{1/3}\)

23. When installing Christmas lights on the outside of your house, you read the warning “Do not string more than four sets of lights together.” This is because the electrical resistance, \(R\), of wire varies directly with the length of the wire, \(l\), and inversely with the square of the diameter of the wire, \(d\).

So the equation for the electrical wire resistance is, \(R = \frac{l}{d^2}\).

a) If you double the wire diameter, what happens to the resistance?

b) If you increase the length of the wire by 25\%, (say going from four to five strings of lights), what happens to the resistance?
Section 3.2 Quadratic Functions

In this section, we will explore the family of 2nd degree polynomials, the quadratic functions. While they share many characteristics of polynomials in general, the calculations involved in working with quadratics is typically a little simpler, which makes them a good place to start our exploration of polynomial functions. In addition, quadratics commonly arise from problems involving area, projectile motion, and business providing some interesting applications.

Example 1
Which of our toolkit functions are quadratic functions?

Solution:
The quadratic \( f(x) = x^2 \) is also considered a power function with a power of two, but we think of this primary as a special type of a polynomial function, which we will explore here. The graph of a quadratic function is called a parabola.

When products are sold with prices affected by supply and demand, the revenue function (the amount of money coming into a company from sales) is often a quadratic or other polynomial function. Suppose the monthly revenue from the sale of televisions is given by the function

\[
R(x) = -0.1x^2 + 600x,
\]

where \( x \) represents the number of televisions sold and \( R(x) \) represents the revenue in dollars. This an example of a quadratic function.

Example 2
A farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose 3 sides, and will put the 4th side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length \( L \).

Solution:
In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, \( W \), to represent the side of fencing parallel to the 4th side or backyard fence.

Since we know we only have 80 feet of fence available, we know that

\[
L + W + L = 80,
\]

or more simply, \( 2L + W = 80 \). This allows us to represent the width, \( W \), in terms of \( L \): \( W = 80 - 2L \).

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

\[
A = LW = L(80 - 2L) = A(L) = 80L - 2L^2.
\]

This formula represents the area of the fence in terms of the variable length \( L \), and is a quadratic equation.
Short run Behavior: Vertex

We now explore the interesting features of the graphs of quadratics. In addition to intercepts, quadratics have an interesting feature where they change direction, called the **vertex**.

**Forms of Quadratic Functions**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>The <strong>standard form</strong> of a quadratic function is ( f(x) = ax^2 + bx + c )</td>
</tr>
</tbody>
</table>
| 2. | The **vertex form** of a quadratic function is \( f(x) = a(x - h)^2 + k \)  
   The **vertex** of the quadratic function is located at the point \((h, k)\). The vertex is often referred to as the turning point of the quadratic (parabolic) function. |
| 3. | The **factored form** of a quadratic function is \( f(x) = a(x - r_1)(x - r_2) \).  
   The \(x\)-intercepts of the graph \( f(x) = a(x - r_1)(x - r_2) \) are the values of \(r_1\) and \(r_2\). |

If the leading coefficient, \(a\), is positive, the parabolic graph of the function will open upward. If the leading coefficient, \(a\), is negative, the parabolic graph of the function will open downward.

**Example 3** (*Video Example Here*)

Write an equation for the quadratic graphed below in vertex form and standard form.

**Solution**: Let’s start with the vertex of the of the parabola, the point \((-2, -3)\). Next substitute these values into vertex form of the equation. (Form #2 above).
\[ g(x) = a(x - (-2))^2 + (-3) \]

**Substitute the vertex**

\[ g(x) = a(x + 2)^2 - 3 \]

**Simplify**

Next, choose another point on the curve [in this case let’s use the point (0, -1)] and substitute these values into the equation for \( x \) and \( y \).

\[-1 = a(0 + 2)^2 - 3 \]

**Substitute the point (0, -1) into the equation.**

\[-1 = 4a - 3 \]

\[ 2 = 4a \]

\[ a = \frac{1}{2} \]

**Simplify.**

Next substitute the value \( a = \frac{1}{2} \) into the equation we determined earlier \( g(x) = a(x + 2)^2 - 3 \).

So, the equation is \( g(x) = \frac{1}{2}(x + 2)^2 - 3 \).

To write this in standard polynomial form, we can expand the formula and simplify terms:

\[ g(x) = \frac{1}{2}(x - 2)(x - 2) - 3 \]

**Rewrite the square as two factors**

\[ g(x) = \frac{1}{2}(x^2 + 4x + 4) - 3 \]

**Double distribute the binomials (FOIL)**

\[ g(x) = \frac{1}{2}x^2 + 2x + 2 - 3 \]

**Distribute the \( \frac{1}{2} \)**

\[ g(x) = \frac{1}{2}x^2 + 2x - 1 \]

**Simplify**

So, the vertex form of the equation is \( g(x) = \frac{1}{2}(x + 2)^2 - 3 \) and the standard form of the equation is \( g(x) = \frac{1}{2}x^2 + 2x - 1 \).

In this example, we see that it is possible to rewrite a quadratic function given in vertex form and rewrite it in standard form by algebraically expanding the formula. We can also see by using Excel, that the two equations do in fact graph the same function!
Example 4: Video Example Here

Write an equation for the quadratic graphed below using factored form, then expand the formula and simplify terms to write the equation in standard form.

Solution: Here we can use the x-intercepts, \(x = -2, \text{ and } x = 6\), to find the factored form of the equation.

\[
f(x) = a(x - r_1)(x - r_2) \quad \text{Vertex form}
\]

\[
f(x) = a(x - (-2))(x - 6) \quad \text{Substitute the roots } r_1 \text{ and } r_2
\]
\[
= a(x + 2)(x - 6) \quad \text{Simplify}
\]

Next we can find the “\(a\)” value by using another point we know is on the curve, (not the x-intercepts), say \((1, -5)\). When \(-5\) is inserted for \(f(x)\) and \(1\) is inserted for “\(x\)” we have,

\[
-5 = a(1 + 2)(1 - 6) = a(3)(-5) = -15a \quad \text{Substitute the values for } x \text{ and } y
\]

\[
\frac{-5}{-15} = \frac{-15a}{-15}
\]

\[
a = \frac{1}{3}
\]

which gives us the factored form of the quadratic now that we have the “\(a\)” value.

\[
f(x) = \frac{1}{3}(x + 2)(x - 6).
\]

To put this expression into standard form, we can expand and simplify terms:

\[
f(x) = \frac{1}{3}(x + 2)(x - 6)
\]
\[
f(x) = \frac{1}{3}(x^2 - 4x - 12) \quad \text{Double distribute (FOIL)}
\]
\[
f(x) = \frac{1}{3}x^2 - \frac{4}{3}x - 4 \quad \text{Distribute the } \frac{1}{3}, \text{ and simplify}
\]
Try it Now

1. A coordinate grid has been superimposed over the quadratic path of a basketball\(^1\). Find an equation for the path of the ball. Does he make the basket?

---

### Finding the Vertex of a Quadratic Function

For a quadratic given in standard form \( f(x) = ax^2 + bx + c \), the vertex \((h, k)\) is located at:

\[
h = -\frac{b}{2a}, \quad k = f(h) = f\left(-\frac{b}{2a}\right)
\]

Therefore, the vertex becomes \((-\frac{b}{2a}, f(-\frac{b}{2a}))\)

---

**Example 5** [*Video Example Here*]

Find the vertex of the quadratic \( f(x) = 2x^2 - 6x + 7 \). Then rewrite the quadratic into vertex form.

**Solution:**

The horizontal coordinate of the vertex will be at \( h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2} \)

The vertical coordinate of the vertex will be at \( f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2} \)

So the vertex of the quadratic is \(\left(\frac{3}{2}, \frac{5}{2}\right)\)

Rewriting into vertex form, \( f(x) = a(x-h)^2 + k \), we have

\[
f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}
\]

---

\(^1\) From [http://blog.mrmeyer.com/?p=4778](http://blog.mrmeyer.com/?p=4778), © Dan Meyer, CC-BY
Try it Now

2. Given the equation \( g(x) = 13 + x^2 - 6x \) write the equation in standard form and then in vertex form.

In addition to enabling us to more easily graph a quadratic written in standard form, finding the vertex serves another important purpose; it allows us to determine the maximum or minimum value of the function, depending on which way the graph opens.

Example 6 (* Video Example Here)

Returning to our backyard farmer from the beginning of the section, what dimensions should she make her garden to maximize the enclosed area?

Solution:
Earlier we determined the area she could enclose with 80 feet of fencing on three sides was given by the equation, \( A(L) = 80L - 2L^2 \). Notice the graph will open downwards, and the vertex will be a maximum value for the area.

In finding the vertex, we take care since the equation is not written in standard polynomial form with decreasing powers. But we know that “a” is the coefficient on the squared term, so \( a = -2 \), \( b = 80 \), and \( c = 0 \).
Finding the vertex:
\[
h = -\frac{b}{2a} = -\frac{80}{2(-2)} = 20, \quad k = A(20) = 80(20) - 2(20)^2 = 800, \quad \text{vertex} \ (20, 800)
\]

The maximum value of the function is an area of 800 square feet, which occurs when \( L = 20 \) feet. When the shorter sides are 20 feet, that leaves 40 feet of fencing for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet, and the longer side parallel to the existing fence has length 40 feet.
Example 7 (* Video Example Here *)

A local newspaper currently has 84,000 subscribers, at a quarterly charge of $30. Market research has suggested that if they raised the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution:
Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the charge per subscription times the number of subscribers. We can introduce variables, \( C \) for charge per subscription and \( S \) for the number subscribers, giving us the equation

\[
Revenue = C \cdot S
\]

Since the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently \( S = 84,000 \) and \( C = 30 \), and that if they raise the price to $32 they would lose 5,000 subscribers, giving a second pair of values, \( C = 32 \) and \( S = 79,000 \), (two points on the line \((30,84000)\) and \((32,79000)\)). From this we can find a linear equation relating the two quantities. Treating \( C \) as the input and \( S \) as the output, the equation will have form

\[
S(C) = mC + b
\]

The slope will be

\[
m = \frac{79000 - 84000}{32 - 30} = \frac{-5000}{2} = -2500
\]

(\text{Number of subscribers}) (\text{Charge in dollars})

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the vertical intercept

\[
S = -2500C + b
\]

84,000 = -2500(30) + b

\( b = 159,000 \)

Plug the slope into the \( S(C) \) equation

Plug in the point \( S = 84,000 \) and \( C = 30 \)

Solve for \( b \)

This gives us the linear equation \( S(C) = -2500C + 159000 \) relating cost and subscribers. We now return to our revenue equation.

\[
Revenue = C \cdot S
\]

\[
Revenue = C(-2500C + 159000) \quad \text{Substituting the equation for } S \text{ from above}
\]

\[
Revenue = -2500C^2 + 159000C \quad \text{Expanding (distributive property)}
\]

We now have a quadratic equation for revenue as a function of the subscription charge. We know the parabola function is opening downward and will have a maximum value because the “\( a \)” value of this quadratic is negative. To find the price that will maximize revenue for the newspaper, we can find the vertex. The “\( h \)” value of the vertex can be found using the formula.

\[
h = -\frac{159000}{2(-2500)} = 31.8
\]

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we can evaluate the revenue equation when \( C = 31.80 \). \( Revenue = -2500C^2 + 159000C \)

\[
Maximum \ Revenue = -2500(31.8)^2 + 159000(31.8) = $2,528,100.00
\]
So the vertex is (31.80, 2528100.00). The revenue is maximized when the subscription charge is $31.80 and the revenue at this point is $2,528,100.00.

**Short run Behavior: Intercepts**

As with any function, we can find the **vertical intercepts** of a quadratic by evaluating the function at an input of zero, i.e. \( f(0) \). We can find the **horizontal intercepts** by solving for when the output will be zero, i.e. \( f(x) = 0 \). Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts. There is always only one vertical intercept for a quadratic function.

![Graph with labeled intercepts](image)

**Video Example 1:** Review of Factoring Trinomials with Leading Coefficient of 1.

**Video Example 2:** Review of Factoring Trinomials with Leading Coefficient not 1.

**Example 8** (*Video Example Here)*

Identify the vertical and horizontal intercepts of the quadratic \( f(x) = 3x^2 + 10x - 8 \).

**Solution:**

We can find the vertical intercept by evaluating the function at an input of zero:

\[
f(0) = 3(0)^2 + 11(0) - 8 = -8 \quad \text{Vertical intercept at } (0, -8)
\]

For the horizontal intercepts, we solve for when the output will be zero, either by hand or by using Excel.

\[
0 = 3x^2 + 10x - 8
\]

In this case, the quadratic can be factored, providing a method for the solution

\[
0 = (3x - 2)(x + 4) \quad \text{Now we use the zero product property to solve each factor (set each factor equal to zero and solve.)}
\]

\[
0 = (3x - 2) \quad \text{or} \quad 0 = (x + 4)
\]

\[
\frac{2}{3} = x \quad -4 = x \quad \text{Solve}
\]

In this case we have two horizontal intercepts. Horizontal intercepts at \((\frac{2}{3}, 0)\) and \((-4,0)\)
Notice that in the standard form of a quadratic, the constant term “c” reveals the vertical intercept of the graph.

Excel can also be used to find the intercepts of a function. Vertical intercept, \( f(0) \) and horizontal intercept, \( f(x) = 0 \). From our table of values, \( f(0) = -8 \), so the vertical intercept is \((0, -8)\).

The table at left below also shows one of the two horizontal, x-intercepts, where the output is zero, at the point \((-4,0)\). The other x-intercept can be found using Solver, and is located somewhere between \( x = 0 \) and \( x = 1 \), as the output changes from negative to positive.

Using Solver (see table to the right) we see the other horizontal intercept is \((0.6666, 0)\) or \((\frac{2}{3}, 0)\).

This gives us the same results that we got above using algebraic methods.
Example 9

Identify the horizontal intercepts of the quadratic \( f(x) = 2x^2 + 4x - 4 \).

Solution:
Again, we will solve for when the output will be zero. So, we solve the equation \( 0 = 2x^2 + 4x - 4 \).

Since the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic into vertex form.

\[
\begin{align*}
h &= -\frac{b}{2a} = -\frac{4}{2(2)} = -1 \\
k &= f(-1) = 2(-1)^2 + 4(-1) - 4 = -6
\end{align*}
\]

\( f(x) = 2(x + 1)^2 - 6 \)

Now we can solve for when the output will be zero
\[
0 = 2(x + 1)^2 - 6
\]

\[6 = 2(x + 1)^2\]

\[3 = (x + 1)^2\]

\[x + 1 = \pm\sqrt{3}\]

\[x = -1 \pm \sqrt{3}\]

The graph has horizontal intercepts at \((-1 - \sqrt{3}, 0)\) and \((-1 + \sqrt{3}, 0)\)

Try it Now

3. In Try it Now problem 2 we found the standard & vertex form for the function, \( g(x) = 13 + x^2 - 6x \). Now find the Vertical & Horizontal intercepts (if any).

There is of course another way to solve quadratic equations. The solutions of any quadratic equation that has real roots, can be found using the quadratic formula.

Finding the Vertex of a Quadratic Function

For a quadratic function given in standard form, \( f(x) = ax^2 + bx + c \), where \( a \neq 0 \), the quadratic formula gives the solutions to \( f(x) = ax^2 + bx + c = 0 \), which would be the \( x \)-coordinate of the horizontal intercepts of the graph of this function, if they exist.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
When using the quadratic formula to solve quadratic equations, the following Excel template is very useful in finding the solutions, as well as the x-coordinate of the vertex, and other interesting information. Follow along with the directions in the example below and create your own template in Excel for you to use with any quadratic equation.

Using Excel for the Quadratic Formula

NOTE: Create your own template in Excel by following these directions!

Example 10

Find the solutions to the equation \( f(x) = 3x^2 + 5x - 2 = 0 \) using Excel. Round answers to four (4) decimal places where necessary.

Solution:

Open an Excel spread sheet, and label cell A1 as “a”, cell B1 as “b”, and cell C1 as “c”.

Next Enter the values for a, b, and c in cells A2, B2, and C2 respectively. These will change depending on the quadratic equation you are given.

\[
\begin{array}{ccc}
1 & a & b & c \\
2 & 3 & 5 & -2 \\
\end{array}
\]

Next, type the following text (not numbers) information into the appropriate cells. (See below)

\[
\begin{array}{cccc}
1 & a & b & c \\
2 & 3 & 5 & -2 & (-b) \\
3 & & & (b^2 - 4ac) & 2a \\
4 & & & & x = (as decimal) \\
5 & & & & x = (as decimal) \\
\end{array}
\]

Now use Absolute Cell reference in column F, next to each quantity, to get the desired values.

In cell F2, \( = -1 \times \$B\!2\). 
In cell F4, \( = ((\$B\!2)^2) - (4 \times \$A\!2 \times \$C\!2) \). 
In cell F6, \( = 2 \times \$A\!2 \). 
In cell F8, \( = ((F2) + SQRT(F4))/(F6) \). 
In cell F10, \( = ((F2) - SQRT(F4))/(F6) \).
Your spreadsheet should look as follows for this particular quadratic equation:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>-2</td>
<td>(- b)</td>
<td></td>
<td>-5</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>(b(^2) - 4ac)</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>2a</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td>(x = \text{(as decimal)})</td>
<td>0.333333</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td>(x = \text{(as decimal)})</td>
<td>-2</td>
<td></td>
</tr>
</tbody>
</table>

The solutions to the quadratic equation are found in cells F8, and F10 as decimals. You can use your knowledge of the formula to find exact solutions by placing your answers to cells F2, F4, and F6 respectively, into the blanks of the formula below, then simplifying.

In Cell H4 = SQRT(F4) you could have Excel calculate the square root of the number under the radical (if it is a perfect square it will be an integer).

You can now use this template to find the solutions to any quadratic equation in standard form by simply changing the values of \(a, b,\) and \(c\) in the first line of the spreadsheet!
Example 11
Find the solutions to the equation $f(x) = -2x^2 - 14x + 16 = 0$ using Excel. Round answers to four (4) decimal places where necessary.

Solution:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>a</td>
<td>-2</td>
<td>-14</td>
<td>16</td>
<td>(- b)</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(b^2 - 4ac)</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>324</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2a</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-4</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x = (as decimal)</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-8</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x = (as decimal)</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

The solutions are $x = -8, x = 1$.

Example 12 Algebraically Solving  (*Video Example Here*)
A ball is thrown upwards from the top of a 40 foot high building at a speed of 80 feet per second. The ball’s height above ground, in feet, can be modeled by the equation, $H(t) = -16t^2 + 80t + 40$, where $t$ represents time in seconds. What is the maximum height of the ball? When does the ball hit the ground?

Solution:
To find the maximum height of the ball, we would need to know the vertex of the quadratic.

$$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2}$$

$$k = H\left(\frac{5}{2}\right) = -16\left(\frac{5}{2}\right)^2 + 80\left(\frac{5}{2}\right) + 40 = 140$$

The ball reaches a maximum height of 140 feet after 2.5 seconds.

To find when the ball hits the ground, we need to determine when the height is zero, that is when $H(t) = 0$. While we could do this using the vertex form of the quadratic, we can also use the quadratic formula:

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32} \quad \text{Substitute values } a,b,c$$

Since the square root does not simplify nicely, we can use a calculator (or Excel) to approximate the values of the solutions:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458 \quad \text{Simplify}$$

The second answer is outside the reasonable domain of our model (since time cannot be negative), so we conclude the ball will hit the ground after about 5.458 seconds.
It would be easier to solve Example 12 using Excel!

Example 12 Solving with Excel

A ball is thrown upwards from the top of a 40 foot high building at a speed of 80 feet per second. The ball’s height above ground, in feet, can be modeled by the equation, \( H(t) = -16t^2 + 80t + 40 \), where \( t \) represents time in seconds. What is the maximum height of the ball? When does the ball hit the ground?

Solution:

To find the maximum height of the ball, we would need to know the vertex of the quadratic. We can use our spreadsheet and add these values:

In cell I6, \( = F2/F6 \).

The y-coordinate of the vertex can be computed by substituting the value we found in cell I6 into the equation of the quadratic.

Type in cell I10, \( = (A2 * (I6^2) + B2 * I6 + C2) \).

This value will give us the maximum height of the ball at 140 feet after 2.5 seconds.

The ball would hit the ground when the output of the function is zero, or at one of the two solutions (\( x \)-intercepts) of the function, which Excel has already found. The negative value does not make sense in this case because the input of the function (the \( t \) value, equivalent to our \( x \) value) is measuring time in seconds. So the only value that makes sense here is the positive value \( t = 5.4580 \) seconds. Notice the answers we found here using Excel are exactly the same as those found algebraically!

Try it Now

4. For these two equations determine if the vertex will be a maximum value or a minimum value.
   
a. \( g(x) = -8x + x^2 + 7 \)
   
b. \( g(x) = -3(3 - x)^2 + 2 \)
## Important Topics of this Section

- Quadratic functions
- Standard form
- Vertex form
- Vertex as a maximum / Vertex as a minimum
- Short run behavior
- Vertex / Horizontal & Vertical intercepts
- Quadratic formula

## Try it Now Answers

1. The path passes through the origin with vertex at \((-4, 7)\).
   
   \[ h(x) = -\frac{7}{16} (x + 4)^2 + 7 \]
   
   To make the shot, \(h(-7.5)\) would need to be about 4.
   
   \[ h(-7.5) \approx 1.64 \]
   
   he doesn’t make it.

2. \(g(x) = x^2 - 6x + 13\) in standard form. \(g(x) = (x - 3)^2 + 4\) in vertex form.

3. Vertical intercept at (0, 13). It has no horizontal intercepts.

4. a. Vertex is a minimum value.
   
   b. Vertex is a maximum value.
Section 3.2 Exercises

Write an equation for the quadratic function graphed. Use the most appropriate formula depending on the information given to you!

For each of the follow quadratic functions, find a) the vertex, b) the vertical intercept, and c) the horizontal intercepts.

7. \( y(x) = 2x^2 + 10x + 12 \)  
8. \( z(p) = 3x^2 + 6x - 9 \)

9. \( f(x) = 2x^2 - 10x + 4 \)  
10. \( g(x) = -2x^2 - 14x + 12 \)

11. \( h(t) = -4t^2 + 6t - 1 \)  
12. \( k(t) = 2x^2 + 4x - 15 \)
Rewrite the quadratic function into vertex form.

13. \( f(x) = x^2 - 12x + 32 \)
14. \( g(x) = x^2 + 2x - 3 \)
15. \( h(x) = 2x^2 + 8x - 10 \)
16. \( k(x) = 3x^2 - 6x - 9 \)

17. Find the values of \( b \) and \( c \) so \( f(x) = -8x^2 + bx + c \) has vertex \((2, -7)\).

18. Find the values of \( b \) and \( c \) so \( f(x) = 6x^2 + bx + c \) has vertex \((7, -9)\).

Write an equation for a quadratic with the given features
19. \( x \)-intercepts \((-3, 0)\) and \((1, 0)\), and \( y \)-intercept \((0, 2)\)
20. \( x \)-intercepts \((2, 0)\) and \((-5, 0)\), and \( y \)-intercept \((0, 3)\)
21. \( x \)-intercepts \((2, 0)\) and \((5, 0)\), and \( y \)-intercept \((0, 6)\)
22. \( x \)-intercepts \((1, 0)\) and \((3, 0)\), and \( y \)-intercept \((0, 4)\)
23. Vertex at \((4, 0)\), and \( y \)-intercept \((0, -4)\)
24. Vertex at \((5, 6)\), and \( y \)-intercept \((0, -1)\)
25. Vertex at \((-3, 2)\), and passing through \((3, -2)\)
26. Vertex at \((1, -3)\), and passing through \((-2, 3)\)

27. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by \( h(t) = -4.9t^2 + 229t + 234 \).
   a. From what height was the rocket launched?
   b. How high above sea level does the rocket reach its peak?
   c. Assuming the rocket will splash down in the ocean, at what time does splashdown occur?

28. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by \( h(t) = -4.9t^2 + 24t + 8 \).
   a. From what height was the ball thrown?
   b. How high above ground does the ball reach its peak?
   c. When does the ball hit the ground?

29. The height of a ball thrown in the air is given by \( h(x) = -\frac{1}{12}x^2 + 6x + 3 \), where \( x \) is the horizontal distance in feet from the point at which the ball is thrown.
   a. How high is the ball when it was thrown?
   b. What is the maximum height of the ball?
   c. How far from the thrower does the ball strike the ground?
30. A javelin is thrown in the air. Its height is given by \( h(x) = -\frac{1}{20}x^2 + 8x + 6 \), where \( x \) is the horizontal distance in feet from the point at which the javelin is thrown.
   a. How high is the javelin when it was thrown?
   b. What is the maximum height of the javelin?
   c. How far from the thrower does the javelin strike the ground?

31. A farmer wishes to enclose two pens with fencing, as shown. If the farmer has 500 feet of fencing to work with, what dimensions will maximize the area enclosed?

32. A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

33. A hot air balloon takes off from the edge of a mountain lake. Impose a coordinate system as pictured and assume that the path of the balloon follows the graph of
   \[ f(x) = -\frac{2}{2500}x^2 + \frac{4}{5}x. \]
   The land rises at a constant incline from the lake at the rate of 2 vertical feet for each 20 horizontal feet. \([UW]\)
   a. What is the maximum height of the balloon above water level?
   b. What is the maximum height of the balloon above ground level?
   c. Where does the balloon land on the ground?
   d. Where is the balloon 50 feet above the ground?

34. A hot air balloon takes off from the edge of a plateau. Impose a coordinate system as pictured below and assume that the path the balloon follows is the graph of
   \[ f(x) = -\frac{4}{2500}x^2 + \frac{4}{5}x. \]
   The land drops at a constant incline from the plateau at the rate of 1 vertical foot for each 5 horizontal feet. \([UW]\)
   a. What is the maximum height of the balloon above plateau level?
   b. What is the maximum height of the balloon above ground level?
   c. Where does the balloon land on the ground?
   d. Where is the balloon 50 feet above the ground?
Section 3.3 Cubic Functions and General Polynomials

In the previous section we explored the short run behavior of quadratics, a special case of polynomials. In this section we will explore the short run behavior of another special polynomial, the cubic function. A cubic function is a polynomial of degree three, that is a polynomial that is composed of the sum and/or difference of power functions that have positive integer exponents, of which the greatest exponent is three. The function \( f(x) = 2x^3 + x^2 - 5x - 14 \) is an example of a cubic polynomial function.

Example 1

Which of our toolkit functions are cubic functions?

Solution: The cubic \( f(x) = x^3 \) is considered a power function (by definition) with highest power of three, but we think of this primary as a special type of a polynomial function, which we will explore here.

<table>
<thead>
<tr>
<th>Polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>A polynomial function has the form:</td>
</tr>
<tr>
<td>( f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 )</td>
</tr>
<tr>
<td>where ( a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0 ) are real numbers (called coefficients) and ( n ) is a nonnegative integer.</td>
</tr>
</tbody>
</table>

DON’T panic here even though this equation looks so daunting!! Let’s look at it more closely and break it down into understandable terms.

Polynomial functions of only one term are called monomials or power functions, some of which we have already looked at in section 3.1. Recall: A power function has the form, \( f(x) = ax^n \) where \( a \) and \( n \) are real numbers.

A polynomial function is the sum of one or more monomials with real coefficients and nonnegative integer exponents.
**Terminology of Polynomial Functions**

A polynomial is a function that can be written as

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 \]

\( n \) is a **nonnegative integer** constant.

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.

The coefficients, \( a_n, a_{x-1}, a_{x-2}, \ldots, a_1, a_0 \), are real numbers. Coefficients can be positive, negative, or zero, and be whole numbers, decimals, or fractions.

A **term** of the polynomial is any one expression of the sum. Each individual term in a polynomial is a power function.

The **leading term** is the term containing the highest power of the variable: the term with the highest degree.

The **leading coefficient** is the coefficient of the leading term.

We often rearrange the terms of polynomials so that the powers of the terms are written in descending order, that is the highest powered term is listed first.

For a polynomial function \( f(x) \), any number \( r \) for which \( f(r) = 0 \) is called a zero or root of the function \( f \). When a polynomial function is completely factored, each of the factors helps identify zeros of the function.

A polynomial \( f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 \) has at most \( n \) real roots (horizontal intercepts). For example, linear functions, which are polynomials of degree one, have at most one horizontal intercept. Quadratic functions, of degree two, have at most two horizontal intercepts, and cubic functions have at most three horizontal intercepts.

\[ f(x) = 10x^3 - 3x^2 + 19x - 7 \] is an example of a third degree polynomial function, otherwise known as a **cubic function**. Cubic functions have at most three horizontal intercepts.

If we compare \( f(x) = 10x^3 - 3x^2 + 19x - 7 \) to the general equation of a polynomial,

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 \]

Then

\[ f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \]

and

\[ a_3 = 10, a_2 = -3, a_1 = 19, a_0 = -7 \]
So maybe that general equation is not too bad after all! The notation is simply to designate the coefficients of each of the terms of the equation and the exponent for each term associated with it.

In section 3.2 we examined polynomial of degree two, quadratic equations. Now let us look more carefully at higher degree polynomials, specifically a polynomial of degree three, cubic equations.

When products are sold with prices affected by supply and demand, the revenue function (the amount of money coming into a company from sales) is often a cubic polynomial function. Suppose the monthly demand price function from the sale of a product is given by the function

\[ D(x) = -0.5x^2 + 60x - 240, \]

where \( x \) represents the number of items sold and \( D(x) \) represents the market demand price in dollars. The revenue function is found by multiplying the demand price by the number of items sold [i.e. \( R(x) = (# \text{ items}) \cdot (\text{price per item}) \)].

Here, in this case, the revenue function would be:

\[ R(x) = (# \text{ items}) \cdot (\text{price per item}) \]

\[ R(x) = (x) \cdot (D(x)) \]

\[ R(x) = (x)(-0.5x^2 + 60x - 240) \]

\[ R(x) = -0.5x^3 + 60x^2 - 240x \]

So, \( R(x) \) is another example of a cubic function.

Example 2 (**Video Example Here**)

Identify the degree, leading term, and leading coefficient of these polynomials:

a) \( f(x) = 3 + 2x^2 - 4x^3 \)

b) \( g(t) = 5t^5 - 2t^3 + 7t \)

c) \( h(p) = 6p - p^3 - 2 \)

Solution:

a) For the function \( f(x) \), the degree is 3, the highest power of \( x \). The leading term is the term containing that power, \(-4x^3\). The leading coefficient is the coefficient of that term, \(-4\). We could always rearrange the function first \( f(x) = -4x^3 + 2x^2 + 3 \)

b) For \( g(t) \), the degree is 5, the leading term is \( 5t^5 \), and the leading coefficient is 5.

c) For \( h(p) \), rearrange first. \( h(p) = -p^3 + 6p - 2 \). The degree is 3, the leading term is \(-p^3\), so the leading coefficient is \(-1\).
Long Run Behavior of Polynomials

For any polynomial, the long run behavior of the polynomial will match the long run behavior of the leading term.

- Polynomials with a leading coefficient that is positive will have right tail \((x \to \infty)\) behavior that tends to positive infinity \((f(x) \to \infty)\).
- Polynomials with a leading coefficient that is negative will have right tail \((x \to \infty)\) behavior that tends to negative infinity \((f(x) \to -\infty)\).
- For polynomials of even degree, both the right and left tail end behavior will be the same. (Either both tending to positive infinity, or both tending to negative infinity.)
- For polynomials of odd degree, the right and left tail behavior will tend in opposite directions. If the right tail behavior tends to infinity, the left tail behavior will tend to negative infinity and vice versa.

So, what is the long run (tail end) behavior of a cubic function?

Example 3: (*Video Example Here)

What can we determine about the long run behavior and degree of the equation for the polynomial graphed here?

![Graph of a cubic function](image)

Solution:
Since the output grows large and positive as the inputs grow large and positive, we describe the long run behavior symbolically by writing: as \(x \to \infty\), \(f(x) \to \infty\). Similarly, as \(x \to -\infty\), \(f(x) \to -\infty\).

In words, we could say that as \(x\) values approach infinity, the function output values approach infinity, and as \(x\) values approach negative infinity the function output values approach negative infinity.

The graph shown is the general shape of a cubic function, a polynomial of degree three. There are three horizontal intercepts. We can tell this graph has the shape of an odd degree power function because the left and right tails head in opposite directions, so the degree of the polynomial creating this graph must be an odd number, and the leading coefficient would be positive.
1. Given the function \( f(x) = 0.2(x - 2)(x + 1)(x - 5) \) use your algebra skills to write the function in standard polynomial form (as a sum of terms) and determine the leading term, degree, and long run behavior of the function.

**Short run Behavior of Polynomials**

As with any function, the vertical intercept can be found by evaluating the function at an input of zero. Since this is evaluation, it is relatively easy to find the vertical intercept for a polynomial of any degree.

To find horizontal intercepts, we need to solve for when the output will be zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and other polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials.

Consequently, we will limit ourselves to three cases:

1) The polynomial can be factored using known methods: greatest common factor (GCF) and trinomial factoring.
2) The polynomial is given in factored form.
3) Technology (Excel) is used to determine the intercepts.

**Example 4**

Find the horizontal intercepts of the cubic function \( f(x) = x^3 - 3x^2 + 2x \).

**Solution:**

Because this function has a GCF, we can attempt to factor this polynomial to find solutions for \( f(x) = 0 \).

\[
x^3 - 3x^2 + 2x.
\]

\[
x(x^2 - 3x + 2) \quad \text{Factor out the greatest common factor of } x.
\]

\[
x(x - 2)(x - 1) \quad \text{Factor the inside as a quadratic.}
\]

\[
x = 0, x = 2 \quad \text{Now use the Zero Product Property.}
\]

\[
x = 0 \quad \text{or}
\]

\[
x = 2 \quad \text{or}
\]

\[
x = 1 \quad \text{Solve each equation.}
\]
This gives us 3 horizontal intercepts.  We could of course do this same problem using Excel and Solver.  Here the horizontal intercepts (x-intercepts) happen to fall on nice integer values so we may not even need to use Solver to find the three intercepts at, \( x = 0, x = 1, \text{ and } x = 2 \). We notice these are the input values that give us an output of zero. So, the horizontal intercepts are \((0, 0), (-1, 0)\) and \((2, 0)\). Remember, we can also find the vertical, or y-intercept easily in Excel, knowing that the vertical intercept is found when the input of the function is zero, \( f(0) \). Here the vertical intercept is the point \((0, 0)\) which just happens to also be a horizontal intercept in this case.

Example 5

Algebraically find the vertical and horizontal intercepts of \( g(t) = (t - 2)^2 (2t + 3) \)

**Solution:** The vertical intercept can be found by evaluating \( g(0) \).

\[
g(0) = (0 - 2)^2 (2(0) + 3) = 12 \quad \text{The vertical intercept is } (0, 12).
\]

The horizontal intercepts can be found by solving \( g(t) = 0 \).

\[
(t - 2)^2 (2t + 3) = 0 \quad \text{This expression is already factored.}
\]

\[
(t - 2)^2 = 0 \quad (2t + 3) = 0
\]

\[
t - 2 = 0 \quad \text{or} \quad t = \frac{-3}{2}
\]

Horizontal intercepts \((2, 0), \left(-\frac{3}{2}, 0\right)\). We can always check our answers are reasonable by graphing the polynomial in Excel! The yellow highlighted values are the horizontal intercepts, the vertical intercept is shown in blue. Notice the cubic “bounces off” the horizontal axis at \( t = 2 \) here which corresponds to the factor that is squared in the equation.
Example 6
Find the horizontal intercepts of \( h(t) = t^3 + 4t^2 + t - 6 \)

Solution:
Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to Excel to find the intercepts. The biggest challenge here however is setting an appropriate domain to see the possible intercepts. Let’s use the domain interval \([-5, 3]\).

Graphing this function, it appears there are horizontal intercepts at \( t = -3, -2, \) and 1.

We could check to see if these are correct by plugging in these values for \( t \) and verifying that \( h(-3) = h(-2) = h(1) = 0 \).
So, the horizontal intercepts are \((-3, 0), (-2, 0), \) and \((1, 0)\).

Using Excel, we can also verify these possibilities. Indeed, the horizontal intercept of this function are at \( t = -3, t = -2, \) and \( t = 1 \). We also notice that the vertical intercept is at the point \((0, -6)\), confirmed by both the graph and the table from in Excel.

Try it Now
2. Find the vertical and horizontal intercepts of the function \( f(t) = t^4 - 4t^2 \).
Example 7

Identify the end behavior, vertical, and horizontal intercepts of \( f(x) = -2(x + 3)^2(x - 5) \).

Solution:
The equation is already in factored form so we can see that this graph will have two horizontal intercepts. Using the Zero Product Property we find horizontal intercepts at \( x = -3 \), where the factor is squared, indicating the graph will bounce at this horizontal intercept. At \( x = 5 \), the factor is not squared, indicating the graph will pass through the axis at this intercept. So, the horizontal intercepts are \((-3, 0)\) and \((5, 0)\). If we algebraically expand this equation we get,

\[
f(x) = -2(x + 3)(x + 3)(x - 5) \quad \text{Write squared factor twice.}
\]

\[
f(x) = -2(x^2 + 6x + 9)(x - 5) \quad \text{Distribute over the first two factors (FOIL).}
\]

\[
f(x) = -2(x^3 + x^2 - 21x - 45) \quad \text{Distribute factors again}
\]

\[
f(x) = -2x^3 - 2x^2 + 42x + 90 \quad \text{Distribute the -2}
\]

We can see the leading term is \(-2x^3\), so the long-run behavior is that of cubic with a negative leading coefficient. Right tail end behavior: as \( x \to \infty \), \( f(x) \to -\infty \). Left tail end behavior: as \( x \to -\infty \), \( f(x) \to \infty \). The y-intercept, \( f(0) = 90 \), is \((0, 90)\).

Using Excel, we can verify that the resulting graph will look like:

Try it Now

3. Given the function \( g(x) = x^3 - x^2 - 6x \) use the methods that we have learned so far to find the vertical & horizontal intercepts, and describe the long run behavior.
Writing Equations using Intercepts

Since a polynomial function written in factored form will have a horizontal intercept where each factor is equal to zero, we can form a function that will pass through a set of horizontal intercepts by introducing a corresponding set of factors.

**Factored Form of Polynomials**

If a polynomial has horizontal intercepts at, \( x = r_1, r_2, \ldots, r_n \), then the polynomial can be written in the factored form as \( f(x) = a(x - r_1)^{p_1}(x - r_2)^{p_2} \ldots (x - r_n)^{p_n} \) where the powers \( p_i \) on each factor can be determined by the behavior of the graph at the corresponding intercept, and the leading coefficient \( a \) can be determined given a value of the function other than the horizontal intercept.

For a cubic polynomial \( f(x) = a(x - r_1)(x - r_2)(x - r_3) \).

**Example 8** (*Video Example Here*)

Find a formula for the cubic function that has horizontal intercepts (-5, 0), (1, 0), (4, 0) and vertical intercept (0, -2).

**Solution:**

Start with the general equation in factored form. \( f(x) = a(x - r_1)(x - r_2)(x - r_3) \). Filling in the horizontal intercepts for the roots we have,

\[
 f(x) = a(x + 5)(x - 1)(x - 4)
 \]

Filling in the roots \( x = -5, x = 1, \) and \( x = 4 \).

\[
 -2 = a(0 + 5)(0 - 1)(0 - 4)
 \]

Enter the vertical intercept in for \( x \) and \( y \).

\[
 -2 = 20a
 \]

Simplify

\[
 \frac{-2}{20} = -\frac{1}{10} = a
 \]

Solve for \( a \)

\[
 f(x) = -\frac{1}{10}(x + 5)(x - 1)(x - 4)
 \]

Rewrite the factored form with the value of ‘\( a \)’ found.

The equation can be left in this form, or expanded if desired.
Estimating Extremes

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using Excel.

Example 9

An open-top box is to be constructed by cutting out squares from each corner of a 14cm by 20cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

Solution:
We will start this problem by drawing a picture, labeling the width of the cut-out squares with a variable, \( w \). Notice that after a square is cut out from each end, it leaves a \((14-2w)\) cm by \((20-2w)\) cm rectangle for the base of the box, and the box will be \( w \) cm tall. This gives the volume:

\[
V(w) = (14 - 2w)(20 - 2w)w = 280w - 68w^2 + 4w^3
\]

Using technology to sketch a graph allows us to estimate the maximum value for the volume, restricted to reasonable values for \( w \): values from 0 to 7.

From this graph, we can estimate the maximum value is approximately 340, and occurs when the value of \( w \) is about 2.75cm. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph.

From this zoomed-in view, we can refine our estimate for the max volume to about 339, when the value of \( w \) is approximately 2.7cm. We of course, could also use the maximum feature in Excel.
Important Topics of this Section
Polynomials
Coefficients
Leading coefficient
Term
Leading Term
Degree of a polynomial
Long run behavior
Short Run Behavior
Intercepts (Horizontal & Vertical)
Methods to find Horizontal intercepts
Factoring Methods
Factored Forms
Technology
Writing equations using intercepts

Try it Now Answers

1. The leading term is $0.2x^3$, so it is a degree 3 polynomial. As $x$ approaches infinity (or gets very large in the positive direction) $f(x)$ approaches infinity; as $x$ approaches negative infinity (or gets very large in the negative direction) $f(x)$ approaches negative infinity. (Basically, the long run behavior is the same as the cubic function).

2. Vertical intercept $(0, 0)$, Horizontal intercepts $(0, 0), (-2, 0), (2, 0)$

3. Vertical intercept $(0, 0)$, Horizontal intercepts $(-2, 0), (0, 0), (3, 0)$
   The leading term is $x^3$ so as $x \to -\infty$, $g(x) \to -\infty$ and as $x \to \infty$, $g(x) \to \infty$


**Section 3.3 Exercises**

Find the $C$ and $t$ intercepts of each function.
1. $C(t) = 2(t - 4)(t + 1)(t - 6)$
2. $C(t) = 3(t + 2)(t - 3)(t + 5)$
3. $C(t) = 4t(t - 2)^2(t + 1)$
4. $C(t) = 2t(t - 3)(t + 1)^2$

Use Excel to solve for the zeros of the function.
5. $f(x) = x^3 - 7x^2 + 4x + 30$
6. $g(x) = x^3 - 6x^2 + x + 28$

Find the long run behavior of each function as $t \to \infty$ and $t \to -\infty$
7. $h(t) = 3(t - 5)(t - 3)(t - 2)$
8. $k(t) = 2(t - 1)^2(t + 2)$
9. $p(t) = -2t(t - 1)(t - 3)$
10. $q(t) = -4t^2(t + 1)$

Expand each equation. Find the end behavior, and the vertical, and horizontal intercepts
11. $f(x) = (x + 3)^2(x - 2)$
12. $g(x) = (x + 4)(x - 1)^2$
13. $p(t) = -2t(t - 2)(t + 1)$
14. $k(x) = (x - 3)^3(x - 2)^2$
15. $m(x) = -2x(x - 1)(x + 3)$
16. $n(x) = -3x(x + 2)(x - 4)$

Write an equation for a polynomial the given features.
17. Degree 3. Zeros at $x = -2$, $x = 1$, and $x = 3$. Vertical intercept at (0, -4)
18. Degree 3. Zeros at $x = -5$, $x = -2$, and $x = 1$. Vertical intercept at (0, 6)

Write a formula for each polynomial function graphed.
22. The data shown describes the sales revenues in thousands of dollars that might be associated with various levels of advertising by a local home improvement store.

<table>
<thead>
<tr>
<th>Advertising ($1000)</th>
<th>Sales ($1000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>71</td>
</tr>
<tr>
<td>10</td>
<td>135</td>
</tr>
<tr>
<td>20</td>
<td>226</td>
</tr>
<tr>
<td>30</td>
<td>324</td>
</tr>
<tr>
<td>40</td>
<td>402</td>
</tr>
<tr>
<td>50</td>
<td>450</td>
</tr>
<tr>
<td>60</td>
<td>476</td>
</tr>
<tr>
<td>70</td>
<td>489</td>
</tr>
</tbody>
</table>

a) Plot the data in Excel and produce a graph with data points. What type of polynomial would be a good model for this data?
b) Use Excel to find the cubic (polynomial of degree three) regression model for the data with coefficients rounded to three decimal places.
c) Using the cubic model, what would you predict the sales revenues to be when spending $100,000 on advertising? When spending $150,000 on advertising? Make sure to use proper units and state your answer in a complete sentence.

23. The table below shows the cost $C$, of traffic accidents, in cents per vehicle-mile, as a function of vehicular speed $s$, on miles per hour, for commercial vehicles driving at night on urban streets.

<table>
<thead>
<tr>
<th>Speed $s$</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost $C$</td>
<td>1.3</td>
<td>0.4</td>
<td>0.1</td>
<td>0.3</td>
<td>0.9</td>
<td>2.2</td>
<td>5.8</td>
</tr>
</tbody>
</table>

a) Use Excel regression to find a cubic model for the data with coefficients rounded to four decimal places.
b) Use this model to find and interpret $C(42)$.
c) On the domain interval from 20 to 50 miles per hour, estimate at what speed is the cost of traffic accidents (for commercial vehicles driving at night on urban streets) at a minimum?
Section 3.4 Rational Functions

In the last few sections, we have built polynomials based on the positive integer power functions. In this section we explore functions based on power functions with negative integer powers, called rational functions. In section 3.1, we talked about the toolkit functions that deal with these power functions, but here we will look at more detailed functions that are formed by dividing polynomial functions. Let’s beginning with a simple example using power functions with negative integer powers.

Example 1
You plan to drive 100 miles. Find a formula for the time the trip will take as a function of the speed you drive.

Solution:
You may recall that multiplying speed by time will give you distance. If we let $t$ represent the drive time in hours, and $v$ represent the velocity (speed or rate) at which we drive, then, $distance = v \cdot t$. Since our distance is fixed at 100 miles, $100 = v \cdot t$. Solving this relationship for time gives us the function we desired:

\[
100 = v \cdot t. \\
t = \frac{100}{v}. \\
\frac{t(v)}{v} = \frac{100}{v} = 100v^{-1}
\]

While this type of relationship can be written using the negative exponent, it is more common to see it written as a fraction.

This particular example is one of an inversely proportional relationship, that is where one quantity is a constant divided by the other variable quantity, like $y = \frac{5}{x}$.

Several natural phenomena, such as gravitational force and volume of sound, behave in a manner inversely proportional to the square of another quantity. For example, the volume, $V$, of a sound heard at a distance $d$ from the source would be related by $V = \frac{k}{d^2}$ for some constant value $k$. These functions are simply the reciprocal squared toolkit function $f(x) = \frac{1}{x^2}$ multiplied by a constant.
We have seen the graphs of the basic reciprocal function and the squared reciprocal function from section 3.1.

**Reciprocal function** \( f(x) = \frac{1}{x} = x^{-1} \)  
**Reciprocal Squared function** \( f(x) = \frac{1}{x^2} = x^{-2} \)

Let’s begin by looking at the reciprocal function, \( f(x) = \frac{1}{x} \). As you well know, dividing by zero is not allowed and therefore zero is not in the domain, and so the function is undefined at an input of zero.

**Short run behavior**

For the function \( f(x) = \frac{1}{x} \), as the input values approach zero from the left side (taking on very small, negative values), the function output values become very large in the negative direction (in other words, they approach negative infinity). We write: As \( x \to 0^- \), \( f(x) \to -\infty \).

As we approach zero from the right side (small, positive input values), the function output values become very large in the positive direction (approaching infinity). We write: As \( x \to 0^+ \), \( f(x) \to \infty \).

This behavior creates a **vertical asymptote**. A vertical asymptote is a vertical line that the graph approaches. In this case the graph is approaching the vertical line \( x = 0 \) as the input becomes close to zero.

**Long run behavior (End behavior)**

We looked at this behavior earlier in this unit but recall the following.  
As the values of \( x \) approach positive infinity, the function output values approach 0.  
As the values of \( x \) approach negative infinity, the function output values approach 0.  
Symbolically: as \( x \to \pm \infty \), \( f(x) \to 0 \)
Based on this long run behavior and the graph we can see that the function approaches 0 but never actually reaches 0, it just “levels off” as the inputs become large. This behavior creates a **horizontal asymptote**. In this case the graph is approaching the horizontal line $f(x) = 0$ as the input becomes very large in the negative and positive directions.

### Vertical and Horizontal Asymptotes

A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends towards positive or negative infinity as the inputs approach $a$. As $x \to a$, $f(x) \to \pm \infty$.

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs get large. As $x \to \pm \infty$, $f(x) \to b$.

---

**Try it Now**

1. Use symbolic notation to describe the long run behavior and short run behavior for the reciprocal squared function.

---

**Example 2**

Find the horizontal and vertical asymptotes of the graph shown, if any.

**Solution:**

The graph also is showing a vertical asymptote at $x = -2$.

As $x \to -2^-$, $f(x) \to -\infty$, and as $x \to -2^+$, $f(x) \to \infty$

As the inputs grow large, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

As $x \to \pm \infty$, $f(x) \to 3$. 
A **rational function** is a function that can be written as the ratio of two polynomials, \( P(x) \) and \( Q(x) \).

\[
f(x) = \frac{P(x)}{Q(x)} \quad \text{OR} \quad f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + bx + b_0}
\]

**Example 3**

A large mixing tank currently contains 100 gallons of water, into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after \( t \) minutes.

**Solution:**

Notice that the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

water = 100 + 10t

sugar = 5 + t

The concentration, \( C \), will be the ratio of pounds of sugar to gallons of water

\[
C(t) = \frac{5 + t}{100 + 10t}
\]

This is an example of a rational function (in this case a linear function divided by a linear function.)
Finding Asymptotes and Intercepts

Given a rational function, as part of investigating the short run behavior we are interested in finding any vertical and horizontal asymptotes, as well as finding any vertical or horizontal intercepts, as we have done in the past.

To find vertical asymptotes, we notice that the vertical asymptotes in our examples occur when the denominator of the function is undefined. **With one exception**, a vertical asymptote will occur whenever the denominator is undefined.

**Example 4** *(Video Example Here)*

Find the vertical asymptotes of the function \( k(x) = \frac{5 + 2x^2}{2 - x - x^2} \)

**Solution:**
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:
\[
2 - x - x^2 = 0 \quad \text{Set the denominator equal to zero.}
\]

\[
(2 + x)(1 - x) = 0 \quad \text{Factor the expression.}
\]

\[
(2 + x)(1 - x) = 0 \quad \text{Use the Zero Product Property to solve for } x.
\]

\[
(2 + x) = 0, x = -2 \quad \text{Solve.}
\]

\[
(1 - x) = 0, x = 1 \quad \text{Vertical asymptotes of } x = -2 \text{ and } x = 1.
\]

This indicates two vertical asymptotes, which a look at a graph confirms. Remember when graphing functions in Excel, we need to eliminate any possible input from the domain that makes the function undefined!

The exception to this rule can occur when both the numerator and denominator of a rational function are zero at the same input. (That is when there is a common linear factor in both the numerator and denominator of the factored rational function).
Example 5 (*Video Example Here)*

Which the vertical asymptotes of the function, \( k(x) = \frac{x-2}{x^2-4} \).

Solution:
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:
\[
x^2 - 4 = 0
\]
\[
(x - 2)(x + 2) = 0
\]
\[
(x - 2) = 0, x = 2; (x + 2) = 0, x = -2
\]

Set the denominator equal to zero.  
Factor the expression.  
Use the zero product property.

However, the numerator of this function is also equal to zero when \( x = 2 \). In factored form, the rational function becomes, \( k(x) = \frac{(x-2)}{(x-2)(x+2)} \). Notice there is a common linear factor in both the numerator and denominator. Because of this, the function will still be undefined at 2, since \( \frac{0}{0} \) is undefined, but the graph will not have a vertical asymptote at \( x = 2 \).

The graph of this function will have the vertical asymptote at \( x = -2 \), but at \( x = 2 \) the graph will have a hole: a single point where the graph is not defined, indicated by an open circle. This “hole” is not seen in Excel unless we edit the domain. Remember when graphing functions in Excel, we need to eliminate any possible input from the domain that makes the function undefined!

Notice: In Excel, the two input values where the function is undefined shows an output “DIV/0!”, implying that the rational function is undefined because the denominator is equal to zero. If we graph the function with these values we produce an incorrect graph!!

**INCORRECT GRAPH**
When we eliminate these values from the data and increase the number of input values (using an increment of 0.1) we obtain a more efficient graph that models the behavior of the graph more correctly.

| x  | y   
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>-0.3333333</td>
</tr>
<tr>
<td>-4.8</td>
<td>-0.3571428</td>
</tr>
<tr>
<td>-4.6</td>
<td>-0.3846153</td>
</tr>
<tr>
<td>-4.4</td>
<td>-0.4166667</td>
</tr>
<tr>
<td>-4.2</td>
<td>-0.4545454</td>
</tr>
<tr>
<td>-4</td>
<td>-0.5</td>
</tr>
<tr>
<td>-3.8</td>
<td>-0.5555556</td>
</tr>
<tr>
<td>-3.6</td>
<td>-0.6</td>
</tr>
<tr>
<td>-3.4</td>
<td>-0.625</td>
</tr>
<tr>
<td>-3.2</td>
<td>-0.6333333</td>
</tr>
<tr>
<td>-3</td>
<td>-1</td>
</tr>
<tr>
<td>-2.8</td>
<td>-1.25</td>
</tr>
<tr>
<td>-2.6</td>
<td>-1.6666667</td>
</tr>
<tr>
<td>-2.4</td>
<td>-2.5</td>
</tr>
<tr>
<td>-2.2</td>
<td>-5</td>
</tr>
<tr>
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<td>-10</td>
</tr>
<tr>
<td>-1.8</td>
<td>-15</td>
</tr>
<tr>
<td>-1.6</td>
<td>-20</td>
</tr>
<tr>
<td>-1.4</td>
<td>1.6666667</td>
</tr>
<tr>
<td>-1.2</td>
<td>1.25</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>-0.8</td>
<td>0.3333333</td>
</tr>
<tr>
<td>-0.6</td>
<td>0.2</td>
</tr>
<tr>
<td>-0.4</td>
<td>0.125</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.5555556</td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

**Corrections Graph**

**Vertical Asymptotes and Holes in Rational Functions**

The **vertical asymptotes** of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.

A **hole** might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. In this case, factor the numerator and denominator and simplify; if the simplified expression still has a zero in the denominator at the original input the original function has a vertical asymptote at the input, otherwise it has a hole.
To find horizontal asymptotes, we are interested in the behavior of the function as the input grows large, so we consider long run behavior of the numerator and denominator separately. Recall that a polynomial’s long run behavior will mirror that of the **leading term**. Likewise, a rational function’s long run behavior will mirror that of the **ratio of the leading terms** of the numerator and denominator functions.

There are three distinct outcomes when this analysis is done:

**Horizontal Asymptote Case 1:** The degree of the denominator > degree of the numerator

Example: \( f(x) = \frac{3x + 2}{x^2 + 4x - 5} \)

In this case, the long run behavior is \( f(x) \approx \frac{3x}{x^2} = \frac{3}{x} \). This tells us that as the inputs grow large, this function will behave similarly to the function \( g(x) = \frac{3}{x} \). This is similar to the toolkit, reciprocal function. As the inputs grow large, the outputs will approach zero, resulting in a horizontal asymptote at \( y = 0 \).

As \( x \to \pm\infty \), \( f(x) \to 0 \)

**Horizontal Asymptote Case 2:** The degree of the denominator < degree of the numerator

Example: \( f(x) = \frac{3x^2 + 2}{x - 5} \)

In this case, the long run behavior is \( f(x) \approx \frac{3x^2}{x} = 3x \). This tells us that as the inputs grow large, this function will behave similarly to the function \( g(x) = 3x \). This is a linear function. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote.

As \( x \to \pm\infty \), \( f(x) \to \pm\infty \), respectively.

Ultimately, if the numerator is larger than the denominator, the long run behavior of the graph will mimic the behavior of the reduced long run behavior fraction. As another example if we had the function \( f(x) = \frac{3x^5 - x^2}{x + 3} \) with long run behavior \( f(x) \approx \frac{3x^5}{x} = 3x^4 \), the long run behavior of the graph would look similar to that of an even polynomial, and as \( x \to \pm\infty \), \( f(x) \to \infty \).

**Horizontal Asymptote Case 3:** The degree of the denominator = degree of the numerator

Example: \( f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5} \)

In this case, the long run behavior is, \( f(x) \approx \frac{3x^2}{x^2} = 3 \). This tells us that as the inputs grow large, this function will behave like the function, \( g(x) = 3 \), which is a horizontal line. As \( x \to \pm\infty \), \( f(x) \to 3 \), resulting in a horizontal asymptote at \( y = 3 \).
The horizontal asymptote of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of denominator > degree of numerator which looks like: \( \frac{\text{Small degree}}{\text{Big degree}} \)
  Horizontal asymptote at \( y = 0 \).

- Degree of denominator < degree of numerator which looks like: \( \frac{\text{Big degree}}{\text{Small degree}} \)
  No horizontal asymptote.

- Degree of denominator = degree of numerator which looks like: \( \frac{\text{Same degree}}{\text{Same degree}} \)
  Horizontal asymptote at ratio of the leading coefficients of the numerator and denominator.

Example 6  

In the sugar concentration problem from earlier, we created the equation \( C(t) = \frac{5 + t}{100 + 10t} \).

Find the horizontal asymptote and interpret it in context of the scenario.

Solution:
Both the numerator and denominator are linear (degree 1), so since the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is \( t \), with coefficient 1. In the denominator, the leading term is \( 10t \), with coefficient 10. The horizontal asymptote will be at the ratio of these values: As \( t \to \infty \), \( C(t) \to \frac{1}{10} \). This function will have a horizontal asymptote at \( y = \frac{1}{10} \).

This tells us that as the input gets large, the output values will approach 1/10. As \( t \to \infty \), \( C(t) \to \frac{1}{10} \)

Interpretation: In context, this means that as more time goes by, the concentration of sugar in the tank will approach one tenth of a pound of sugar per gallon of water or 1/10 pounds per gallon.
Example 7 (*Video Example Here)  
Find the horizontal and vertical asymptotes of the function  
\[ f(x) = \frac{(x - 2)(x + 3)}{(x - 1)(x + 2)(x - 5)} \]  
**Solution:**  
First, note this function has no inputs that make both the numerator and denominator zero, so there are no potential holes. The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at \( x = 1, -2, \) and \( 5, \) indicating vertical asymptotes at these values.  
The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as \( x \to \pm \infty, \ f(x) \to 0. \) This function will have a horizontal asymptote at \( y = 0. \)  

Try it Now  
2. Find the vertical and horizontal asymptotes of the function  
\[ f(x) = \frac{(2x - 1)(2x + 1)}{(x - 2)(x + 3)} \]  

**Intercepts**  
As with all functions, a rational function will have a vertical intercept when the input is zero, if the function is defined at zero. It is possible for a rational function to not have a vertical intercept if the function is undefined at zero.  
Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be zero (unless that input corresponds to a hole). It is possible there are no horizontal intercepts. **Since a fraction is only equal to zero when the numerator is zero**, horizontal intercepts will occur when the numerator of the rational function is equal to zero.  

Example 8 (*Video Example Here)*  
What are the intercepts of \( f(x) = \frac{(x - 2)(x + 3)}{(x - 1)(x + 2)(x - 5)} \)  
**Solution:**  
We can find the vertical intercept by evaluating the function at zero
Vertical intercept \( (0, -\frac{3}{5}) \)

The horizontal intercepts will occur when the function is equal to zero:

\[
\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} = 0
\]

Set equal to zero.

\[(x - 2) (x + 3) = 0\]

This equation is zero when the numerator is zero.

\[(x - 2) = 0, x = 2; \]
\[(x + 3) = 0, x = -3\]

\[(2,0) \text{and } (-3,0)\]

Horizontal intercepts

These can, of course be found using Excel, where we can check our answers.

Writing Rational Functions from Intercepts and Asymptotes

If a rational function has horizontal intercepts at \( x = r_1, r_2, \ldots, r_n \), and vertical asymptotes at \( x = v_1, v_2, \ldots, v_m \) then the function can be written in the form

\[
f(x) = a \frac{(x-r_1)(x-r_2)\ldots(x-r_n)}{(x-v_1)(x-v_2)\ldots(x-v_n)}
\]

where the behavior of the graph at the corresponding intercept or asymptote, and the leading coefficient \( a \) can be determined given a value of the function other than the horizontal intercept, or by the horizontal asymptote if it is nonzero.

Video Link: Writing Equations of Rational Functions Using Asymptotes and Intercepts
Important Topics of this Section
Inversely proportional; Reciprocal toolkit function
Reciprocal squared toolkit function
Horizontal Asymptotes
Vertical Asymptotes
Rational Functions
  Finding intercepts, asymptotes, and holes.
  Identifying a function from its graph

Try it Now Answers

1. Long run behavior, as \( x \to \pm \infty \), \( f(x) \to 0 \)
   Short run behavior, as \( x \to 0 \), \( f(x) \to \infty \) (there are no horizontal or vertical intercepts)

2. Vertical asymptotes at \( x = 2 \) and \( x = -3 \); horizontal asymptote at \( y = 4 \).
Section 3.4 Exercises

For each function, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal asymptote, if they exist.

1. \( p(x) = \frac{2x - 3}{x + 4} \)

2. \( q(x) = \frac{x - 5}{3x - 1} \)

3. \( s(x) = \frac{4}{(x - 2)(x + 3)} \)

4. \( r(x) = \frac{5}{(x - 5)(x + 1)} \)

5. \( f(x) = \frac{3x^2 - 14x - 5}{3x^2 + 8x - 16} \)

6. \( g(x) = \frac{2x^2 + 7x - 15}{3x^2 - 14 + 15} \)

7. \( a(x) = \frac{x^2 + 2x - 3}{x^2 - 1} \)

8. \( b(x) = \frac{x^2 - x - 6}{x^2 - 4} \)

9. \( h(x) = \frac{2x^2 + x - 1}{x - 4} \)

10. \( k(x) = \frac{2x^2 - 3x - 20}{x - 5} \)

11. \( n(x) = \frac{3x^2 + 4x - 4}{x^3 - 4x^2} \)

12. \( m(x) = \frac{5 - x}{2x^2 + 7x + 3} \)

For each function graphed, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal asymptote, if they exist.

19. [Graph of function 19]

20. [Graph of function 20]
25. A street light is 10 feet north of a straight bike path that runs east-west. Olav is bicycling down the path at a rate of 15 miles per hour. At noon, Olav is 33 feet west of the point on the bike path closest to the street light. (See the picture). The relationship between the intensity $C$ of light (in candlepower) and the distance $d$ (in feet) from the light source is given by $C = \frac{k}{d^2}$, where $k$ is a constant depending on the light source.

a. From 20 feet away, the street light has an intensity of 1 candle. What is $k$?

b. Find a function which gives the intensity of the light shining on Olav as a function of time, in seconds.

c. When will the light on Olav have maximum intensity?

d. When will the intensity of the light be 2 candles
Section 4.1 Exponential Functions

India is the second most populous country in the world, with a population in 2019 of about 1.39 billion people. The population of India is growing by about 1.1% each year. We might ask if we can find a formula to model the population, \( P \), as a function of time, \( t \), in years after 2019, if the population continues to grow at this rate.

In linear growth, we had a constant rate of change – a constant number that the output increased for each increase in input. For example, in the equation \( f(x) = 3x + 4 \), the slope tells us the output increases by 3 each time the input increases by 1. This population scenario involving the population of India is different – we have a constant percent increase in people rather than a constant number of people as our rate of change. To see the significance of this difference, consider these two companies:

Company A has 200 stores, and expands by opening 50 new stores each year.

Company B has 200 stores, and expands by increasing the number of stores by 50% of their total each year.

Looking at a few years of growth for company A:

<table>
<thead>
<tr>
<th>Year</th>
<th># Stores Company A</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>200 stores</td>
<td>Starts with 200 stores.</td>
</tr>
<tr>
<td>1</td>
<td>200 + 50 = 250 stores</td>
<td>Increases by 50 stores in the first year.</td>
</tr>
<tr>
<td>2</td>
<td>250 + 50 = 300 stores</td>
<td>Increases by 50 stores in second year.</td>
</tr>
<tr>
<td>3</td>
<td>300 + 50 = 350 stores</td>
<td>Increases by 50 stores in third year.</td>
</tr>
</tbody>
</table>

Looking at a few years of growth for company B:

<table>
<thead>
<tr>
<th>Year</th>
<th>Stores, company B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>200 stores</td>
<td>Starts with 200 stores.</td>
</tr>
<tr>
<td>1</td>
<td>200 + 50% of 200</td>
<td>Increases by 100 stores in the first year.</td>
</tr>
<tr>
<td></td>
<td>=200 + 0.50(200)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>=200 + 100 = 300 stores</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>300 + 50% of 300</td>
<td>Increases by 150 stores in the second year.</td>
</tr>
<tr>
<td></td>
<td>= 300 + 0.50(300)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>= 300 + 150 = 450 stores</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>450 + 50% of 450</td>
<td>Increases by 225 stores in third year.</td>
</tr>
<tr>
<td></td>
<td>= 450 + 0.50(450)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>= 450 + 225 = 675 stores</td>
<td></td>
</tr>
</tbody>
</table>

Notice that with the percent growth, each year the company grows by 50% of the current year’s total. So as the company grows larger, the number of stores added in a year grows as well.
There is an alternate, simpler way to do the calculations that we performed on the previous page.

Notice that after 1 year, the number of stores for company B was:

\[ 200 + 0.50(200) = 250 \]

But, we could perform that calculation in a different manner which will, ultimately, make these calculations easier. Observe that there is a common factor of 200 in both terms.

Since there is a common factor of 200, that means that we can factor out the 200 from each term, and rewrite the expression as follows:

This expression can be simplified as follows:

In summary, observe that “50% more than 200” is the same as calculating “200(1.50)”. A simple way to see that this is definitely true is to observe that multiplying 200 by 1.50 is the same as finding “150% of 200.” It is (100% of 200) plus and additional (50% of 200).

We can think of this as “the new number of stores is the original 100% of the stores plus another 50% of that.”

### One-Step Process to Finding a New Amount after a Percent Increase

\[ P \text{ percent more than a number, } A, \text{ is } (100+P)\% \text{ of } A \]

Examples:
- 25% more than 60 = 125% of 60 = (1.25)(60) = 75
- 72% more than 1200 = 172% of 1200 = (1.72)(1200) = 2100
- 41% more than 675 = 141% of 675 = (1.41)(675) = 951.75
Now let’s revisit the calculations using this simplified calculation method:

To start, Company B had 200 stores.

After 1 year, Company B had $200(1.50) = 300$ stores.

After 2 years, Company B had $200(1.50)(1.50) = 200(1.50)^2 = 450$ stores

After 3 years, Company B had $200(1.50)(1.50)(1.50) = 200(1.50)^3 = 675$ stores

From this, we can generalize. Notice that to calculate a 50\% increase, each year we multiply by a factor of (1.50). So, our general equation would be

$$B(n) = 200(1.50)^n$$

where $n$ is the number of years that have passed, and $B(n)$ is the number of stores that Company B has that year.

In this equation, the 200 represented the initial quantity of stores, and the 0.50 was the percent growth rate.
Generalizing further, we arrive at the general form of exponential functions.

**Exponential Functions**

The equation of an exponential function can be written in the form

\[ f(x) = a(1 + r)^x \quad \text{or} \quad f(x) = a(b)^x \]

Where \( a \) is positive, and \( r \) is a decimal between 0 and 1.

\( a \) is the initial or starting value of the function when \( x = 0 \).

*The y-intercept is \((0, a)\).*

\( r \) is the **percent growth rate or percent decay rate, written as a decimal.** As \( x \) increases by 1 unit, \( y \) increases or decreases by the percentage rate.

*When \( r \) is positive, then the factor \( b \) will be more than 1, and the function \( f(x) \) is increasing.*

*When \( r \) is negative, then the factor \( b \) will be less than 1, and the function \( f(x) \) is decreasing.*

\( b \) is the **factor.** As \( x \) increases by 1 unit, \( y \) is multiplied by \( b \).

*When \( b \) is more than 1, then \( r \) is positive, and the function \( f(x) \) is increasing.*

*When \( b \) is less than 1, then \( r \) is negative, and the function \( f(x) \) is decreasing.*

**Example 1** (*Video Example Here*)

At the beginning of this section, we were given India’s population of 1.39 billion in the year 2019 and a percent growth rate of 1.1% each year. Write an exponential function for India’s population, and use it to predict the population in 2030.

**Solution:**

Using 2019 as our starting time \((t = 0)\), our initial population will be 1.39 billion. Since the percent growth rate was 1.1%, our value for \( r \) is 0.011, which means the factor, \( b \), is \( 1+0.011 = 1.011 \).

Using the basic formula for exponential growth \( f(x) = a(1 + r)^t \) we can write the formula,

\[ f(t) = 1.39(1 + 0.011)^t \]

which simplifies to \( f(t) = 1.39(1.011)^t \) where \( t \) is the number of years since 2019, and \( f(t) \) is the population of India in billions.

Notice that the initial value in the equation is 1.39, representing the “initial” population in 2019 being 1.39 billion. The 1.011 is the factor, which means the population of India is multiplied by 1.011 each year. The factor is greater than 1, which indicates that the population is increasing.

To estimate the population in 2030, we evaluate the function at \( t = 11 \), since 2030 is 11 years after 2019.

\[ f(11) = 1.39(1.011)^{11} \approx 1.57 \]

So, we predict that there will be 1.57 billion people in India in 2030.
Try it Now
1. Given the three statements below, identify which represent exponential functions. Explain how you know.
   A. The cost of living allowance for state employees increases salaries by 3.1% each year.
   B. State employees can expect a $300 raise each year they work for the state.
   C. Tuition costs have increased by 2.8% each year for the last 3 years.

2. Looking at these two equations $A(t) = 1000(1.05)^t$ and $B(t) = 900(1.075)^t$ that represent the balance in two different savings accounts after t years…
   (a) Identify the initial value, the percent rate, and the factor of each function.
   (b) Identify the amount in each account after 3 years.

In all the preceding examples, we saw exponential growth. Exponential functions can also be used to model quantities that are decreasing at a constant percent rate. An example of exponential decay is radioactive decay, a process in which radioactive isotopes of certain atoms transform to an atom of a different type, causing a percentage decrease of the original material over time.

Example 2 (*Video Example Here)
Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?

Solution:
With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is $a = 100$ mg, and our growth rate will be negative 13%, since we are decreasing: $r = -0.13$. This gives the equation:

$$Q(d) = 100\left(1 + (-0.13)\right)^d$$

$$Q(d) = 100(1 - 0.13)^d$$

$$Q(d) = 100(0.87)^d$$

This can also be explained by recognizing that if 13% of the isotope decays, then 87% of the isotope remains.

After one week, which is 7 days, the quantity remaining would be $Q(7) = 100(0.87)^7 = 37.73$ mg of Bismuth-210 remains.
A population of 1000 people is decreasing at a rate of 3% each year.
(a) Write an exponential function, \( P(t) \) that gives the population after \( t \) years.
(b) Identify the initial value, the constant rate, and the factor.
(c) Find the population in 30 years.

**Example 3**

\( T(q) \) represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured quarterly since January 1, 2010. Interpret all of the component of the equation \( T(2) = 86(1.64)^2 = 231.3056 \) in real-world, contextual terms.

**Solution:**
Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2010 this region had 86,000 Android smart phone contracts.

The factor is 1.64. This means that the number of smart phone contract is multiplied by 1.64 every quarter.

The rate is 0.64 = 64%. This means that the number of smart phone contracts grows by 64% every quarter.

\( T(2) = 231.3056 \). This means that in the 2\(^{nd} \) quarter (or at the end of the second quarter) there were approximately 231,305 Android smart phone contracts.
Finding Equations of Exponential Functions

In the previous examples, we were able to write equations for exponential functions since we knew the initial quantity and the growth rate. If we do not know the growth rate, but instead know only some input and output pairs of values, we can still construct an exponential function.

Example 4 (*Video Example Here*)

In 2002, 80 deer were reintroduced into a wildlife refuge area from which the population had previously been hunted to elimination. By 2008, the population had grown to 180 deer. If this population grows exponentially, find a formula, \( f(t) \), that gives the number of deer at \( t \) years.

Solution:
We will define \( t \) to be the years after 2002. That means that when \( t = 0 \), there were 80 deer. So, we know that the y-intercept of \( f(t) \) is the point (0, 80).

We know the initial value of the function is 80, and we know this is an exponential function which can be written in the form \( f(t) = a(b)^t \). Therefore, we can say that our equation is \( f(t) = 80(b)^t \) where we still need to find the value of \( b \) since it was not given in the problem.

How will we find the value of the factor, \( b \)? Notice that the information given in the problem also tells us that when \( t = 6 \), \( f(t) = 180 \). So, the point (6, 180) is on the graph of the function. To find the value of \( b \), we will use the fact that (0,80) and (6, 180) are both points on the graph.

Substituting in our second input-output pair allows us to solve for \( b \):

\[
f(t) = 80(b)^t \\
180 = 80b^6 \\
b^6 = \frac{180}{80} = \frac{9}{4} \\
b = \left(\frac{9}{4}\right)^{1/6}
\]

\( b \approx 1.44714243 \) \( \text{Solve using Excel.} \)

This gives us our equation for the population: \( f(t) \approx 80(1.44714243)^t \)

We can interpret this to mean that the population is growing by about 44.71% each year.
In the previous example, we were able to “give” ourselves the initial value by clever definition of our input variable. Next, we consider a situation where we are not given that information.

**Example 5** *(Video Example Here)*

Find a formula for an exponential function, \( f(x) \), passing through the points \((-2,6)\) and \((2,1)\).

**Solution:**

We will show two different methods for finding the value of the factor, \( b \). You can use whichever method makes sense to you!

**Method 1:**

The equation has the form \( f(x) = a(b)^x \).

Substituting the first point into the equation, we can say that \( f(x) = 6 = a(b)^{-2} \).

Substituting the second point into the equation, we can say that \( f(x) = 1 = a(b)^{2} \). We then solve each equation for \( a \):

\[
6 = a(b)^{-2} \\
\frac{6}{b^{-2}} = a \\
1 = a(b)^{2} \\
\frac{1}{b^{2}} = a \\
\text{Divide by } b^{-2} \text{ on both sides} \\
\text{Divide by } b^{-2} \text{ on both sides} \\
\text{Since both values equal } a, \text{ we can combine the two in order to solve for } b: \\
6 = b^{-2} \\
\frac{6}{b^{-2}} = a = \frac{1}{b^{2}} \\
\text{Divide both sides by } b^{2} \\
6b^{2} = 1b^{-2} \\
6 = \frac{b^{-2}}{b^{2}} \\
\text{Quotient rule: subtract exponents} \\
6 = b^{-2-2} \\
6 = b^{-4} \\
\text{Raise both sides to the } -\frac{1}{4} \text{ power} \\
6^{-1/4} = b \\
(6)^{-1/4} = b \\
b \approx 0.638943104
\]

**Method 2:**

The equation has the form \( f(x) = a(b)^x \). We also know that exponential functions have the following pattern:

every time \( x \) increases by 1, \( y \) is multiplied by \( b \).

Since this equation passes through the points \((-2, 6)\) and \((2, 1)\), we know that to travel from the first point to the second point the \( x \)-value increased from \( x = -2 \) up to \( x = 2 \) which is an increase of 4 units in the \( x \)-value.

Since the \( x \)-value increases by 4 units, that means that the \( y \)-value was multiplied by the factor, \( b \), a total of 4 times. So, we know that the \( y \)-value started at \( y = 6 \), and was then multiplied by \( b \) a total of 4 times, and ended at a \( y \)-value of 1. As an equation, that can be written as follows:

\[
6(b)^{4} = 1 \\
y-value \text{ starts at 6} \\
\text{Multiply by } b \text{ four times} \\
y-value \text{ ends at 1} \\
6(b)^{4} = 1 \\
\text{Divide both sides by } 6 \\
b^{4} = \frac{1}{6} \\
\text{Raise both sides to the } \frac{1}{4} \text{ power} \\
b = (\frac{1}{6})^{1/4} \\
b \approx 0.638943104
\]

\[
b \approx 0.638943104
\]
Lastly, we solve for the value of $a$, the initial value, by substituting one of the points on the graph into the equation and solving for $a$. It doesn’t matter which point you use: you can plug in (-2, 6), or you can plug in (2, 1).

$$f(x) \approx a(0.638943104)^x$$

Plug in one of the points on the graph. We will use (2, 1). So $x=2$ and $y=1$.

$$1 \approx a(0.638943104)^2$$

Calculate the value of $(0.638943104)^2$.

$$1 \approx a(0.40824829)$$

Divide both sides by 0.40824829.

$$a \approx 2.449489743$$

Putting this together gives the equation $f(x) \approx 2.4495(0.63894)^x$

---

**Try it Now**

4. Given the two points (1, 3) and (2, 4.5) find the equation of an exponential function that passes through these two points.
Example 6

Find an equation for the exponential function graphed below.

Solution:
The initial value for the function is not clear in this graph, so we will instead work using two clearer points. There are three fairly clear points: (−1, 1), (1, 2), and (3, 4). As we saw in the last example, two points are sufficient to find the equation for a standard exponential, so we will use the latter two points.

Method 1:
Substituting in (1,2) gives $2 = ab^1$
Substituting in (3,4) gives $4 = ab^3$

Solving the first equation for $a$ gives $a = 2/b$.
Solving the second equation for $a$ gives $a = 4/b^3$.
Setting the equations equal to each other gives $2/b = a = 4/b^3$.

Solving for $b$:
$2/b = 4/b^3$
Cross multiply
$2b^3 = 4b$
Divide to get b’s on one side, constants on opposite side
$b^3 = 4/b$
Simplify
$b^2 = 2$
Raise both sides to the $1/2$ power
$b = 2^{1/2} \approx 1.414213562$

Method 2:
We know that to travel from the first point, (1,2), to the second point (3, 4), the x-value increased 2 units. That means that the y-value was multiplied by the factor, b, a total of 2 times. So, we know that the y-value started at $y=2$, and was then multiplied by b a total of 2 times, and ended at a y-value of 4. As an equation, that can be written as follows:

$2(b)^2 = 4$

Now we can solve this equation for b.
$2(b)^2 = 4$
Divide both sides by 2
$b^2 = \frac{4}{2} = 2$
Raise both sides to the $1/2$ power
$b = (2)^{1/2} \approx 1.414213562$

Lastly, we can then go back and find a by plugging in one of the points on the graph and solving for $a$:

$f(x) = a(1.414213562)^x$
$2 = a(1.414213562)^1$
$a \approx 1.414213562$
So, $f(x) = 1.414213562(1.414213562)^x$
Common Error: Power Function vs. Exponential Function

Be careful to avoid the common error of mixing up power functions and exponential functions!

Power Functions have the form \( f(x) = a(x)^p \) where \( a \) and \( p \) are both real numbers.

Exponential Functions have the form \( f(x) = a(b)^x \) where \( a \) and \( b \) are both real numbers (with particular characteristics).

Example 7

For each function, identify each as either an exponential function or a power function. Also identify the basic features of each function.

(a) \( f(x) = 7(2)^x \)
(b) \( f(x) = 7(x)^2 \)
(c) \( f(x) = 85(x)^{0.25} \)
(d) \( f(x) = 85(0.25)^x \)

Solutions:

(a) \( f(x) = 7(2)^x \) is an exponential function of the form \( f(x) = a(b)^x \) where \( a = 7 \) and \( b = 2 \). This function is increasing since the factor, \( b=2 \), is greater than 1. The y-intercept is \((0, 7)\). The rate of growth is 100%.

(b) \( f(x) = 7(x)^2 \) is a power function of the form \( f(x) = a(x)^p \) where \( a = 7 \) and \( p = 2 \). This is a 2\textsuperscript{nd} degree power function, otherwise known as a quadratic function. It’s vertex is the same as its y-intercept, which is the point \((0,0)\), and that is also the only x-intercept.

(c) \( f(x) = 85(x)^{0.25} \) is a power function of the form \( f(x) = a(x)^p \) where \( a = 85 \) and \( p = 0.25 \). The function could also be written as \( f(x) = 85(x)^{1/4} = 85 \cdot \sqrt[4]{x} \). This function passes through the points \((0,0)\) and \((1,85)\).

(d) \( f(x) = 85(0.25)^x \) is an exponential function of the form \( f(x) = a(b)^x \) where \( a = 85 \) and \( b = 0.25 \). This function is decreasing since the factor, \( b = 0.25 \), is less than 1. The y-intercept is \((0,85)\). The rate of decrease is 75%.
Try it Now Answers

1. A & C are exponential functions because they grow by a constant percent rate.

2. (a) Account A has an initial starting amount of $1000, a growth rate of 5% per year, and the factor is 1.05 which means that the money in the account is multiplied by 1.05 each year. Account B has an initial starting amount of $900, a growth rate of 7.5% per year, and the factor is 1.075 which means that the money in the account is multiplied by 1.075 each year.
   (b) $A(3) = 1000(1.05)^3 = 1157.63$ and $B(3) = 900(1.075)^3 = 1118.07$. So, account A has $1,157.63 after 3 years and Account B has $1,118.07 after 3 years.

3. (a) $P(t) = 1000\left(1 + (-0.03)\right)^t$
   $P(t) = 1000(0.97)^t$
   (b) The initial value is 1000. The constant rate is $r = -3\% = -0.03$. The factor is 0.97.
   (c) $P(30) = 1000(0.97)^{30} = 401.0071$. There would be about 401 people after 30 years.

4. $f(x) = 2(1.5)^x$
Section 4.1 Exercises

1. Fill in the blank for each.
   a) 45% more than A is the same as multiplying \((\_\_\_\_\_\_\_\_\_) \cdot (A)\).
   b) 1.5% more than B is the same as multiplying \((\_\_\_\_\_\_\_\_) \cdot (B)\).
   c) 45% less than C is the same as multiplying \((\_\_\_\_\_\_\_\_) \cdot (C)\).
   d) 1.5% more than D is the same as multiplying \((\_\_\_\_\_\_\_\_) \cdot (D)\).

2. Fill in the blank for each, and also identify if it is “more” or “less” for each by circling either “more” or “less.”
   a) The calculation \((1.06) \cdot A\) will give a final answer that is \(\_\_\_\_\_\_\_\_\%\) \underline{more/less} \(\) than A.
   b) The calculation \((0.721) \cdot B\) will give a final answer that is \(\_\_\_\_\_\_\_\_\%\) \underline{more/less} \(\) than B.
   c) The calculation \((0.84) \cdot C\) will give a final answer that is \(\_\_\_\_\_\_\_\_\%\) \underline{more/less} \(\) than C.
   d) The calculation \((1.1) \cdot D\) will give a final answer that is \(\_\_\_\_\_\_\_\_\%\) \underline{more/less} \(\) than D.

For each table below, could the table represent a function that is linear, exponential, or neither?

3. \[
\begin{array}{cccc}
  x & 1 & 2 & 3 & 4 \\
  f(x) & 70 & 40 & 10 & -20 \\
\end{array}
\]

4. \[
\begin{array}{cccc}
  x & 1 & 2 & 3 & 4 \\
  h(x) & 70 & 49 & 34.3 & 24.01 \\
\end{array}
\]

5. \[
\begin{array}{cccc}
  x & 1 & 2 & 3 & 4 \\
  m(x) & 80 & 61 & 42.9 & 25.61 \\
\end{array}
\]

6. \[
\begin{array}{cccc}
  x & 1 & 2 & 3 & 4 \\
  g(x) & 40 & 32 & 26 & 22 \\
\end{array}
\]

7. \[
\begin{array}{cccc}
  x & 1 & 2 & 3 & 4 \\
  k(x) & 90 & 80 & 70 & 60 \\
\end{array}
\]

8. \[
\begin{array}{cccc}
  x & 1 & 2 & 3 & 4 \\
  n(x) & 90 & 81 & 72.9 & 65.61 \\
\end{array}
\]

9. A population numbers 11,000 organisms initially and grows by 8.5% each year. Write an exponential model for the population.

10. A population is currently 6,000 and has been increasing by 1.2% each day. Write an exponential model for the population.

11. The fox population in a certain region has an annual growth rate of 9 percent per year. It is estimated that the population in the year 2010 was 23,900. Estimate the fox population in the year 2018.

12. The amount of area covered by blackberry bushes in a park has been growing by 12% each year. It is estimated that the area covered in 2009 was 4,500 square feet. Estimate the area that will be covered in 2020.
13. A vehicle purchased for $32,500 depreciates at a constant rate of 5% each year. Determine the approximate value of the vehicle 12 years after purchase.

14. A business purchases $125,000 of office furniture which depreciates at a constant rate of 12% each year. Find the residual value of the furniture 6 years after purchase.

15. Find a formula for an exponential function passing through the two points. Round to four places or more if necessary.
   a) (0, 6) and (3, 750)
   b) (0, 3) and (2, 75)
   c) (7, 2000) and (10, 1400)
   d) (5, 9000) and (11, 9800)

23. A radioactive substance decays exponentially. A scientist begins with 100 milligrams of a radioactive substance. After 35 hours, 50 mg of the substance remains. How many milligrams will remain after 54 hours?

24. A radioactive substance decays exponentially. A scientist begins with 110 milligrams of a radioactive substance. After 31 hours, 55 mg of the substance remains. How many milligrams will remain after 42 hours?

25. A house was valued at $110,000 in the year 1985. The value appreciated to $145,000 by the year 2005. What was the annual growth rate between 1985 and 2005? Assume that the house value continues to grow by the same percentage. What did the value equal in the year 2010?

26. An investment was valued at $11,000 in the year 1995. The value appreciated to $14,000 by the year 2008. What was the annual growth rate between 1995 and 2008? Assume that the value continues to grow by the same percentage. What did the value equal in the year 2012?

27. A car was valued at $38,000 in the year 2003. The value depreciated to $11,000 by the year 2009. Assume that the car value continues to drop by the same percentage. What will the value be in the year 2013?

28. A car was valued at $24,000 in the year 2006. The value depreciated to $20,000 by the year 2009. Assume that the car value continues to drop by the same percentage. What will the value be in the year 2014?
29. In 1968, the U.S. minimum wage was $1.60 per hour. In 1976, the minimum wage was $2.30 per hour. Assume the minimum wage grows according to an exponential model $w(t)$, where $t$ represents the time in years after 1960.
   a. Find a formula for $w(t)$.
   b. What does the model predict for the minimum wage in 1960?
   c. If the minimum wage was $5.15 in 1996, is this above, below or equal to what the model predicts?
   d. Use Excel to find when the model predicts the minimum wage to be $15.00 per hour.

30. For each function, identify if it is an exponential function or a power function. Also identify the basic features of each function.
   a. $f(x) = 1200x^3$
   b. $f(x) = 950(1.03)^x$
   c. $f(x) = 395(x)^{0.8}$
   d. $f(x) = 41(0.9)^x$
Section 4.2 Graphs of Exponential Functions

Like with linear functions, the graph of an exponential function is determined by the values for the parameters in the function’s formula.

To get a sense for the behavior of exponentials, let us begin by looking more closely at the function \( f(x) = 2^x \). Listing a table of values for this function:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = 2^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3)</td>
<td>( 2^{-3} = \frac{1}{2^3} = 1/8 = 0.125 )</td>
</tr>
<tr>
<td>(-2)</td>
<td>( 2^{-2} = \frac{1}{2^2} = 1/4 = 0.25 )</td>
</tr>
<tr>
<td>(-1)</td>
<td>( 2^{-1} = \frac{1}{2^1} = 1/2 = 0.5 )</td>
</tr>
<tr>
<td>(0)</td>
<td>( 2^0 = 1 )</td>
</tr>
<tr>
<td>(1)</td>
<td>( 2^1 = 2 )</td>
</tr>
<tr>
<td>(2)</td>
<td>( 2^2 = 4 )</td>
</tr>
<tr>
<td>(3)</td>
<td>( 2^3 = 8 )</td>
</tr>
</tbody>
</table>

Notice that:
1) The y-values of the function are positive for all values of \( x \).
2) As \( x \) decreases, the y-values grow smaller, approaching zero, but they never become negative.
3) This is an example of exponential growth.

Looking at the function \( g(x) = (0.5)^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) = (0.5)^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3)</td>
<td>( (0.5)^{-3} = 2^3 = 8 )</td>
</tr>
<tr>
<td>(-2)</td>
<td>( (0.5)^{-2} = 2^2 = 4 )</td>
</tr>
<tr>
<td>(-1)</td>
<td>( (0.5)^{-1} = 2^1 = 2 )</td>
</tr>
<tr>
<td>(0)</td>
<td>( (0.5)^0 = 1 )</td>
</tr>
<tr>
<td>(1)</td>
<td>( (0.5)^1 = 0.5 )</td>
</tr>
<tr>
<td>(2)</td>
<td>( (0.5)^2 = 0.25 )</td>
</tr>
<tr>
<td>(3)</td>
<td>( (0.25)^3 = 0.125 )</td>
</tr>
</tbody>
</table>

Notice that:
1) The y-values of the function are positive for all values of \( x \).
2) As \( x \) increases, the y-values grow smaller, approaching zero, but they never become negative.
3) This is an example of exponential decay.
The graphs of $f(x)$ and $g(x)$ are shown below:

Consider a function of the form $f(x) = ab^x$. Since $a$, which we called the initial value in the last section, is the function value at an input of zero, $a$ will give us the vertical intercept of the graph. From the graphs above, we can see that an exponential graph will have a horizontal asymptote on one side of the graph, and can either increase or decrease, depending upon the factor. This horizontal asymptote will also help us determine the long run behavior and is easy to determine from the graph.

The graph will grow when the growth rate is positive, which will make the growth factor $b$ larger than one. When the growth rate is negative, the growth factor will be less than one.

**Graphical Features of Exponential Functions**

Graphically, in the exponential function $f(x) = a(b)^x$ where $a$ is positive:

- $(0, a)$ is the vertical intercept of the graph.
- $b$ is the factor. As $x$ increases by 1 unit, the $y$-value is multiplied by $b$.
  * The function will increase if $b > 1$.
    That is, as $x \to \infty$, $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to 0$.
  * The function will decrease if $0 < b < 1$.
    That is, as $x \to \infty$, $f(x) \to 0$, and as $x \to -\infty$, $f(x) \to \infty$.
- The graph will have a horizontal asymptote at $y = 0$.
  * If $f(x)$ is increasing then $f(x)$ has a horizontal asymptote of $y = 0$ is on the left tail.
  * If $f(x)$ is decreasing, then $f(x)$ has a horizontal asymptote of $y = 0$ is on the right tail.
- The graph will be concave up if $a > 0$.
- The domain of the function is all real numbers.
- The range of the function is $(0, \infty)$. 
Example 1

Sketch a graph of \( f(x) = 4(0.33)^x \).

**Solution:**

This graph will have a vertical intercept at \((0, 4)\).
The graph is decreasing since the factor, \( b=0.33 \), is between 0 and 1.
The percent rate of decrease is \( r = -67\% \).

As \( x \) increases by 1 unit, the \( y \)-value is multiplied by 0.33.
As \( x \) increases by 1 unit, the \( y \)-value decreases by 67%.

As \( x \to \infty \), \( f(x) \to 0 \), and as \( x \to -\infty \), \( f(x) \to \infty \). This means that there is a horizontal asymptote of \( y=0 \) on the right tail.

Example 2

To get a better feeling for the effect of \( a \) and \( b \) on the graph, examine the sets of graphs below. The first set shows various graphs, where \( a \) remains the same and we only change the value for \( b \).

In the graph to the left, the \( y \)-intercept of each function is \((0,1)\), but the factor, \( b \), is different for each function.

Notice that the closer the value of \( b \) is to 1, the less “steep” the graph will be.

In the graph to the left, the factor of each function is \( b=1.2 \), but the \( y \)-intercept is different for each function.

Notice that the closer the value of \( b \) is to 1, the less “steep” the graph will be.
Match each equation with its graph.

\[ f(x) = 2(1.3)^x \]
\[ g(x) = 2(1.8)^x \]
\[ h(x) = 4(1.3)^x \]
\[ k(x) = 4(0.7)^x \]

**Solution:**
The graph of \( k(x) \) is the easiest to identify, since it is the only equation with a growth factor less than one, which will produce a decreasing graph. The graph of \( h(x) \) can be identified as the only growing exponential function with a vertical intercept at (0,4). The graphs of \( f(x) \) and \( g(x) \) both have a vertical intercept at (0,2), but since \( g(x) \) has a larger growth factor, we can identify it as the graph increasing faster.

---

**Try it Now**

1. Sketch the following functions all together on the same axis by hand:
\[ f(x) = (2)^x, \quad g(x) = 2(2)^x, \quad h(x) = 2(1/2)^x. \]
Important Topics of this Section
Graphs of exponential functions

Try it Now Answers

1. 

\[ f(x) = 2^x \]
\[ g(x) = 2 \left( \frac{1}{2} \right)^x \]
\[ h(x) = 2 \left( \frac{1}{2} \right)^x \]
Section 4.2 Exercises

Match each function with one of the graphs below.

1. \( f(x) = 2(0.69)^x \)
2. \( f(x) = 2(1.28)^x \)
3. \( f(x) = 2(0.81)^x \)
4. \( f(x) = 4(1.28)^x \)
5. \( f(x) = 2(1.59)^x \)
6. \( f(x) = 4(0.69)^x \)

If all the graphs to the right have equations with form \( f(x) = ab^x \),

7. Which graph has the largest value for \( b \)?
8. Which graph has the smallest value for \( b \)?
9. Which graph has the largest value for \( a \)?
10. Which graph has the smallest value for \( a \)?

11. The population of a certain town is given by the function \( P(t) = 75,000(1.032)^t \) where \( t \) is the number of years after 2000. Answer the following questions.
   a) Identify the \( y \)-intercept, and write a sentence to interpret the contextual meaning of the \( y \)-intercept.
   b) Is the function increasing or decreasing? What is the percent rate of increase or decrease? Write a sentence to interpret the contextual meaning of the percent rate.
   c) Use Excel to create a graph of \( P(t) \) for the years 2000 up to 2020.
   d) Use Excel to find \( P(13) \). Write a sentence in context to interpret the contextual meaning.
   e) Use Excel to find \( P(t) = 100,000 \). Write a sentence in context to interpret the contextual meaning.

12. A new book is released to the public, and the number sold \( w \) weeks after its release is given by the function \( S(w) = 8000(0.65)^w \). Answer the following questions.
   a) Identify the \( y \)-intercept, and write a sentence to interpret the contextual meaning of the \( y \)-intercept.
   b) Is the function increasing or decreasing? What is the percent rate of increase or decrease? Write a sentence to interpret the contextual meaning of the percent rate.
   c) Use Excel to create a graph of \( S(w) \) for the first 15 weeks of the book’s sales.
   d) Use Excel to find \( S(7) \). Write a sentence in context to interpret the contextual meaning.
   e) Use Excel to find \( S(w) = 1 \). Write a sentence in context to interpret the contextual meaning.
Section 4.3 Compounding

Typically bank accounts and other savings instruments in which earnings are reinvested, such as mutual funds and retirement accounts, utilize compound interest. The term *compounding* comes from the behavior that interest is earned not on the original value, but on the total accumulated, compounded, value of the account.

Often a bank will advertise an annual interest rate, usually called the **nominal rate (the in-name-only rate)** or **annual percentage rate (APR)**. In the fine print, it is important to identify the number of times that the interest is compounded per year. If it is compounded monthly, then the annual rate is divided by 12 to find the monthly interest. If the interest is compounded daily, then the annual rate is divided by 365 (if we assume 365 days in a year) to find the daily interest.

Generalizing this, we can now give the general formula for compound interest. If the APR is written in decimal form as $r$, and there are $n$ compounding periods per year, then the interest per compounding period will be $r/n$. Likewise, if we are interested in the value after $t$ years, then there will be $n \cdot t$ compounding periods in that time.

<table>
<thead>
<tr>
<th>Compound Interest Formula</th>
</tr>
</thead>
</table>

| **Compound Interest** can be calculated using the formula $A(t) = a \left(1 + \frac{r}{n}\right)^{nt}$ where |
| $A(t)$ is the accumulated account value after $t$ years have passed. |
| $t$ is the number of years the money remains in the account |
| $a$ is the starting amount of the account, often called the principal |
| $r$ is the annual percentage rate (APR) for the year, also called the nominal rate |
| $n$ is the number of compounding periods in one year |

**Example 1** (*Video Example Here*)

If you invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

**Solution:**

Since we are starting with investing $3000, we know that $a = 3000$.  
Our annual interest rate is 3%, so $r = 0.03$.  
Since we are compounding quarterly, we are compounding 4 times per year, so $n = 4$.  
We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$A(10) = 3000 \left(1 + \frac{0.03}{4}\right)^{4 \cdot 10}$$

The account will be worth $4045.05 in 10 years.
Example 2

A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested. If a bank offers a 24-month CD with an annual interest rate of 1.2% compounded monthly, how much will a $1000 investment be worth at the end of the 24 months?

Solution:
In this case, the initial value is \( a = 1000 \), the APR is \( r = 0.12 \), the time invested is \( t = 2 \) years, and the number of compounding times per year is \( n = 12 \).

\[
A(t) = 1000 \left( 1 + \frac{0.12}{12} \right)^{12 \times 2}
\]

After 24 months, the account will have grown to $1024.28.

Example 3

A 529 plan is a college savings plan in which a relative can invest money to pay for a child’s later college tuition, and the account then grows tax free. If Lily wants to set up a 529 account for her new granddaughter, and she wants the account to grow to a total of $40,000 over 18 years, and she believes the account will earn 6% APR compounded semi-annually (twice a year), how much will Lily need to invest in the account now?

Solution:
The APR on the account is \( r = 0.06 \), and the time invested is \( t = 18 \) years, and the interest is compounded twice per year, which means \( n = 2 \).

In this problem, we don’t know the value of \( a \), which is the amount we are starting with, we do know the accumulated final value, \( A(18) = 40,000 \). So, we will be solving for \( a \), the initial amount needed. So, we need to solve for \( a \) in the following equation:

\[
40,000 = a \left( 1 + \frac{0.06}{2} \right)^{2 \times 18}
\]

Calculate the value of \( \left( 1 + \frac{0.06}{2} \right)^{2 \times 18} \)

\[
40,000 = a(2.898278328)
\]

Divide both sides by 2.898278328

\[
13,801.30 = a
\]

Lily will need to invest $13,801.30 to have $40,000 in 18 years.
Try it Now

1. If you invest $5000 for 20 years in an account that has a 3.2% APR that is compounded monthly, how much will you have in the account after 20 years?

2. Suppose you invest money in an account for 8 years, and the account has a 4.1% APR compounded daily. If you have a total of $30,000 in the account after 8 years, then how much did you originally invest in the account? Assume there are 365 days in the compounding year.

Because of compounding throughout the year, the interest you gain will be added to your account each compounding period, and then the following compounding period will calculate interest using the new total amount in your account. Each compounding period, you gain interest on your original investment, but also gain interest on your earned interest. Compounding the interest results in you gaining more than the advertised nominal interest rate (more than the advertised APR).

If $1,000 were invested at a 10% APR, the table below shows the value after 1 year at different compounding frequencies:

<table>
<thead>
<tr>
<th>Frequency of compounding</th>
<th>Value of the account after 1 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually compounding</td>
<td>(1000 \left(1 + \frac{0.1}{1}\right)^{1\times1} \approx $1100)</td>
</tr>
<tr>
<td>Semiannually compounding</td>
<td>(1000 \left(1 + \frac{0.1}{2}\right)^{2\times1} \approx $1102.50)</td>
</tr>
<tr>
<td>Quarterly compounding</td>
<td>(1000 \left(1 + \frac{0.1}{4}\right)^{4\times1} \approx $1103.81)</td>
</tr>
<tr>
<td>Monthly compounding</td>
<td>(1000 \left(1 + \frac{0.1}{12}\right)^{12\times1} \approx $1104.71)</td>
</tr>
<tr>
<td>Daily compounding</td>
<td>(1000 \left(1 + \frac{0.1}{365}\right)^{365\times1} \approx $1105.16)</td>
</tr>
</tbody>
</table>

Now consider the actual percentage increase that occurred for each compounding frequency listed above. Recall that percent increase is calculated as \(\frac{\text{amount of increase}}{\text{original amount}}\).
This actual percent increase over the course of one year is called the **annual percentage yield** (APY).

### Annual Percentage Yield (APY)

The **annual percentage yield** (APY) is the actual percent a quantity increases in one year. It can be calculated as \( \text{APY} = \left(1 + \frac{r}{k}\right)^k - 1 \)

Notice this is equivalent to finding the value of $1 after 1 year, and subtracting the original dollar.

### Example 4

Use the annual percentage yield formula to re-calculate the APY for each problem from the previous page (fill in the column on the right).

<table>
<thead>
<tr>
<th>Frequency of compounding</th>
<th>Value of the account after 1 year</th>
<th>APY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually compounding</td>
<td>$1100</td>
<td></td>
</tr>
<tr>
<td>Semiannually compounding</td>
<td>$1102.50</td>
<td></td>
</tr>
<tr>
<td>Quarterly compounding</td>
<td>$1103.81</td>
<td></td>
</tr>
<tr>
<td>Monthly compounding</td>
<td>$1104.71</td>
<td></td>
</tr>
<tr>
<td>Daily compounding</td>
<td>$1105.16</td>
<td></td>
</tr>
</tbody>
</table>

**Solution:**

<table>
<thead>
<tr>
<th>Frequency of compounding</th>
<th>Value of the account after 1 year</th>
<th>Percent increase over 1 year (from starting value of $1000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually compounding</td>
<td>$1100</td>
<td>( \left(1 + \frac{0.1}{1}\right)^1 - 1 = 0.1 = 10%</td>
</tr>
<tr>
<td>Semiannually compounding</td>
<td>$1102.50</td>
<td>( \left(1 + \frac{0.1}{2}\right)^2 - 1 = 0.1025 = 10.25%</td>
</tr>
<tr>
<td>Quarterly compounding</td>
<td>$1103.81</td>
<td>( \left(1 + \frac{0.1}{4}\right)^4 - 1 = 0.10381 = 10.381%</td>
</tr>
<tr>
<td>Monthly compounding</td>
<td>$1104.71</td>
<td>( \left(1 + \frac{0.1}{12}\right)^{12} - 1 = 0.10471 = 10.471%</td>
</tr>
<tr>
<td>Daily compounding</td>
<td>$1105.16</td>
<td>( \left(1 + \frac{0.1}{365}\right)^{365} - 1 = 0.10516 = 10.516%</td>
</tr>
</tbody>
</table>

These are the same values that we found on the previous page when we calculated the percent increase over the course of one year.
Example 5  [*Video Example Here]*

Bank A offers an account paying 1.2% compounded quarterly. Bank B offers an account paying 1.1% compounded monthly. Which is a better offer for the customer?

Solution:
We can compare these rates using the annual percentage yield, which is the actual percent increase in a year.

Bank A:

\[
APY = \left(1 + \frac{0.012}{4}\right)^4 - 1 = 0.012054 = 1.2054\%
\]

Your investment in Bank A will increase 1.2054% over the course of 1 year.

Bank B:

\[
APY = \left(1 + \frac{0.011}{12}\right)^{12} - 1 = 0.011056 = 1.1056\%
\]

Your investment in Bank B will increase 1.1056% over the course of 1 year.

Bank B’s monthly compounding is not enough to catch up with Bank A’s better APR. It is better for the customer to invest in Bank A.

---

Try it Now

3. If you invest $5000 for 20 years in an account that has a 3.2% APR that is compounded monthly, then what is the actual percentage that this investment will increase over the course of one year?

4. Suppose you invest money in an account for 8 years, and the account has a 4.1% APR compounded daily. What is the APY for this investment? Assume there are 365 days in the compounding year.
Important Topics of this Section

- Compound interest
- Nominal rate, annual percentage rate (APR)
- Annual Percent Yield

Try it Now Answers

1. $9,474.33 in the account at the end of the 8 years

2. You’d need to invest $21,611.29.

3. The APY, which is the percent it will increase in a year, is 3.247%.

4. The APY is 4.185% increase per year.
Section 4.3 Exercises

1. If $4,000 is invested in a bank account at an interest rate of 7 per cent per year…
   a. Find the amount in the bank after 9 years if interest is compounded annually. Also find the APY.
   b. Find the amount in the bank after 9 years if interest is compounded quarterly. Also find the APY.
   c. Find the amount in the bank after 9 years if interest is compounded daily. Also find the APY.

2. Suppose $6,000 is in an account with an APR of 9%. If the money had been held in the account for 5 years, the…
   a. Find the amount that was originally invested in the account 5 years ago, if the interest had been compounded annually.
   b. Find the amount that was originally invested in the account 5 years ago, if the interest had been compounded quarterly.
   c. Find the amount that was originally invested in the account 5 years ago, if the interest had been compounded daily.

3. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 3% compounded quarterly.

4. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 5% compounded monthly.
Section 4.4 Continuous Compounding

A Limit to Compounding
As we saw in the previous section, the amount we earn in an investment account increases as we increase the compounding frequency. The table below, though, shows that the increase from annual to semi-annual compounding (from $n = 1$ to $n = 2$) is a larger increase than the increase from monthly to daily compounding (from $n = 12$ to $n = 365$). In general, increasing the frequency of compounding will increase our result, but there is an upper limit to this process.

To see this, let us examine the value of $1$ invested at 100% interest for 1 year.

<table>
<thead>
<tr>
<th>Frequency of compounding</th>
<th>Value in account after 1 year (100% interest, starting with $1$ investment)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual</td>
<td>$2$</td>
</tr>
<tr>
<td>Semiannually</td>
<td>$2.25$</td>
</tr>
<tr>
<td>Quarterly</td>
<td>$2.441406$</td>
</tr>
<tr>
<td>Monthly</td>
<td>$2.613035$</td>
</tr>
<tr>
<td>Daily</td>
<td>$2.714567$</td>
</tr>
<tr>
<td>Hourly</td>
<td>$2.718127$</td>
</tr>
<tr>
<td>Once per minute</td>
<td>$2.718279$</td>
</tr>
<tr>
<td>Once per second</td>
<td>$2.718282$</td>
</tr>
</tbody>
</table>

As the number of compounding periods per year increases, the amount in the account increases, but the amount of increase is getting smaller and smaller. These values do indeed appear to be approaching an upper limit. This value, which is approximately 2.71828, ends up being so important that it gets represented by its own mathematical constant, $e$, where $e \approx 2.718282$.

Euler’s Number, $e$

- $e$ is the letter used to represent the value that $\left(1 + \frac{1}{n}\right)^n$ approaches as $n$ gets large ($\to \infty$).
- $e$ is an irrational number: $e \approx 2.718282$
- The compounding function $A(t) = 1 \left(1 + \frac{1}{n}\right)^{nt}$ begins to approach $A(t) = 1e^t$ as the compounding period, $n$, gets larger and larger.
- The compounding function $A(t) = a \left(1 + \frac{r}{n}\right)^{nt}$ begins to approach $A(t) = ae^{rt}$ as the compounding period is “continuous”, we say that it is **compounding continuously**, and we use the formula $A(t) = ae^{rt}$.
- As we compound more and more frequently, we get closer and closer to “compounding continuously.”
**Continuous Compounding Functions**

Continuous Compounding can be calculated using the formula $f(x) = ae^{rx}$ where

- $a$ is the initial amount. $(0, a)$ is the y-intercept.
- $r$ is the continuous growth rate.
  *When $r$ is positive, the function is increasing.*
  *When $r$ is negative, the function is decreasing.*

We say that the function is “growing/decaying at a continuously compounding rate” when using this formula.

---

**Try it Now**

1. Write a complete sentence to interpret each value in the function $S(t) = 20e^{0.12t}$ if $S(t)$ represents the growth of a substance in grams, and time is measured in days.

Because $e$ is a commonly used base for exponential functions, most scientific and graphing calculators have a button that can calculate powers of $e$, usually labeled $e^x$. Microsoft Excel calculations use the $=\text{EXP}( )$ formula to calculate expressions involving $e$.

---

**Example 1**

Use Excel to perform the following calculations.

a) $e^3$

b) $500e^{-0.04*15}$

**Solution:**

a) In Excel, to perform the calculation $e^3$, we need to type $=\text{EXP}(3)$. We find that $e^3 \approx 20.08553692$.  

b) In Excel, to perform the calculation $500e^{-0.04*15}$, we need to type $=\text{EXP}(-0.04 * 15)$. We find that $500e^{-0.04*15} \approx 274.405818$. 

---
The continuously compounding equation, \( f(x) = ae^{rx} \), is commonly used when describing quantities that change more or less continuously, like chemical reactions, growth of large populations, and radioactive decay.

**APY for Continuous Compounding**

To calculate the APY for continuous compounding: \( APY = e^r - 1 \)

**Example 2 ([Video Example Here])**

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

**Solution:**
First it is important to note that the words “continuous rate” were used. That automatically tells us that we need to use the function \( f(x) = ae^{rx} \).
*In this case, the continuous rate is given “per day” which tells us that \( x \) represents days.  
*The initial value is 100.
* Since the substance is decaying, we know the growth rate will be negative: \( r = -0.173 \)

\[ f(3) = 100e^{-0.173(3)} \approx 59.512 \text{ mg of Radon-222 will remain after 3 days.} \]

**Example 3 ([Video Example Here - uses calculator])**

If $1000 is invested in an account earning 7.2% APR compounded continuously, find the value after 8 years, and find the APY.

**Solution:**
We have a continuous growth rate, and so we need to use the formula \( f(x) = ae^{rx} \).
The rate is \( r = 7.2\% = 0.072 \).
The initial value is \( a = 1000 \).
To find the value after 8 years, we have \( f(8) = 1000e^{0.072\times8} \approx 1,778.91 \)

The APY is \( e^{0.072} - 1 \approx 0.074655 = 7.4655\% \)

There is $1,778.91 in the account after 8 years. The account is growing with an annual percentage yield of 7.4655% each year.
Try it Now

2. Suppose a population starts with 40,000 people and grows at a continuously compounding rate of 2.7% each year. Find a formula for the population, \( P(t) \), after \( t \) years. Find the population after 10 years.

3. Suppose a company is able to “sell” 60,000 items when they give away the items for free. Also, suppose that the number of items sold decreases at a continuously compounding rate of 18% for every dollar that they increase the price. Find a formula, \( Q(p) \), that gives the number of items sold when they charge \( p \) dollars per item. Find the number of items they would sell if they charged $4.25 per item.

### Approximating \( f(x) = ae^{rx} \) in the form \( f(x) = a(b)^x \)

The continuously compounding function \( f(x) = ae^{rx} \) can be approximated in the form \( f(x) = a(b)^x \) by noting the following:

\[
\begin{align*}
   f(x) &= ae^{rx} \\
   f(x) &= a(e^r)^x \quad \text{(power rule of exponents)} \\
   f(x) &= a(e^r)^x \quad \text{(calculate the value of } e^r \text{ and round it many decimal places)} \\
   f(x) &= a(e^r)^x \approx a(b)^x \quad \text{where } b \approx e^r
\end{align*}
\]

### Example 4

Rewrite each continuously compounding form in the form \( f(x) = a(b)^x \) where \( b \) is approximated to at least 6 places.

a. \( f(x) = 1350e^{0.075x} \)

b. \( g(x) = 12,300e^{-0.09x} \)

Solution:

a. \( f(x) = 1350e^{0.075x} \)

\[
\begin{align*}
   f(x) &= 1350(e^{0.075})^x \\
   f(x) &\approx 1350(1.077884)^x \quad \text{where } 1.077884 \approx e^{0.075}
\end{align*}
\]

\[
\begin{array}{c|c}
   x & 1.077884151 \\
   \hline
   1 & 1.077884151
\end{array}
\]

b. \( g(x) = 12,300e^{-0.09x} \)

\[
\begin{align*}
   g(x) &= 12,300(e^{-0.09})^x \\
   g(x) &\approx 12,300(0.917431)^x \quad \text{where } 0.917431 \approx e^{-0.09}
\end{align*}
\]
b. \( f(x) = 12,300e^{-0.09x} \)

\[
f(x) = 12,300(e^{-0.9})^x
\]

(power rule of exponents)

\[
f(x) = 12,300(e^{-0.09})^x \approx 12,300(0.913931)^x
\]

(calculate the value of \( e^{0.075} \) and round it 6 decimal places)

Important Topics of this Section
Continuous compounding
APY for continuous compounding
Approximating \( a e^{rx} \) in the form \( a(b)^x \)

Try it Now Answers
1. An initial substance weighing 20g is growing at a continuous rate of 12% per day.

2. \( P(t) = 40,000e^{0.027t} \)

\[
P(10) = 40,000e^{0.027 \times 10} \approx 52,399.
\]

There are about 52,399 people after 10 years.

3. \( Q(p) = 60,000e^{-0.18p} \)

\[
Q(4.25) = 60,000e^{-0.18 \times 4.25} \approx 27,920.04.
\]

If they charged $4.25 per item, then they would sell about 27,920 items.
Section 4.4 Exercises

1. A population numbers 11,000 organisms initially and grows at a continuously compounding rate of 8.5% each year.
   a. Write an exponential model for the population, P(t), and identify the meaning of t and P(t) in your model.
   b. Find P(7) and interpret in context.
   c. Use Excel to solve P(t)=20,000. Interpret in context.
   d. Re-write the function in the form \( P(t) = a(b)^t \) where b is rounded to six decimal places.

2. A population is currently 6,000 and has been increasing at a continuously compounding rate of 1.2% each day. Write an exponential model for the population.
   a. Write an exponential model for the population, P(t), and identify the meaning of t and P(t) in your model.
   b. Find P(13) and interpret in context.
   c. Use Excel to solve P(t)=8,000. Interpret in context.
   d. Re-write the function in the form \( P(t) = a(b)^t \) where b is rounded to six decimal places.

3. The fox population in a certain region is decreasing at a continuously compounding rate of 7.3% per year, and the population in the year 2010 was 23,900.
   a. Write an exponential model for the population, P(t), and identify the meaning of t and P(t) in your model.
   b. Find the fox population in the year 2018.
   c. Use Excel to solve P(t)=10,000. Interpret in context.
   d. Re-write the function in the form \( P(t) = a(b)^t \) where b is rounded to six decimal places.

4. The amount of area covered by blackberry bushes in a park has been growing by 12% continuously compounding each year, and the area covered in 2009 was 4,500 square feet.
   a. Write an exponential model for the population, A(t), and identify the meaning of t and A(t) in your model.
   b. Find A(11) and interpret in context.
   c. Use Excel to solve A(t)=7,000. Interpret in context.
   d. Re-write the function in the form \( A(t) = a(b)^t \) where b is rounded to six decimal places.

5. A vehicle purchased for $32,500 depreciates at a continuously compounding rate of 5% each year. Determine the approximate value of the vehicle 12 years after purchase.

6. A business purchases $125,000 of office furniture which depreciates at a continuously compounding rate of 12% each year. Find the residual value of the furniture 6 years after purchase.
7. If $4,000 is invested in a bank account at an interest rate of 7% continuously compounding per year, find the amount in the bank after 9 years.

8. If $6,000 is invested in a bank account at an interest rate of 9% continuously compounding per year, find the amount in the bank after 5 years.

9. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 3% compounded continuously.

10. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 5% compounded continuously.

11. A population of bacteria is growing according to the equation $P(t)=1600e^{0.21t}$, with $t$ measured in years. Use Excel to find when the population will exceed 7569.

12. A population of bacteria is growing according to the equation $P(t)=1200e^{0.17t}$, with $t$ measured in years. Use Excel to find when the population will exceed 3443.

For problems #13 - #18, match each function with one of the graphs below.

13. $f(x) = 7e^{-0.31x}$

14. $f(x) = 7e^{0.28x}$

15. $f(x) = 7e^{-0.19x}$

16. $f(x) = 9e^{0.28x}$

17. $f(x) = 7e^{0.59x}$

18. $f(x) = 9e^{0.69x}$
Section 4.5 Fitting Exponential Models to Data

Recall that in Chapter 2 we were able to use Excel to find a best-fit linear equation for data that appeared to be somewhat linear in its pattern. We can perform a similar procedure in Excel to find a best-fit exponential equation for data that appears to be somewhat exponential in its pattern.

Excel is able to produce a best-fit equation using whichever type of function the user selects. But that does not mean that the type of equation selected by the user is actually a good fit. It is important for us to first examine the pattern in the data in order to determine which type of function makes the most sense to use as a model.

The functions that we have already discussed so far in this class are also available in Excel as best-fit functions. Recall the function basic patterns we have considered so far:

- Linear Functions
- Quadratic Functions (2nd degree polynomials, called “2nd order polynomials” in Excel)
- Cubic Functions (3rd degree polynomials, called “3rd order polynomials” in Excel)
- Power Functions
- Exponential Functions

Make sure that you always carefully examine the pattern of the data in the scatterplot and compare it to the general shape of these functions in order to determine which function makes the most sense as a best-fit model. Also make sure that you carefully consider the real-world, contextual meaning of the data and match the best-fit function behavior to that real-world scenario as best as possible.

Example 1
Given the Excel scatterplot shown below, which type of function or functions would make sense to use as a best-fit model?

Solution:
This data is increasing and concave up. An “opening upward” quadratic function would behave in that manner on the right side of the parabola. An exponential function would behave in that manner if it were an increasing exponential function. Since we have no evidence that this data was first decreasing and then increasing (like a parabola), we conclude that a good first function to try as a best-fit model is the exponential function. However, a quadratic function (a 2nd degree polynomial) may be the best fit if we also point out that the quadratic best-fit model only makes sense as a model on a restricted domain of values.
Example 2  

The table below shows the population of a certain city over a period of time. Use Excel to plot this data, determine the type of pattern the data appears to have, and find an equation that best fits the data.

<table>
<thead>
<tr>
<th>Years since 2000</th>
<th>0</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>11</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>population</td>
<td>1,300,852</td>
<td>533,620</td>
<td>420,851</td>
<td>140,290</td>
<td>111,655</td>
<td>23,384</td>
</tr>
</tbody>
</table>

Solution:

1) In Excel, first create the input-output table, and plot using the scatterplot feature. The scatterplot appears to be decreasing and concave up. Since it has that shape, it makes sense to find an exponential function to best model the data.

2) To find the best-fit exponential equation, click on one of the points in the scatterplot. Then right-click and select “Add Trendline…”

3) Select “Exponential” under Trendline Options.

4) At the bottom of the Trendline Options, select “Display Equation on Chart.”

5) Click on the equation that is now displayed within the scatterplot so that the textbox containing the equation is selected. Right click on that equation textbox, and select “Format Trendline Label…”.

6) Under “Label Options”...

   Select “Number”...

   Enter the number of decimal places that you’d like the values in the equation to display.

(continued on next page…)
The best fit exponential equation is 
\[ P(t) = 1,297,824.47234e^{-0.22306t} \]
where \( t \) is the number of years since 2000 and \( P(t) \) is the population of the city.

Notice that the best-fit equation does not output exactly the same values that were observed in reality.

For example, \( P(0) \approx 1,297,824 \). So, according to the model, the population of the city would be about 1,297,824 people in the year 2000. But, in reality, the population was 1,300,852 in the year 2000 (from the actual data in the table).

Remember that the best-fit equation is only a model. It can be used to make predictions, but we always must recognize that the predictions will likely have error in them. We just want to do the best we can to select a model that resembles reality as closely as possible so that the predictions come as close to reality as possible.
The table below gives the number of items sold, $Q$, by a certain company when they price the items at $p$ dollars each. Use Excel to find the best-fit exponential function for the given data. Write the equation in both $f(x) = a(b)^x$ form, and also in $f(x) = ae^{rx}$ form. Then write contextual sentences to explain the parameters in each function.

<table>
<thead>
<tr>
<th>price per item, $p$</th>
<th>Number sold, $Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6340</td>
</tr>
<tr>
<td>2</td>
<td>2200</td>
</tr>
<tr>
<td>5</td>
<td>600</td>
</tr>
<tr>
<td>8</td>
<td>150</td>
</tr>
<tr>
<td>9</td>
<td>95</td>
</tr>
<tr>
<td>12</td>
<td>18</td>
</tr>
<tr>
<td>15</td>
<td>7</td>
</tr>
</tbody>
</table>

**Solution:**

The first step is to enter the data in Excel and create a scatterplot:

Next, we click on one of the points in the scatterplot, right click and select “Add Trendline.” Since this data is decreasing concave up, an exponential function makes sense as a best-fit model.

So, we select “Exponential” in Excel, and we also select “Display Equation on chart.”

(continued on next page)
Clicking on the given equation and selecting “Format Trendline Label”, we change the rounding to 3 decimal places within the equation. This gives a best-fit equation with the formula

\[ Q(p) = 5,892.121e^{-0.461p} \]

where \( p \) represents the price per item, and \( Q(p) \) represents the number sold.

In the equation \( Q(p) = 5,892.121e^{-0.461p} \), the y-intercept is \((0, 5892.121)\) which tells us that they predict sales to be about 5,892 items sold if they give the items away for free (0 dollars per item). We also see that this model predicts that sales will decrease at a continuously compounding rate of 46.1% per dollar increase in the price of each item.

We can rewrite the formula \( Q(p) = 5,892.121e^{-0.461p} \) in the form \( Q(p) = a(b)^p \) by using Excel to estimate the value of \( e^{-0.461} \).

So, we can rewrite the function formula as \( Q(p) = 5,892.121(0.6307)^p \). This has the same y-intercept. But, the 0.6307 in the formula tells us that increasing the price by $1 per item will result in selling 63.07% of the items they sold at the previous price. This is a rate of −36.93% which means that increasing the price by $1 per item will result in selling 36.93% fewer items.
Section 4.5 Exercises

1. Use Excel to find the best-fit exponential equation for each data set below, and then answer the questions for each. Round the values in the equation to five decimal places.

   a) Find a best-fit exponential equation, \( P(t) \), for the data below. Then answer the questions about the function.

<table>
<thead>
<tr>
<th>Years since 2000</th>
<th>Population of a town in thousands</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>438.7</td>
</tr>
<tr>
<td>2</td>
<td>219.6</td>
</tr>
<tr>
<td>5</td>
<td>82.4</td>
</tr>
<tr>
<td>8</td>
<td>23.1</td>
</tr>
<tr>
<td>13</td>
<td>6.9</td>
</tr>
</tbody>
</table>

   i) Find, and give the real world meaning of \( P(0) \). Explain why it is different than the value given in the table.
   ii) Identify the continuously compounding percentage rate for \( P(t) \), and write a sentence to give the real-world, contextual meaning of that rate.
   iii) Rewrite the function in the form \( y = a(b)^x \) and write a contextual sentence to interpret the meaning of \( b \) and the meaning of the rate for this function.
   iv) Use the model to predict the population in the year 2010. Do you trust the model is probably giving a fairly accurate prediction of the population in 2010? Explain.
   v) Use the model to predict the population in the year 1950. Do you trust the model is probably giving a fairly accurate prediction of the population in 1950? Explain.

   b) Find a best-fit exponential equation, \( C(y) \), for the data below. Then answer the questions about the function.

<table>
<thead>
<tr>
<th>Years since 1900</th>
<th>Cost of milk in dollars/gallon</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>0.47</td>
</tr>
<tr>
<td>50</td>
<td>0.83</td>
</tr>
<tr>
<td>70</td>
<td>1.32</td>
</tr>
<tr>
<td>85</td>
<td>2.20</td>
</tr>
</tbody>
</table>

   i) Find, and give the real world meaning of \( C(0) \). Explain why it is different from the value given in the table.
   ii) Identify the continuously compounding percentage rate for \( C(y) \), and write a sentence to give the real-world, contextual meaning of that rate.
   iii) Rewrite the function in the form \( y = a(b)^x \) and write a contextual sentence to interpret the meaning of \( b \) and the meaning of the rate for this function.
   iv) Use the model to predict the price of milk in the year 2020. Do you trust the model is probably giving a fairly accurate prediction of the population in 2020? Explain.
   v) Use the model to predict the price of milk in the year 1780. Do you trust the model is probably giving a fairly accurate prediction of the population in 1780? Explain.
2. For each data set given below, identify the type of function that should be used as the best-fit equation (linear, quadratic, cubic, power, or exponential), and then use Excel to find that best fit equation. Round the values in the equation to five decimal places.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>245.6</td>
</tr>
<tr>
<td>14</td>
<td>243.3</td>
</tr>
<tr>
<td>20</td>
<td>240.5</td>
</tr>
<tr>
<td>35</td>
<td>233.1</td>
</tr>
<tr>
<td>54</td>
<td>223.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>330</td>
</tr>
<tr>
<td>4</td>
<td>600</td>
</tr>
<tr>
<td>8</td>
<td>1500</td>
</tr>
<tr>
<td>13</td>
<td>6100</td>
</tr>
<tr>
<td>18</td>
<td>21500</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3020</td>
</tr>
<tr>
<td>7</td>
<td>5888</td>
</tr>
<tr>
<td>9</td>
<td>4990</td>
</tr>
<tr>
<td>13</td>
<td>5760</td>
</tr>
<tr>
<td>18</td>
<td>16950</td>
</tr>
<tr>
<td>23</td>
<td>46020</td>
</tr>
</tbody>
</table>
Section 5.1 Composition of Functions

Suppose we wanted to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The temperature depends on the day, and the cost depends on the temperature.

The first function, \( C(T) \), gives the cost \( C \) of heating a house when the average daily temperature is \( T \) degrees Celsius. The second function, \( T(d) \), gives the average daily temperature (in degrees Celsius) on day \( d \) of the year in some city.

If we wanted to determine the cost of heating the house on the 5th day of the year, we could evaluate \( T(5) \) to determine the average daily temperature on the 5th day of the year. Then we could then use that temperature, \( T(5) \) as the input to the \( C(T) \) function to find the cost to heat the house on the 5th day of the year: \( C(T(5)) \). We can use the output of the first function as the input of the second function.

<table>
<thead>
<tr>
<th>Composition of Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>When the output of one function is used as the input of another, we call the entire operation a composition of functions. We write ( f(g(x)) ), and read this as “( f ) of ( g ) of ( x ).”</td>
</tr>
</tbody>
</table>

An alternate notation for composition uses the composition operator, \( \circ \)

\[
(f \circ g)(x) \text{ is read “} f \text{ of } g \text{ of } x \text{” just like } f(g(x)).
\]

Video Link: An Overview of Function Composition.

Example 1

Suppose \( c(s) \) gives the number of calories burned doing \( s \) sit-ups, and \( s(t) \) gives the number of sit-ups a person can do in \( t \) minutes. Interpret \( c(s(3)) \) in real-world, contextual terms.

Solution:
The inside expression in the composition is \( s(3) \). Since the input to the \( s \) function is time, \( t \), we know that the \( 3 \) is representing 3 minutes. The output, \( s(3) \), is the number of sit-ups that can be done in 3 minutes.

Taking this output number of sit-ups, \( s(3) \), and using it as the input to the \( c(s) \) function will gives us \( c(s(3)) \). It will input the value \( s(3) \), which is inputting the number of sit-ups, and it will output the value \( c(s(3)) \), which is the number of calories burned.

Therefore, \( c(s(3)) \) will output the number of calories that can be burned by this person doing sit-ups for 3 minutes.
Note that it is essential that the meaning and the units on the output of the inside function match the meaning and units on the input to the outside function. In other words, the inside function needs to output the exact thing (unit) that is input to the outside function.

**Example 2**

Suppose $f(x)$ gives the total miles that can be driven in $x$ hours, and suppose that $g(y)$ gives the gallons of gas used in driving $y$ miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

**Solution:**
The expression $g(y)$ inputs miles, and outputs number of gallons.

The function $f(x)$ inputs number of hours, and outputs number of miles.

We cannot take the output of $g(y)$, which are gallons, and use it as the input of $f(x)$, which are hours. So, $f(g(y))$ does not make sense.

We can take the output of $f(x)$, which are miles, and use it as the input of $g(y)$, which are also miles. So, $g(f(x))$ makes sense. The function $g(f(x))$ would allow us to input hours driven, and ultimately output gallons used.

**Try it Now**
1. In a department store you see a sign that says 50% off clearance merchandise. So final cost $C$ depends on the clearance price, $p$, according to the function $C(p)$. Clearance price, $p$, depends on the original discount, $d$, given to the clearance item, $p(d)$. Interpret $C(p(d))$ in context.
Composition of Functions using Tables and Graphs

When working with functions given as tables and graphs, we can look up values for the functions using a provided table or graph. We start evaluation from the provided input, and first evaluate the inside function. We can then use the output of the inside function as the input to the outside function. To remember this, always work from the inside out.

Example 3 (*Video Example Here)

Using the tables below, evaluate \( f(g(3)) \) and \( g(f(4)) \)

\[
\begin{array}{c|c}
 x & f(x) \\
 1 & 6 \\
 2 & 8 \\
 3 & 3 \\
 4 & 1 \\
\end{array}
\quad
\begin{array}{c|c}
 x & g(x) \\
 1 & 3 \\
 2 & 5 \\
 3 & 2 \\
 4 & 7 \\
\end{array}
\]

Solution:
To evaluate \( f(g(3)) \), we start from the inside with the input value 3. We evaluate the inside expression \( g(3) \) using the table which tells us that \( g(3) = 2 \).

We can then use that result as the input to the \( f \) function.

So \( g(3) \) is replaced by the equivalent value 2: \( f(g(3)) = f(2) \)

Lastly, we evaluate \( f(2) \) using the table which tells us that \( f(2) = 8 \).

\( f(g(3)) = f(2) = 8 \).

So \( f(g(3)) = 8 \).

To evaluate \( g(f(4)) \), we start from the inside with the input value 4. We evaluate the inside expression \( f(4) \) using the table which tells us that \( f(4) = 1 \).

We can then use that result as the input to the \( g \) function.

So \( f(4) \) is replaced by the equivalent value 1: \( g(f(4)) = g(1) \)

Lastly, we evaluate \( g(1) \) using the table which tells us that \( g(1) = 3 \).

\( g(f(4)) = g(1) = 3 \).

So \( g(f(4)) = 3 \).

Try it Now

2. Using the tables from the example above, evaluate \( f(g(1)) \) and \( g(f(3)) \).
Example 4: Using the graphs below, evaluate \( f(g(1)) \).

\[ f(x) \quad g(x) \]

**Solution:**
To evaluate \( f(g(1)) \), we start from the inside with the input value 1.
We evaluate the inside expression \( g(1) \) using the graph which tells us that \( g(1) = 3 \).
We can then use that result as the input to the \( f \) function.
So \( g(1) \) is replaced by the equivalent value 3: \( f(g(1)) = f(3) \)
Lastly, we evaluate \( f(3) \) using the graph which tells us that \( f(3) = 6 \).
\( f(g(1)) = f(3) = 6 \).
So \( f(g(1)) = 6 \).

**Try it Now**
3. Using the graphs from the previous example, evaluate \( g(f(2)) \).
Composition using Formulas

When evaluating a composition of functions where we have either created or been given formulas, the concept of working from the inside out remains the same. First, we evaluate the inside function using the input value provided, then use the resulting output as the input to the outside function.

Example 5

Given \( f(t) = t^2 - t \) and \( h(x) = 3x + 2 \), evaluate \( f(h(1)) \).

Solution:
Since the inside evaluation is \( h(1) \) we start by evaluating the \( h(x) \) function at 1:
\[
h(1) = 3(1) + 2 = 5
\]
Then \( f(h(1)) = f(5) \), so we evaluate the \( f(t) \) function at an input of 5:
\[
f(h(1)) = f(5) = 5^2 - 5 = 20
\]
So \( f(h(1)) = 20 \).

Try it Now

4. Using the functions from the example above, evaluate \( h(f(-2)) \).
While we can compose the functions as above for each individual input value, sometimes it would be really helpful to find a single formula for the composition function \( f(g(x)) \). To do this, we will extend our idea of function evaluation. Recall that when we evaluate a function, we put whatever value is inside the parentheses into the formula wherever we see the input variable. But, we are not limited, to using a numerical value as the input to the function. We can put anything into the function: a value, a different variable, or even an algebraic expression, provided we use the input expression everywhere we see the input variable.

**Example 6**

Suppose that \( f(t) = t^2 - 3t + 7 \). Evaluate \( f(x + 2) \).

**Solution:**

Wherever there is a \( t \) in the formula of \( f(t) \), we would replace the \( t \) with the input expression \( (x + 2) \).

\[
f(x + 2) = (x + 2)^2 - 3(x + 2) + 7
\]

We could simplify this expression further if we wanted to:

\[
\begin{align*}
  f(x + 2) &= (x^2 + 4x + 4) - 3x - 6 + 7 \\
  &= x^2 + x + 5
\end{align*}
\]

\( \text{(FOIL on first term, and distributed -3 to second term)} \)

\( \text{(combine like terms to simplify)} \)

**Try it Now**

5. Given \( g(x) = 3x - \sqrt{x} \), evaluate \( g(t - 2) \).

This now allows us to find a simplified formula for a composition of functions. If we want to find a formula for \( f(g(x)) \). We can evaluate the function \( f(x) \) by replacing the input variable \( x \) with the \( g(x) \) expression.
Example 7 (*Video Example Here)

Let \( f(x) = 3x^2 - 5x + 7 \) and let \( g(x) = 4x + 9 \). Find \( f(g(x)) \) and \( g(f(x)) \).

Solution:
To find \( f(g(x)) \), we replace the \( x \) in the \( f(x) \) function with the function \( g(x) \).
\[
\begin{align*}
    f(g(x)) &= 3(g(x))^2 - 5(g(x)) + 7 \\
    f(g(x)) &= 3(4x + 9)^2 - 5(4x + 9) + 7 \\
    f(g(x)) &= 3(16x^2 + 72x + 81) - 20x - 45 + 7 \\
    f(g(x)) &= 48x^2 + 216x + 243 - 20x - 45 + 7 \\
    f(g(x)) &= 48x^2 + 196x + 205
\end{align*}
\]

To find \( g(f(x)) \), we replace the \( x \) in the \( g(x) \) function with the function \( f(x) \).
\[
\begin{align*}
    g(f(x)) &= 4(f(x)) + 9 \\
    g(f(x)) &= 4(3x^2 - 5x + 7) + 9 \\
    g(f(x)) &= 12x^2 - 20x + 28 + 9 \\
    g(f(x)) &= 12x^2 - 20x + 37
\end{align*}
\]

Try it Now
6. Let \( f(x) = x^3 + 3x \) and \( g(x) = \sqrt{x} \), find \( f(g(x)) \) and \( g(f(x)) \).

Video Link: Two Application Problems-Writing Composite Function Notation.

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Example 8

A city manager determines that the tax revenue, \( R \), in millions of dollars collected on a population of \( p \) thousand people is given by the formula \( R(p) = 0.03p + \sqrt{p} \). Also, that city’s population, in thousands, is predicted to follow the formula \( p(t) = 60 + 2t + 0.3t^2 \), where \( t \) is measured in years after 2010. Find a formula for the tax revenue as a function of the year.

Solution:
Since we want tax revenue as a function of the year, we want a function \( R(t) \) that inputs years since 2010, and outputs the tax revenue that year.

We can get a function \( R(t) \) from creating the composition function \( R(p(t)) \).

\[
R(p(t)) = R(60 + 2t + 0.3t^2) = 0.03(60 + 2t + 0.3t^2) + \sqrt{60 + 2t + 0.3t^2}
\]

This composition gives us a single formula which can be used to predict the tax revenue during a given year, without needing to find the intermediary population value.

For example, to predict the tax revenue in 2017, when \( t = 7 \), we get

\[
R(p(7)) = 0.03(60 + 2(7) + 0.3(7)^2) + \sqrt{60 + 2(7) + 0.3(7)^2} \approx 12.079 \text{ million dollars.}
\]

So, the tax revenue in 2017 is 12.079 million dollars.

Try it Now

7. Suppose the radius of a circular puddle is increasing at a constant rate of 2 inches per hour, and it has a radius of 7 inches when we first observe it.

a) Write the formula for \( A(r) \) that gives the area of the puddle when it has a radius of \( r \) inches.
b) Write the formula for \( r(t) \) that gives the area of the puddle \( t \) hours after we first observe it.
c) Use function composition to find a formula for \( A(t) \) that gives the area of the puddle after \( t \) hours.
Decomposing Functions

In some cases, it is desirable to decompose a function – to write it as a composition of two simpler functions. If you go on to Calculus, you will be asked to regularly decompose function in order to use “the chain rule”.

Example 9 (*Video Example Here)

Write \( f(x) = 3 + \sqrt{5-x^2} \) as the composition of two functions, \( g(x) \) and \( h(x) \).

Solution:
We are looking for two functions, \( g \) and \( h \), so that \( f(x) = g(h(x)) \).

To do this, we first look for an “inside function,” \( h(x) \).
As one possibility, we might notice that \((5 - x^2)\) is the inside of the square root.
If we call that the inside function, then we have \( h(x) = 5 - x^2 \) which would mean
\[
\begin{align*}
  f(x) &= 3 + \sqrt{5-x^2} \\
  f(x) &= 3 + \sqrt{h(x)}
\end{align*}
\]

We could then decompose the function as:
\( h(x) = 5 - x^2 \) and \( g(x) = 3 + \sqrt{x} \)

We can check our answer by recomposing the functions:
\[
g(h(x)) = g(5-x^2) = 3 + \sqrt{5-x^2}
\]

Example 10

Write \( f(x) = 750e^{-3x^2+8x-9} \) as the composition of two functions, \( g(x) \) and \( h(x) \).

Solution:
We are looking for two functions, \( g \) and \( h \), so that \( f(x) = g(h(x)) \).

To do this, we first look for an “inside function,” \( h(x) \).
As one possibility, we might notice that \((-3x^2 + 8x - 9)\) is in the exponent of the exponential.
If we call that the inside function, then we have \( h(x) = -3x^2 + 8x - 9 \) which would mean
\[
\begin{align*}
  f(x) &= 750e^{-3x^2+8x-9} \\
  f(x) &= 750e^{h(x)}
\end{align*}
\]

We could then decompose the function as:
\( h(x) = -3x^2 + 8x - 9 \) and \( g(x) = 750e^x \)

We can check our answer by recomposing the functions:
\[
g(h(x)) = g(-3x^2 + 8x - 9) = 750e^{-3x^2+8x-9}
\]
Important Topics of this Section

Definition of Composition of Functions
Compositions using:
  Words
  Tables
  Graphs
  Equations
Domain of Compositions
Decomposition of Functions

Try it Now Answers
1. The final cost, \( C \), depends on the clearance price, \( p \), which is based on the original discount, \( d \). (Or the original discount \( d \), determines the clearance price and the final cost is half of the clearance price.)

2. \( f(g(1)) = f(3) = 3 \) and \( g(f(3)) = g(3) = 2 \)

3. \( g(f(2)) = g(5) = 3 \)

4. \( h(f(-2)) = h(6) = 20 \) did you remember to insert your input values using parentheses?

5. \( g(t-2) = 3(t-2)-\sqrt{(t-2)} \)

6. \( f(g(x)) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^3 + 3\sqrt{x} \)

7. \( g(f(x)) = g\left(x^3 + 3x\right) = \sqrt{x^3 + 3x} \)

7. 
   a) \( A(r) = \pi r^2 \)
   b) \( r(t) = 7 + 2t \)
   c) \( A(t) = \pi(7 + 2t)^2 \)
Section 5.1 Exercises

Given each pair of functions, calculate \( f(g(0)) \) and \( g(f(0)) \).

1. \( f(x) = 4x + 8, \ g(x) = 7 - x^2 \)
2. \( f(x) = 5x + 7, \ g(x) = 4 - 2x^2 \)

Use the table of values to evaluate each expression

3. \( f(g(8)) \)
4. \( f(g(5)) \)

<table>
<thead>
<tr>
<th>x</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Use the graphs to evaluate the expressions below.

5. \( f(g(3)) \)
6. \( f(g(1)) \)
7. \( g(f(1)) \)
8. \( g(f(0)) \)
For each pair of functions, find \( f(g(x)) \) and \( g(f(x)) \). Simplify your answers.

9. \( f(x) = x^2 + 1, \ g(x) = \sqrt{x+2} \)  
10. \( f(x) = \sqrt{x+2}, \ g(x) = x^2 + 3 \)

11. \( f(x) = |x|, \ g(x) = 5x + 1 \)  
12. \( f(x) = 3\sqrt{x}, \ g(x) = \frac{x+1}{x^3} \)

13. The function \( D(p) \) gives the number of items that will be demanded when the price is \( p \). The production cost, \( C(x) \) is the cost of producing \( x \) items. To determine the cost of production when the price is $6, you would do which of the following:
   a. Evaluate \( D(C(6)) \)  
   b. Evaluate \( C(D(6)) \)
   c. Solve \( D(C(x)) = 6 \)  
   d. Solve \( C(D(p)) = 6 \)

14. The function \( A(d) \) gives the pain level on a scale of 0-10 experienced by a patient with \( d \) milligrams of a pain reduction drug in their system. The milligrams of drug in the patient’s system after \( t \) minutes is modeled by \( m(t) \). To determine when the patient will be at a pain level of 4, you would need to:
   a. Evaluate \( A(m(4)) \)  
   b. Evaluate \( m(A(4)) \)
   c. Solve \( A(m(t)) = 4 \)  
   d. Solve \( m(A(d)) = 4 \)

15. The radius \( r \), in inches, of a spherical balloon is related to the volume, \( V \), by \( r = \frac{3V}{4\pi} \).
   Air is pumped into the balloon, so the volume after \( t \) seconds is given by \( V(t) = 10 + 20t \).
   a. Find the composite function \( r(V(t)) \).
   b. Find the radius after 20 seconds.

16. The number of bacteria in a refrigerated food product is given by \( N(T) = 23T^2 - 56T + 1 \), \( 3 < T < 33 \), where \( T \) is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by \( T(t) = 5t + 1.5 \), where \( t \) is the time in hours.
   a. Find the composite function \( N(T(t)) \)
   b. Find the bacteria count after 4 hours.
Find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x) = f(g(x))$.

17. $h(x) = (x + 2)^2$  
18. $h(x) = (x - 5)^3$

19. $h(x) = \frac{3}{x - 5} + 9$  
20. $h(x) = \frac{4}{(x + 2)^2} - 15$

21. $h(x) = 3 + \sqrt{x - 2}$  
22. $h(x) = 4 + \sqrt[3]{x}$

23. $h(x) = 150(1.05)^x$  
24. $h(x) = 5000e^{-3x + 5}$

25. $h(x) = 6(x - 7)^2 - (x - 7) + 9$  
26. $h(x) = \frac{7}{(x+1)^2} + \frac{19}{x+1} + 12$
Section 5.2 Inverse Functions

A fashion designer is travelling to Milan for a fashion show. He asks his assistant, Betty, what 27 degrees Celsius is in Fahrenheit. After a quick search on Google, she finds the formula $C(F) = \frac{5}{9}(F - 32)$. Why didn’t Betty just use a Google search to find a simple conversion calculator? Because this is a math book and we’re making Betty do it by hand.

Betty uses the formula to solve for the unknown value of $F$:

\[27 = \frac{5}{9}(F - 32)\]

Multiply both sides by $\frac{9}{5}$

\[\frac{9}{5} \cdot 27 = \frac{9}{5} \cdot \frac{5}{9}(F - 32)\]

Simplify

\[48.6 = F - 32\]

Add 32 on both sides

\[48.6 + 32 = F - 32 + 32\]

Simplify

\[80.6 = F\]

So, Betty knows that 27 degrees Celsius is the same as 80.6 degrees Fahrenheit.

The next day, the designer sends Betty the week’s weather forecast for Milan, and asks her to convert each of the temperatures to Fahrenheit.

At first, Betty might consider re-using the formula she has already found to do the conversions. After all, she knows her algebra, and can repeatedly solve the equation for $F$ after substituting in each value of $C$.

After considering this option for a moment, she realizes that solving the equation for each of the temperatures would get awfully tedious, and realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula. It would be easier to have a formula that inputs Celsius temperature and outputs the Fahrenheit temperature. This is the idea of an inverse function, where the input becomes the output and the output becomes the input.
### Inverse Function

If \( f(x) \) and \( g(x) \) are inverse functions, then:

* The two functions have inputs and outputs that are exactly flip-flopped.
* If points \((a, b)\) are on the function \( f(x) \), then points \((b, a)\) are on the function \( g(x) \).
* The domain of \( f(x) \) is the range of the function \( g(x) \).
* The range of \( f(x) \) is the domain of the function \( g(x) \).
* The inverse function, \( g(x) \), is typically notated \( f^{-1}(x) \), which is read “\( f \) inverse of \( x \)”.
* \( f(a) = b \) means \( f^{-1}(b) = a \).

The composition of two inverse functions, where \((a, b)\) is on \( f(x) \), produces the following results:

\[
\begin{align*}
  f(f^{-1}(b)) &= f(a) = b \\
  f^{-1}(f(a)) &= f^{-1}(b) = a
\end{align*}
\]

**Important Note:**

The raised \(-1\) used in the notation for inverse functions, \( f^{-1}(x) \), is simply a notation, and **does not designate an exponent or power of \(-1\)**.

### Example 1 (*Video Example Here*)

If for a particular function, \( f(2) = 4 \), what do we know about the inverse?

**Solution:**

The inverse function reverses which quantity is input and which quantity is output, so if \( f(2) = 4 \), then \( f^{-1}(4) = 2 \).

We know that the point \((2,4)\) is on the graph of \( f(x) \), and we know that the point \((4,2)\) is on the graph of \( f^{-1}(x) \).

### Try it Now

1. Given that \( h^{-1}(6) = 2 \), what do we know about the original function \( h(x) \)?

**Video Link:** Finding Inverse Function Values given a Formula for the Function \( f \).
Example 2 (Video Example Here)
For each scenario below, write a sentence to interpret the real-world, contextual meaning of the given functional statement.

a) The function \( f(x) \) gives the number of items sold, in thousands of items, when the items are priced at \( x \) dollars each. Interpret the contextual meaning of \( f(3) = 70 \) and also \( f^{-1}(9) = 40 \).

b) The function \( g(x) \) gives the average cost per item, in dollars per item, when producing \( x \) items. Interpret the contextual meaning of \( g(300) = 200 \) and \( g^{-1}(150) = 400 \).

Solution:

a) The point \((3, 70)\) is on the graph of \( f(x) \), and it means that they will sell 70,000 items when they price the items at $3.00 each. The point \((9, 40)\) is on the graph of \( f^{-1}(x) \), so the point \((40, 9)\) is on the graph of \( f(x) \), and it means that they will sell 9,000 items when they price the items at $40.00 each.

b) The point \((300, 200)\) is on the graph of \( g(x) \), and it means that the average cost per item is $200 per item when they are producing 300 items. The point \((150, 400)\) is on the graph of \( g^{-1}(x) \), so the point \((400, 150)\) is on the graph of \( g(x) \), and it means that the average cost per item is $150 per item when they are producing 400 items.

Try it Now

2. The number of grey mittens in the supply warehouse is given by the function \( f(x) \), where \( x \) is the number of days after November 1\(^{st}\). Write a real-world, contextual meaning of \( f^{-1}(25) = 30 \).
Notice that an original function and its inverse function undo each other. If \( f(a) = b \), then \( f^{-1}(b) = a \), returning us to the original input. More simply put, if you compose these functions together you get the original input as your answer.

\[
f^{-1}(f(a)) = a \quad \text{and} \quad f(f^{-1}(b)) = b
\]

Since the outputs of the function \( f \) are the inputs to \( f^{-1} \), the range of \( f \) is also the domain of \( f^{-1} \). Likewise, since the inputs to \( f \) are the outputs of \( f^{-1} \), the domain of \( f \) is the range of \( f^{-1} \). Basically, like how the input and output values switch, the domain & ranges switch as well. But be careful, because sometimes a function doesn’t even have an inverse function, or only has an inverse on a limited domain. For example, the inverse of \( f(x) = \sqrt{x} \) is \( f^{-1}(x) = x^2 \), since a square “undoes” a square root, but it is only the inverse of \( f(x) \) on the domain \([0,\infty)\), since that is the range of \( f(x) = \sqrt{x} \).

Example 3

The function \( f(x) = 2^x \) has domain \((-\infty, \infty)\) and range \((0, \infty)\), what would we expect the domain and range of \( f^{-1} \) to be?

Solution:
We would expect \( f^{-1}(x) \) to swap the domain and range of \( f(x) \), so \( f^{-1}(x) \) would have domain \((0, \infty)\) and range \((-\infty, \infty)\).

Example 4 [Video Example Here]

A function \( f(t) \) is given as a table below, showing distance in miles that a car has traveled in \( t \) minutes. Find and interpret \( f^{-1}(70) \).

<table>
<thead>
<tr>
<th>( t ) (minutes)</th>
<th>30</th>
<th>50</th>
<th>70</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) ) (miles)</td>
<td>20</td>
<td>40</td>
<td>60</td>
<td>70</td>
</tr>
</tbody>
</table>

Solution:
The inverse function, \( f^{-1}(x) \), takes an output of \( f(x) \) and returns an input for \( f(x) \). So, in the expression \( f^{-1}(70) \), the 70 is an output value of \( f(x) \), representing 70 miles. \( f^{-1}(70) \) will return the value \( t = 90 \) minutes. So \( f^{-1}(70) = 90 \) and \( f(90) = 70 \). Both equations tell us that a 90-minute drive will correspond with 70 miles traveled.
Try it Now

3. The table below gives the number of miles traveled in t minutes.

<table>
<thead>
<tr>
<th>t (minutes)</th>
<th>30</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(t) (miles)</td>
<td>20</td>
<td>40</td>
<td>50</td>
<td>60</td>
<td>70</td>
</tr>
</tbody>
</table>

Find and write a complete sentence to interpret the real-world, contextual meaning of the point.

a. \( f(60) \)

b. \( f^{-1}(60) \)

---

Example 5 (*Video Example Here*)

A function \( g(x) \) is given as a graph below. Find \( g(3) \) and \( g^{-1}(3) \)

Solution:

To evaluate \( g(3) \), we find \( x = 3 \) on the horizontal axis and find the corresponding output value on the vertical axis. The point \((3, 1)\) tells us that \( g(3) = 1 \)

To evaluate \( g^{-1}(3) \), recall that by definition \( g^{-1}(3) \) means \( g(x) = 3 \). By looking for the output value \( y = 3 \) on the vertical axis we find the point \((5, 3)\) on the graph, which means \( g(5) = 3 \), so by definition \( g^{-1}(3) = 5 \).

---

Try it Now

4. Using the graph in Example 4 above to approximate each.

   a. find \( g^{-1}(1) \)
   b. estimate \( g^{-1}(4) \)
Returning to our designer’s assistant, find a formula for the inverse of the function

\[ C(F) = \frac{5}{9}(F - 32) \]

Solution:
To find an inverse of the function, we need to find a function that reverses the input and output. So, we need to find a function that inputs \( C \) and outputs \( F \). To do this, we need to re-write the equation so that it is solved for \( F \).

\[ C = \frac{5}{9}(F - 32) \]

Multiply both sides by \( \frac{9}{5} \)

\[ \frac{9}{5} \cdot C = \frac{9}{5} \cdot \frac{5}{9}(F - 32) \]

Simplify

\[ \frac{9}{5} \cdot C = F - 32 \]

Add 32 on both sides

\[ \frac{9}{5} \cdot C + 32 = F - 32 + 32 \]

Simplify

\[ \frac{9}{5} \cdot C + 32 = F \]

So, \( C(F) = \frac{5}{9}(F - 32) \), then \( F(C) = \frac{9}{5} \cdot C + 32 \). If we used \( f(x) \) and \( f^{-1}(x) \) notation to describe these functions, then we could say the following:

\( f(x) = \frac{5}{9}(x - 32) \) where \( x \) is the temperature in Fahrenheit, and \( f(x) \) is the temperature in Celsius, and \( f^{-1}(x) = \frac{9}{5}x + 32 \) where \( x \) is the temperature in Celsius, and \( f^{-1}(x) \) is the temperature in Fahrenheit.

It is important to note that not all functions will have an inverse function. Since the inverse \( f^{-1}(x) \) takes an output of \( f \) and returns an input of \( f \), in order for \( f^{-1} \) to itself be a function, then each output of \( f \) (input to \( f^{-1} \)) must correspond to exactly one input of \( f \) (output of \( f^{-1} \)) in order for \( f^{-1} \) to be a function. This is the definition of a one-to-one function.

**One-to-one Functions, and Existence of Inverse Functions**

A function is one-to-one if each output \( y \)-value corresponds to only one unique input \( x \)-value. A function is one-to-one if and only if its graph passes the horizontal line test.

In order for a function to have an inverse, it must be a one-to-one function.
In some cases, it is desirable to have an inverse for a function even though the function is not one-to-one. In those cases, we can often limit the domain of the original function to an interval on which the function is one-to-one, then find an inverse only on that interval.

**Example 7** (*Video Example Here*)

The quadratic function \( h(x) = x^2 \) is not one-to-one. Explain how we know that. Then find a domain on which this function is one-to-one, and find the inverse on that domain.

**Solution:**
The function \( h(x) = x^2 \) is not one-to-one because there are \( y \)-values on the function which correspond with more than one \( x \)-value. For example, the \( y \)-value \( y = 4 \) corresponds with both \( x = 2 \) and also \( x = -2 \). This function fails the horizontal line test because a horizontal line, for example, would intersect both \((-2, 4)\) and \((2, 4)\).

If we limit the domain to \([0, \infty)\) then the resulting graph of \( h(x) = x^2 \) on the interval \( x \geq 0 \) would now be one-to-one, and so it would have an inverse function. This restricted-domain graph of \( h(x) \) is pictured to the right.

To find the inverse function, \( h^{-1}(x) \), we need to start with the equation \( y = x^2 \) and solve the equation for \( x \).

\[
y = x^2 \quad \rightarrow \quad \sqrt{y} = \sqrt{x^2} \quad \text{Take the square root of both sides}
\]

\[
\sqrt{y} = x \quad \text{Simplify. Only consider the } x \geq 0 \text{ since we are restricting the domain.}
\]

So the inverse of the function \( h(x) = x^2 \), when restricted to \( x \geq 0 \) is the function \( h^{-1}(x) = \sqrt{x} \).

Note that the domain and range of the square root function do correspond with the range and domain of the quadratic function on the limited domain. In fact, if we graph \( h(x) \) on the restricted domain and \( h^{-1}(x) \) on the same axes, we can notice symmetry: the graph of \( h^{-1}(x) \) is the graph of \( h(x) \) reflected over the line \( y = x \).
**Important Topics of this Section**

- Definition of an inverse function
- Composition of inverse functions yield the original input value
- Not every function has an inverse function
- To have an inverse a function must be one-to-one
- Restricting the domain of functions that are not one-to-one.

**Try it Now Answers**

1. \( g(2) = 6 \)

2. The point (30, 25) is on the graph of \( f(x) \). This means that there are 25 mittens in the warehouse 30 days after November 1st.

3.
   a. \( f(60) = 50 \). In 60 minutes, 50 miles are traveled.
   b. \( f^{-1}(60) = 70 \). To travel 60 miles, it will take 70 minutes.

4. a. \( g^{-1}(1) = 3 \)
   b. \( g^{-1}(4) = 5.5 \) (this is an approximation – answers may vary slightly)
Section 5.2 Exercises

Assume that the function \( f \) is a one-to-one function.

1. If \( f(6) = 7 \), find \( f^{-1}(7) \)  
2. If \( f(3) = 2 \), find \( f^{-1}(2) \)  
3. If \( f^{-1}(-4) = -8 \), find \( f(-8) \)  
4. If \( f^{-1}(-2) = -1 \), find \( f(-1) \)  
5. If \( f(5) = 2 \), find \( (f(5))^{-1} \)  
6. If \( f(1) = 4 \), find \( (f(1))^{-1} \)

Write a real-world, contextual meanings for each of the given functional statements below:

7. The function \( f(x) \) gives the Fahrenheit temperature, where \( x \) is the temperature in Celsius. Interpret the meaning of \( f^{-1}(32) = 0 \) in context.

8. The function \( g(x) \) gives the amount of fencing required to enclose a circular field, where \( x \) is the area of that circular field. Interpret the meaning of \( g^{-1}(25.1) = 50 \) in context.

9. The function \( w(x) \) gives the weight of an object in kg, where \( x \) is the weight of the object in pounds. Interpret the meaning of \( w^{-1}(454) = 1000 \) in context.

10. The number of people who purchase tickets to an event is given by the function \( A(x) \), where \( x \) is the price per ticket. Interpret the meaning of \( A^{-1}(80) = 90 \) in context.

7. Using the graph of \( f(x) \) shown
   a. Find \( f(0) \)
   b. Solve \( f(x) = 0 \)
   c. Find \( f^{-1}(0) \)
   d. Solve \( f^{-1}(x) = 0 \)

8. Using the graph shown
   a. Find \( g(1) \)
   b. Solve \( g(x) = 1 \)
   c. Find \( g^{-1}(1) \)
   d. Solve \( g^{-1}(x) = 1 \)
9. Use the table below to find the indicated quantities.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>8</td>
<td>0</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>9</td>
<td>1</td>
</tr>
</tbody>
</table>

a. Find $f(1)$

b. Solve $f(x) = 3$

c. Find $f^{-1}(0)$

d. Solve $f^{-1}(x) = 7$

10. Use the table below to fill in the missing values.

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>h(t)</td>
<td>6</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

a. Find $h(6)$

b. Solve $h(t) = 0$

c. Find $h^{-1}(5)$

d. Solve $h^{-1}(t) = 1$

For each table below, create a table for $f^{-1}(x)$.

11. | x  | 3 | 6 | 9 | 13 | 14 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>16</td>
</tr>
</tbody>
</table>

12. | x  | 3 | 5 | 7 | 13 | 15 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>2</td>
<td>6</td>
<td>9</td>
<td>11</td>
<td>16</td>
</tr>
</tbody>
</table>
For each function below, find a formula for $f^{-1}(x)$.

13. $f(x) = x + 3$
14. $f(x) = x + 5$
15. $f(x) = 2 - x$
16. $f(x) = 3 - x$
17. $f(x) = 11x + 7$
18. $f(x) = 9 + 10x$

For each function, find a domain on which $f$ is one-to-one and non-decreasing, then find the inverse of $f$ restricted to that domain.

19. $f(x) = (x + 7)^2$
20. $f(x) = (x - 6)^2$
21. $f(x) = x^2 - 5$
22. $f(x) = x^2 + 1$

23. If $f(x) = x^3 - 5$ and $g(x) = \sqrt[3]{x + 5}$, find
   a. $f(g(x))$
   b. $g(f(x))$
   c. What does this tell us about the relationship between $f(x)$ and $g(x)$?

24. If $f(x) = \frac{x}{2 + x}$ and $g(x) = \frac{2x}{1 - x}$, find
   a. $f(g(x))$
   b. $g(f(x))$
   c. What does this tell us about the relationship between $f(x)$ and $g(x)$?
Section 5.3 Introduction to Logarithmic Functions

A population of 50 flies is expected to double every week, leading to a function of the form \( f(x) = 50(2)^x \) where \( x \) represents the number of weeks that have passed. When will this population reach 500?

We could solve this problem on Excel using Solver as follows:

```
1  x  y=50(2)^x
2  0   50
3  1  100
4  2  200
5  3  400
6  3.321929  500
7  4  800
8  5  1600
9  6  3200
10
11
12
13
```

We see from the Excel Solver solution above that the function \( f(x) = 50(2)^x \) reaches an output value of \( y = 500 \) at approximately \( x \approx 3.321929 \). So, there would be 500 flies after about 3.3 weeks.

Alternatively, we could try to solve the problem by hand. To set this up, we set the output value of the function \( f(x) = 500(2)^x \) equal to 500:

\[
500 = 50(2)^x
\]

\[
\text{Dividing both sides by 50 to isolate the exponential}
\]

\[
10 = (2)^x
\]

Unfortunately, none of the algebraic tools discussed so far in this book are sufficient to solve the exponential equation \( 10 = (2)^x \). Consider the following:

- When solving \( 10 = (2)^x \), dividing both sides by 2 won’t work since \( x \) is not being multiplied by 2.
- When solving \( 10 = (2)^x \), subtracting both sides by 2 won’t work since \( x \) since \( x \) is not being added to 2.
- When solving \( 10 = (2)^x \), taking the square root of both sides will not isolate the \( x \), since \( x \) is not being squared. Remember, the inverse of the square root of \( x \) would be \( x^2 \) which is different than \( 2^x \)!

So, what operation will isolate the \( x \) when solving the equation \( 10 = (2)^x \)? What operation will “un-do” the operation \( 2^x \) in order to isolate \( x \)?
Consider the equation $2^x = 10$ equation that we were trying to solve above. None of the functions we have already discussed in the book would serve as an inverse function to isolate $x$, and so we must introduce a new function, named \textbf{log} as the inverse of an exponential function. \textbf{Log} is short for \textbf{logarithm}.

<table>
<thead>
<tr>
<th>The Logarithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textbf{The logarithm} (base (b)) function, written (\log_b(x)), is the inverse of the exponential function (base (b)), (b^x).</td>
</tr>
<tr>
<td>The input of the log function is the output of the exponential function, and vice versa.</td>
</tr>
<tr>
<td>\textbf{The statement} (b^a = c) \textbf{is equivalent to the statement} (\log_b(c) = a)</td>
</tr>
</tbody>
</table>

We can always re-write logarithmic equations into exponential form, and vice versa. Just keep in mind that the input and output of the functions are reversed!

\[\text{Video Link Can you evaluate a log for a negative number?}\]

**Example 1** (*Video Example Here) (*Video Example Here)

Rewrite each equation in the alternate form, either exponential or logarithmic. Identify the input and output of each equation.

- a) \(y = \log_2(x)\)
- b) \(y = 7^x\)
- c) \(10^{-3} = 0.001\)
- d) \(\log_7(1) = 0\)

\textbf{Solutions:}

- a) \(y = \log_2(x)\), pronounced “log base 2 of \(x\)” is the inverse of the function \(x = 2^y\).
  Notice that the input to the logarithm is \(x\), and the output is \(y\).
  Notice that the output to the exponential is \(x\), and the input is \(y\).

- b) \(x = \log_7(y)\), pronounced “log base 7 of \(y\)” is the inverse of the function \(y = 7^x\).
  Notice that the input to the logarithm is \(y\), and the output is \(x\).
  Notice that the output to the exponential is \(y\), and the input is \(x\).

- c) \(10^{-3} = 0.001\) is exponential form, and it is equivalent to saying that \(\log_{10}(0.001) = -3\)
  Notice that the input to the logarithm is 0.001, and the output is \(-3\).
  Notice that the output to the exponential is 0.001, and the input is \(-3\).

- d) \(\log_7(1) = 0\) is logarithmic form, and it is equivalent to saying that \(7^0 = 1\)
  Notice that the input to the logarithm is 1, and the output is 0.
  Notice that the output to the exponential is 1, and the input is 0.
Try it Now

1. Re-write each equation in the alternate form (either exponential or logarithmic).
   a) \(4^2 = 16\)
   b) \(12^2 = 144\)
   c) \(\log_7(343) = 3\)
   d) \(\log_4(0.25) = -1\)

On the previous pages, we asked how to solve for when the function \(f(x) = 50(2)^x\) would reach a height of 500. Solving this equation would tell us when there would be 500 flies. We reduced the equation to solving \(10 = (2)^x\). Now that we have the new “log” function, we can now re-write that equation as \(x = \log_2(10)\).

Unfortunately, answering the fly question by saying

“It will take \(\log_2(10)\) weeks for there to be 500 flies.”

is not very helpful. Instead, we need to find a way to convert the \(\log_2(10)\) into a decimal.

The Logarithm

To calculate logarithmic values in Excel, use the following Excel function:

\[= \text{LOG(value, base)}\]
Example 2

Use Excel to calculate the following values. Round the answers to 4 places if necessary.

a) \( \log_2(8) \)

b) \( \log_7(1) \)

c) \( \log_3(10) \)

Solutions:

a) First, let’s consider the answer without using Excel. \( \log_2(8) \) is equal to some value, let’s call it \( x \). That means that \( \log_2(8) = x \).

Re-writing the equation in exponential form, we get \( 2^x = 8 \).

Notice that this problem is easy enough to solve without Excel. The answer is \( x = 3 \) since \( 2^3 = 8 \).

In Excel: we enter \( =\log(8,2) \) and it returns an answer of 3.

Therefore, \( \log_2(8) = 3 \).

b) First, let’s consider the answer without using Excel. \( \log_7(1) \) is equal to some value, let’s call it \( x \). That means that \( \log_7(1) = x \).

Re-writing the equation in exponential form, we get \( 7^x = 1 \).

Notice that this problem is easy enough to solve without Excel. The answer is \( x = 0 \) since \( 7^0 = 1 \).

In Excel: we enter \( =\log(1,7) \) and it returns an answer of 0.

Therefore, \( \log_7(1) = 0 \).

c) First, let’s consider the answer without using Excel. \( \log_3(10) \) is equal to some value, let’s call it \( x \). That means that \( \log_3(10) = x \).

Re-writing the equation in exponential form, we get \( 3^x = 10 \).

Notice that this problem is NOT easy enough to solve without Excel.

We can estimate the answer as being some value between 2 and 3.

We know this because \( 3^2 = 9 \) and \( 3^3 = 27 \).

So, some value of \( x \) between 2 and 3 will solve the equation.

In Excel: we enter \( =\log(10,3) \) and it returns an answer of approximately 2.0959.

Therefore, \( \log_3(10) \approx 2.0959 \).
We now have the ability to solve the original fly question that was posed at the start of this section. We were trying to solve for when the function $f(x) = 50(2)^x$ will reach a height of 500. We now have the skills to complete this problem:

$$500 = 50(2)^x$$

*Divide both sides by 50.*

$$10 = (2)^x$$

*Re-write the exponential equation in logarithmic form.*

$$\log_2(10) = x$$

*Approximate the value of $x$ on Excel.*

So, there will be 500 flies at 3.3219 weeks.

Instead of using “log base 2”, it is much more common to see the “common log” and the “natural log” used in the world. Calculators generally have a common log button and a natural log button.

---

The Common Log and the Natural Log

<table>
<thead>
<tr>
<th>The common log</th>
<th>The natural log</th>
</tr>
</thead>
<tbody>
<tr>
<td>The <strong>common log</strong> is the logarithm with base 10, and is written $\log(x)$.</td>
<td>The <strong>natural log</strong> is the logarithm with base $e$, and is written $\ln(x)$.</td>
</tr>
<tr>
<td>In other words, $\log_{10}(x) = \log(x)$.</td>
<td>In other words, $\log_e(x) = \ln(x)$.</td>
</tr>
<tr>
<td><em>When working with the common log, we do not write the 10 in the base.</em></td>
<td>*When working with the natural log, we do not write the log base $e$; instead we write $\ln$.</td>
</tr>
<tr>
<td>*Excel calculates a common log using the function =$LOG$(value)</td>
<td>*Excel calculates a natural log using the function =$LN$(value)</td>
</tr>
</tbody>
</table>

Note: an interesting historical fact is that we use “$\ln$” instead of “$\text{nl}$” to represent the “natural log” because the Latin name of “natural log” is “logarithmus naturali.”
Example 3

Re-write each exponential equation in logarithmic form, and then calculate the value of the unknown using Excel. Round to 4 places.

a) \(1.04^x = 7\)

b) \(95 = e^x\)

c) \(450 = 10^x\)

Solutions:

a) \(1.04^x = 7\) is the same as \(\log_{1.04}(7) = x\). In Excel we get the following:

And so \(x \approx 49.6144\).

b) \(95 = e^x\) is the same as \(\ln(95) = x\). The “natural log of 95” is calculated in Excel with the LN function. In Excel we get the following:

And so \(x \approx 4.5539\).

c) \(450 = 10^x\) is the same as \(\log(450) = x\). The “log of 450,” which is the same as “the common log of 450” or “log base 10 of 450” is calculated in Excel with the LOG function. In Excel we get the following:

And so \(x \approx 2.6532\).

Try it Now

2. Re-write each exponential equation in logarithmic form, and then calculate the value of the unknown using Excel. Round to 4 places.

a) \(e^x = 150\)

b) \(640 = 10^x\)

c) \(1230 = 4^x\)
Example 4

The population of a country in billions, \( t \) years after 2008, is given by the function \( P(t) = 1.14(1.0134)^t \).

If the population continues following this trend, when will the population reach 2 billion?

**Solution:**

We need to solve for the \( t \) to solve the equation \( 2 = 1.14(1.0134)^t \).

\[
2 = 1.14(1.0134)^t \\
\frac{2}{1.14} = (1.0134)^t \\
\log_{1.0134} \left( \frac{2}{1.14} \right) = t
\]

*Divide both sides by 1.0134.*

*Rewrite the exponential equation in logarithmic form.*

*Use Excel to find the decimal approximation for \( t \).*

\[
\begin{array}{c|c}
A & B \\
1 & \text{LOG}(2/1.14,1.0134) \\
2 & 42.22960893
\end{array}
\]

\( t \approx 42.2296 \)

This means it is 42.2296 years after 2008.

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

**Try it Now**

3.  
   a) Solve \( 5(0.93)^t = 10 \) by hand and then use Excel to approximate that value.  
   b) Solve \( 800 = 450(10)^x \) by hand and then use Excel to approximate that value.  
   c) Solve \( 95 = 17e^x \) by hand and then use Excel to approximate that value.
Example 5  (*Video Example Here*)

Re-write each logarithmic equation in exponential form, and then calculate the value of the unknown using Excel. Round to 4 places.

a) \( \log_5(x) = 1.7 \)

b) \( \ln(x) = 7 \)

c) \( \log(x) = 0.25 \)

Solutions:

a) \( \log_5(x) = 1.7 \) is the same as \( 5^{1.7} = x \). In Excel, we calculate \( 5^{1.7} \) and find that \( x \approx 15.4259 \).

b) \( \ln(x) = 7 \) is the same as \( e^7 = x \). In Excel, we calculate \( e^7 \) by typing

\[ =\text{EXP}(7) \]

and find that \( x \approx 1096.6332 \).

c) \( \log(x) = 0.25 \) is the same as \( 10^{0.25} = x \). In Excel, we calculate \( 10^{0.25} \) and find that \( x \approx 1.7783 \).

Try it Now

4.

a) Solve \( 40 \ln(x) = 80 \) by hand and then use Excel to approximate that value.

b) Solve \( 635 = 144 \log(x) \) by hand and then use Excel to approximate that value.

c) Solve \( 10,200 \log_8(x) = 50,800 \) by hand and then use Excel to approximate that value.

Having the logarithmic function allows us to convert exponential functions in the form \( f(x) = a(b)^x \) into the continuously compounding form, \( f(x) = ae^{rx} \). When making this conversion, make sure to remember that whenever you use a calculator to get a decimal, you are introducing error into the value. So, when re-writing formulas using these decimals, you are introducing error into the formula itself.
Example 6 (* Video Example Here) 

Re-write the exponential function \( f(x) = 700(1.04)^x \) in the form \( f(x) = ae^{rx} \). Then interpret the parameters in the function in both forms. Round values to 5 decimal places if necessary.

Solution:
We need to convert \( f(x) = 700(1.04)^x \) into the form \( f(x) = ae^{rx} \). First recall that the 700 in the equation \( f(x) = 700(1.04)^x \) is the y-intercept, and also recall that the \( a \) in the equation \( f(x) = ae^{rx} \) is the y-intercept.

So, we know that the function can be written as \( f(x) = 700(1.04)^x = 700e^{rx} \). We need to solve for the value of \( r \), the continuously compounding rate. We set the expressions equal to one another and solve as follows:

\[
700(1.04)^x = 700e^{rx}
\]

Divide both sides by 700.

\( (1.04)^x = e^{rx} \)

On the right side, \( e^{rx} = (e^r)^x \) by the power rule of exponents

\( (1.04)^x = (e^r)^x \)

The base of each exponential expression is the same. Isolate the base.

\( 1.04 = e^r \)

Rewrite the exponential equation in logarithmic form.

\( \ln(1.04) = r \)

Use Excel to calculate the decimal approximation.

\( r \approx 0.03922 \)

And so, \( f(x) = 700(1.04)^x \) and \( f(x) \approx 700e^{0.03922x} \).

Interpretations of the parameters in the equation: The y-intercept of the function is \((0,700)\). The function is increasing because the factor, 1.04, is greater than 1. We can also see that the function is increasing because the continuously compounding rate, 0.03922, is positive. Every time \( x \) increases by 1 unit, the y-value is multiplied by 1.04. Every time \( x \) increases by 1 unit, the y-value increases by 4%. The function is increasing at a continuously compounding rate of approximately 3.922%. 
Example 7

Re-write the exponential function \( f(x) = 8000(0.93)^x \) in the form \( f(x) = ae^{rx} \). Then interpret the parameters in the function in both forms. Round values to 5 decimal places if necessary.

**Solutions:**

a) We need to convert \( f(x) = 8000(0.93)^x \) into the form \( f(x) = ae^{rx} \). Recall that the 8000 in the equation \( f(x) = 8000(0.93)^x \) is the y-intercept, and also recall that the \( a \) in the equation \( f(x) = ae^{rx} \) is the y-intercept.

So, we know that the function can be written as \( f(x) = 8000(0.93)^x = 8000e^{rx} \). We need to solve for the value of \( r \), the continuously compounding rate. We set the expressions equal to one another and solve as follows:

\[
8000(0.93)^x = 8000e^{rx}
\]

Divide both sides by 8000.

\[
(0.93)^x = e^{rx}
\]

On the right side, \( e^{rx} = (e^r)^x \) by the power rule of exponents.

\[
(0.93)^x = (e^r)^x
\]

The base of each exponential expression is the same. Isolate the base.

\[
0.93 = e^r
\]

Rewrite the exponential equation in logarithmic form.

\[
\ln(0.93) = r
\]

Use Excel to calculate the decimal approximation.

\[
r \approx -0.07257
\]

And so, \( f(x) = 8000(0.93)^x \) and \( f(x) \approx 8000e^{-0.07257x} \).

Interpretations of the parameters in the equation: The y-intercept of the function is (0,8000). The function is decreasing because the factor, 0.93, is less than 1. We can also see that the function is decreasing because the continuously compounding rate, \(-0.07257\), is negative. Every time \( x \) increases by 1 unit, the y-value is multiplied by 0.93. Every time \( x \) increases by 1 unit, the y-value decreases by 7%. The function is decreasing at a continuously compounding rate of approximately 7.257%.

Try it Now

5. Re-write the exponential function \( f(x) = 75(0.85)^x \) in the form \( f(x) = ae^{rx} \). Then interpret the parameters in the function in both forms. Round values to 5 decimal places if necessary.
Important Topics of this Section

The Logarithmic function/operation is the inverse of the exponential function/operation
Converting log equations ↔ exponential equations
Common log is log base 10
Natural log is log base e
Solving exponential equations of the form $a(b)^x = c$ and $log_m(w) = a$

Try it Now Answers

1.  
   a) $log_4(16) = 2$
   b) $log_{12}(144) = 2$
   c) $log_7(343) = 3$
   d) $log_4(0.25) = -1$

2.  
   a) $\ln(150) = x$ and so $x \approx 5.0106$
   b) $x = \log(640)$ and so $x \approx 2.8062$
   c) $log_4(1230) = x$ and so $x \approx 5.1322$

3.  
   a) $x = log_{0.93}(2)$. In Excel, we type =LOG(2, 0.93), and so $x \approx -9.5513$.
   b) $x = log\left(\frac{800}{450}\right)$. In Excel, we type =LOG(800/450), and so $x \approx 0.2499$.
   c) $x = ln\left(\frac{95}{17}\right)$. In Excel, we type =LN(95/17), and so $x \approx 1.7207$.

4.  
   a) $x = e^2$. In Excel we type =EXP(2). $x \approx 7.3891$
   b) $x = 10^{\frac{635}{144}}$. In Excel we type =10^(635/144) $x \approx 15,687.5227$.
   c) $x = 8^{\frac{50,800}{10,200}}$. In Excel we type =8^(50800/10200). $x \approx 31,458.8098$.

5. $f(x) = 75(0.85)^x$ is approximately $f(x) \approx 75e^{-0.16252x}$. The y-intercept of the function is (0,75).
   The function is decreasing because the factor, 0.85, is less than 1. We can also see that the function is decreasing because the continuously compounding rate, −0.16252, is negative. Every time x increases by 1 unit, the y-value is multiplied by 0.85. Every time x increases by 1 unit, the y-value decreases by 15%. The function is decreasing at a continuously compounding rate of approximately 16.252%.
Section 5.3 Exercises

Rewrite each equation in exponential form.
1. \( \log_b q = m \)  
2. \( \log_b t = k \)  
3. \( \log_a b = c \)  
4. \( \log_p z = u \)  
5. \( \log(v) = t \)  
6. \( \log(r) = s \)  
7. \( \ln(w) = n \)  
8. \( \ln(x) = y \)

Rewrite each equation in logarithmic form.
9. \( 4^x = y \)  
10. \( 5^y = x \)  
11. \( e^d = k \)  
12. \( n^z = L \)  
13. \( 10^a = b \)  
14. \( 10^b = v \)  
15. \( e^h = k \)  
16. \( e^x = x \)

Solve for \( x \). Find the exact solution, and then use Excel to find the decimal value (round to 4 places if necessary).
17. \( \log_3(x) = 2 \)  
18. \( \log_4(x) = 3 \)  
19. \( \log_2(x) = -3 \)  
20. \( \log_5(x) = -1 \)  
21. \( \log(x) = 3 \)  
22. \( \log(x) = 5 \)  
23. \( \ln(x) = 2 \)  
24. \( \ln(x) = -2 \)

Find the exact value of each expression by calling the value \( x \), and then re-writing the equation in exponential form, and then identifying the answer. Excel is not necessary for these problems since they each have answers that you can do without a calculator.
25. \( \log_5(25) \)  
26. \( \log_2(8) \)  
27. \( \log_3\left(\frac{1}{27}\right) \)  
28. \( \log_6\left(\frac{1}{36}\right) \)  
29. \( \log_6(\sqrt[6]{6}) \)  
30. \( \log_5(\sqrt[5]{5}) \)  
31. \( \log(10000) \)  
32. \( \log(100) \)  
33. \( \log(0.001) \)  
34. \( \log(0.00001) \)  
35. \( \ln(e^{-2}) \)  
36. \( \ln(e^3) \)

Solve each equation for the variable. Find the exact solution, and also use Excel to find the decimal answer (round to 4 places if necessary).
37. \( 5^x = 14 \)  
38. \( 3^x = 23 \)  
39. \( 7^x = \frac{1}{15} \)  
40. \( 3^x = \frac{1}{4} \)  
41. \( 1000(1.03)^x = 5000 \)  
42. \( 200(1.06)^x = 550 \)  
43. \( 10 - 8\left(\frac{1}{2}\right)^x = 5 \)  
44. \( 100 - 100\left(\frac{1}{4}\right)^x = 70 \)
Re-write the exponential function in the form \( f(x) = ae^{rx} \). Then interpret the parameters in the function in both forms. Round values to 5 decimal places if necessary.

45. \( f(t) = 300(0.91)^t \)  
46. \( f(t) = 120(0.07)^t \)

47. \( f(t) = 10(1.04)^t \)  
48. \( f(t) = 1400(1.12)^t \)

Convert the equation into annual growth form, \( f(t) = a(b)^t \). Round values to 5 decimal places if necessary.

49. \( f(t) = 150e^{0.06t} \)  
50. \( f(t) = 100e^{0.12t} \)

51. \( f(t) = 50e^{-0.012t} \)  
52. \( f(t) = 80e^{-0.85t} \)

53. The population of Kenya was 39.8 million in 2009 and has been growing by about 2.6% each year. If this trend continues, when will the population exceed 45 million? Solve by hand (not using Solver).

54. The population of Algeria was 34.9 million in 2009 and has been growing by about 1.5% each year. If this trend continues, when will the population exceed 45 million? Solve by hand (not using Solver).

55. The population of Seattle grew from 563,374 in 2000 to 608,660 in 2010. If the population continues to grow exponentially at the same rate, when will the population exceed 1 million people? Solve by hand (not using Solver).

56. The median household income (adjusted for inflation) in Seattle grew from $42,948 in 1990 to $45,736 in 2000. If it continues to grow exponentially at the same rate, when will median income exceed $50,000? Solve by hand (not using Solver).

57. A scientist begins with 100 mg of a radioactive substance. After 4 hours, it has decayed to 80 mg. How long after the process began will it take to decay to 15 mg? Solve by hand (not using Solver).

58. A scientist begins with 100 mg of a radioactive substance. After 6 days, it has decayed to 60 mg. How long after the process began will it take to decay to 10 mg? Solve by hand (not using Solver).

59. If $1000 is invested in an account earning 3% compounded monthly, how long will it take the account to grow in value to $1500? Solve by hand (not using Solver).

60. If $1000 is invested in an account earning 2% compounded quarterly, how long will it take the account to grow in value to $1300? Solve by hand (not using Solver).
Section 5.4 Properties of Logarithms

In this section we will learn to use a few properties of logs which will allow us to solve exponential and logarithmic equations by hand. The properties below apply for logarithms of all bases, but we will only work with the common log and the natural log from this point forward.

<table>
<thead>
<tr>
<th>Properties of Logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Power Rule:</strong></td>
</tr>
<tr>
<td>$\log(a^b) = b \cdot \log(a)$ and $\ln(a^b) = b \cdot \ln(a)$</td>
</tr>
<tr>
<td><strong>Product Rule:</strong></td>
</tr>
<tr>
<td>$\log(a \cdot b) = \log(a) + \log(b)$ and $\ln(a \cdot b) = \ln(a) + \ln(b)$</td>
</tr>
<tr>
<td><strong>Quotient Rule:</strong></td>
</tr>
<tr>
<td>$\log(a/b) = \log(a) - \log(b)$ and $\ln(a/b) = \ln(a) - \ln(b)$</td>
</tr>
</tbody>
</table>

To help understand why these new properties are true, we offer the following proof of the product rule:

Let $m = \log(A)$ and $n = \log(C)$.
Then we can rewrite these in exponential form as $10^m = A$ and $10^n = C$.
Using these expressions, we can now say that $AC = 10^m \cdot 10^n$.
Using exponent rules on the right, $AC = 10^{m+n}$.
Taking the log of both sides, and utilizing the inverse property of logs, we can say $\log(AC) = \log(10^{m+n})$.
And the right side of this is equal to $m + n$ since $\log(10^{m+n}) = m + n$.
Therefore, $\log(AC) = m + n$
Replacing $m$ and $n$ with their definition establishes the result $\log(AC) = \log(A) + \log(B)$

The proof for the quotient rule and power rule are very similar.

Video Link: Sum and Difference Properties of Logarithms.
**Example 1**

Use the power rule to rewrite each logarithmic expression.

a) \( \log(3^x) \)

b) \( \ln(1.04^x) \)

c) \( 4 \log(2x) \)

d) \( 8 \ln(x + 1) \)

**Solutions:**

a) \( \log(3^x) = x \log(3) \)

b) \( \ln(1.04^x) = x \ln(1.04) \)

c) \( 4 \log(2x) = \log((2x)^4) = \log(2^4x^4) = \log(16x^4) \)

d) \( 8 \ln(x + 1) = \ln((x + 1)^8) \)

**Example 2** *(Video Example Here)*

Use the product rule and/or quotient rule to rewrite each logarithmic expression.

a) \( \log(3x) \)

b) \( \ln \left( \frac{8}{x} \right) \)

c) \( \log \left( \frac{7x}{11} \right) \)

d) \( \ln(x + 1) - \ln(x) + \ln(y) \)

e) \( \log(x) - \log(4 + x) + \log(9) \)

**Solutions:**

a) \( \log(3x) = \log(3) + \log(x) \) \hspace{1cm} \text{product rule}

b) \( \ln \left( \frac{8}{x} \right) = \ln(8) - \ln(x) \) \hspace{1cm} \text{quotient rule}

c) \( \log \left( \frac{7x}{11} \right) = \log(7) + \log(x) - \log(11) \) \hspace{1cm} \text{product and quotient rule}

d) \( \ln(x + 1) - \ln(x) + \ln(y) = \ln \left( \frac{(x+1)y}{x} \right) = \ln \left( \frac{xy+y}{x} \right) \) \hspace{1cm} \text{quotient and product rule}

e) \( \log(x) - \log(4 + x) + \log(9) = \log \left( \frac{x^9}{4+x} \right) = \log \left( \frac{9x}{4+x} \right) \) \hspace{1cm} \text{quotient and product rule}
Try it Now

1. Rewrite each expression using properties of logarithms
   a) \( \ln(0.85^x) \)
   b) \( \log(7x) \)
   c) \( \ln(8) - \ln(x) \)
   d) \( x\ln(4) \)

It’s just as important to know what properties logarithms do not exist as to memorize the valid properties listed above. There are several extremely common errors that students make in regards to logarithmic properties, and it is important to be mindful of these errors in order to avoid them:

<table>
<thead>
<tr>
<th>Common Errors to Avoid with Logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\log(a)}{\log(b)} \neq \frac{a}{b} )</td>
</tr>
<tr>
<td>( \log(a + b) \neq \log(a) + \log(b) )</td>
</tr>
<tr>
<td>( \log(a \cdot b) \neq \log(a) \cdot \log(b) )</td>
</tr>
</tbody>
</table>

Example 3

If possible, re-write each expression using properties of logarithms. If not possible, then say so.
   a) \( \log(x^7) \)
   b) \( \ln(7 + x) \)
   c) \( \log(7x) \)
   d) \( \ln(7 - x) \)

Solutions:
   a) \( \log(7^x) = x\log(7) \)  \( \text{Power rule} \)
   b) \( \ln(7 + x) \)  \( \text{Not possible to rewrite since there is no property for this} \)
   c) \( \log(7x) = \log(7) + \log(x) \)  \( \text{Product rule} \)
   d) \( \ln(7 - x) \)  \( \text{Not possible to rewrite since there is no property for this} \)
Using the logarithm properties is often necessary when solving problems involving exponentials and logarithms.

**General Order for Solving Exponential and Logarithmic Equations**

**Solving Exponential Equations:**
1) First get the equation in the exponential form $b^{expression} = number$.
2) Take the log of both sides, or take the ln of both sides:
   
   $\log (b^{expression}) = \log (number)$ or $\ln (b^{expression}) = \ln (number)$
3) Use the power rule to bring the exponent down:
   
   $(expression) \cdot \log (b) = \log (number)$ or $(expression) \cdot \ln (b) = \ln (number)$
4) Isolate the unknown variable using the usual algebraic solving tools (+, −, ×, ÷)

**Solving Logarithmic Equations:**
1) First use properties of logs to rewrite the equation in the logarithmic form
   
   $\log (expression) = \log (number)$ or $\ln (expression) = \log (number)$
2) Use the inverse property to rewrite the equation in exponential form:
   
   $10^{number} = expression$ or $e^{number} = expression$
3) Isolate the unknown variable using the usual algebraic solving tools (+, −, ×, ÷)

*Video Link: Another Example of Solving an Exponential Equation*

**Example 4 (Video Example Here)**

Solve the equation $150(1.04)^x = 800$ by hand and then use Excel to approximate that value.

**Solution:**

$150(1.04)^x = 800$

**Divide both sides by 150 in order to get the equation in the form $b^{expression} = number$**

$(1.04)^x = \frac{800}{150}$

**Take the log of both sides and use the Power rule**

$x \cdot \log(1.04) = \log(800/150)$

**Divide both sides by the number $\log(1.04)$ in order to isolate $x$**

$x = \frac{\log(800/150)}{\log(1.04)}$

Use Excel to get decimal approximation.

$x \approx 42.6809$

**CHECK:** $150(1.04)^{42.6809} \approx 799.9991$ (Correct. This is very close to 800.)
2. Solve each equation by hand and then use Excel to approximate that value.
   a) \(45 = 7(1.09)^x\)
   b) \(3500 = 6000(0.85)^x\)
   c) \(123(1.4)^x + 500 = 2000\)

Example 5
Solve the equation \(75 \log(7x + 1) = 200\)

Solution:
\[
75 \log(7x + 1) = 200
\]
\[
\log(7x + 1) = \frac{200}{75}
\]
\[
10^{\frac{200}{75}} = 7x + 1
\]
\[
10^{\frac{200}{75}} - 1 = 7x
\]
\[
\frac{10^{\frac{200}{75}} - 1}{7} = x
\]

Use Excel to find the decimal approximation.

And so, the solution is \(x \approx 66.16555\).

Check: \(75 \log(7 \times 66.16555 + 1) \approx 199.9999977\) (Correct. This is very close to 200.)

3. Solve each equation by hand and then use Excel to approximate that value.
   a) \(4.8 = 9\ln(x - 5)\)
   b) \(60 \log(4 - x) - 12 = 20\)
   c) \(1.5 \ln(5x) = 30\)
Example 6

In 2008, the population of Kenya was approximately 38.8 million, and was growing by 2.64% each year, while the population of Sudan was approximately 41.3 million and growing by 2.24% each year\(^1\). If these trends continue, when will the population of Kenya match that of Sudan?

Solution:
We start by writing an equation for each population in terms of \(t\), the number of years after 2008. \(K(t)\) will be the population of Kenya, and \(S(t)\) will be the population of Sudan. So, we know that

\[
K(t) = 38.8(1.0264)^t \quad \text{and} \quad S(t) = 41.3(1.0224)^t.
\]

To find when the populations will be equal, we can set the equations equal to one another:

\[
38.8(1.0264)^t = 41.3(1.0224)^t.
\]

\[
\left(\frac{1.0264}{1.0224}\right)^t = \left(\frac{41.3}{38.8}\right)
\]

\[
\log\left(\frac{1.0264}{1.0224}\right)^t = \log\left(\frac{41.3}{38.8}\right)
\]

\[
t \cdot \log\left(\frac{1.0264}{1.0224}\right) = \log\left(\frac{41.3}{38.8}\right)
\]

\[
t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)}
\]

So, it will be about 15.991 years until the populations will be equal. That means that at the end of the year 2023, it is predicted that Kenya and Sudan will have the same population.

\(^1\) World Bank, World Development Indicators, as reported on [http://www.google.com/publicdata](http://www.google.com/publicdata), retrieved August 24, 2010
Try it Now

4. Tank A contains 10 liters of water, and 35\% of the water evaporates each week. Tank B contains 30 liters of water, and 50\% of the water evaporates each week. In how many weeks will the tanks contain the same amount of water?

Interesting History:

Solving exponential equations was not the reason logarithms were originally developed. Historically, up until the advent of calculators and computers, the power of logarithms was that these log properties reduced multiplication, division, roots, or powers to be evaluated using addition, subtraction, division and multiplication, respectively, which are much easier to compute without a calculator. Large books were published listing the logarithms of numbers, such as in the table to the right. To find the product of two numbers, the product rule for logs was used.

Suppose for example we didn’t know the value of 2 times 3. Using the product rule of logs:

\[
\log(2 \cdot 3) = \log(2) + \log(3)
\]

Using the log table,

\[
\log(2 \cdot 3) = \log(2) + \log(3) = 0.3010300 + 0.4771213 = 0.7781513
\]

We can then use the table again in reverse, looking for 0.7781513 as an output of the logarithm. From that we can determine:

\[
\log(2 \cdot 3) = 0.7781513 = \log(6).
\]

By doing addition and the table of logs, we were able to determine \(2 \cdot 3 = 6\).

Although these calculations are simple and insignificant they illustrate the same idea that was used for hundreds of years as an efficient way to calculate the product, quotient, roots, and powers of large and complicated numbers, either using tables of logarithms or mechanical tools called slide rules.

Slide rules were used in place of log tables, but served the same purpose.
Important Topics of this Section
Product rule for logs
Quotient rule for logs
Power rule for logs
Solving exponential equations
Solve logarithmic equations

Try it Now Answers

1.  
(a) $x \ln(0.85)$  
(b) $\log(7) + \log(x)$  
(c) $\ln\left(\frac{8}{7}\right)$  
(d) $\ln(4^x)$

2.  
(a) $x = \frac{\log(45/7)}{\log(1.09)} \approx 21.592041$  
(b) $x = \frac{\log(3500/6000)}{\log(0.85)} \approx 3.316515$  
(c) $x = \frac{\log(1500/123)}{\log(1.4)} \approx 7.433113$

3.  
(a) $x = e^{4.8/9} + 5 \approx 6.704605$  
(b) $x = 4 - 10^{32} \approx 0.585451$  
(c) $x = \frac{e^2}{5} \approx 1.477811$

4.  4.1874 weeks
Section 5.4 Exercises

If possible, simplify to a single logarithm of the form \( \log(\text{expression}) \) or \( \ln(\text{expression}) \). If the expression is already in simplified form, then say so.

1. \( \log(4) + \log(x) \)
2. \( \ln(4) - \ln(x + 5) \)
3. \( 5\ln(x + 2) \)
4. \( 4\log(x) \)
5. \( \log(4 + x) \)
6. \( \ln(x) \cdot \ln(5) \)

Solve each equation by hand.

7. \( 540 = 20(1.04)^x \)
8. \( 140e^x = 750 \)
9. \( 250\log(x) = 500 \)
10. \( 25 = \ln(x) + 20 \)
11. \( 3(0.72)^x = 1.2 \)
12. \( 50,000 = 800(1.6)^x \)
13. \( 64 = 100\ln(x) \)
14. \( 14 - \log(x) = 9 \)

Solve each equation by hand.

15. \( 17(1.14)^x = 19(1.16)^x \)
16. \( 5e^{0.12t} = 10e^{0.08t} \)
17. \( 2\ln(3x) + 3 = 1 \)
18. \( \log(x) + \log(x + 3) = 3 \)
19. \( \log(x + 4) - \log(3) = 1 \)
20. \( 20(1.07)^x = 8(1.13)^x \)
21. \( 3e^{0.09t} = e^{0.14t} \)
22. \( 4\ln(5x) + 5 = 2 \)
23. \( \log(x + 4) + \log(x) = 9 \)
24. \( \log(5) - \log(x + 2) = 2 \)

26. Suppose the price of milk increases by 3% each year. How many years will it take until the price of milk doubles? Solve by hand (not using solver).

27. Suppose a disease strikes, and a population of trees begins to die off at a rate of 8% per year. How many years will it take for there to be \( \frac{1}{2} \) the number of trees that there were to start? Solve by hand (not using solver).
**Section 5.5 Graphs of Log Functions**

Recall that the exponential function \( f(x) = 10^x \) produces this table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = 10^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>( 10^{-3} = .001 )</td>
</tr>
<tr>
<td>-2</td>
<td>( 10^{-2} = .01 )</td>
</tr>
<tr>
<td>-1</td>
<td>( 10^{-1} = .1 )</td>
</tr>
<tr>
<td>0</td>
<td>( 10^0 = 1 )</td>
</tr>
<tr>
<td>1</td>
<td>( 10^1 = 10 )</td>
</tr>
<tr>
<td>2</td>
<td>( 10^2 = 100 )</td>
</tr>
<tr>
<td>3</td>
<td>( 10^3 = 1000 )</td>
</tr>
</tbody>
</table>

Features of \( f(x) = 10^x \):
- As \( x \) increases by 1, the y-value is multiplied by 10.
- The domain is all real numbers, \( x \).
- The range is all positive numbers, \( y \).

Recall that inverse functions have the property of flip-flopping the x and y-values. Since the logarithm function \( g(x) = \log(x) \) is the inverse of the function \( f(x) = 10^x \), we know the following:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) = \log(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.001</td>
<td>( \log(.001) = -3 )</td>
</tr>
<tr>
<td>.01</td>
<td>( \log(.001) = -3 )</td>
</tr>
<tr>
<td>.1</td>
<td>( \log(.01) = -3 )</td>
</tr>
<tr>
<td>1</td>
<td>( \log(1) = 0 )</td>
</tr>
<tr>
<td>10</td>
<td>( \log(10) = 1 )</td>
</tr>
<tr>
<td>100</td>
<td>( \log(100) = 2 )</td>
</tr>
<tr>
<td>1000</td>
<td>( \log(1000) = 3 )</td>
</tr>
</tbody>
</table>

Features of \( f(x) = \log(x) \)
- As the x-value is multiplied by 10, the x-value increases by 1.
- The domain is all positive numbers, \( x \).
- The range is all real numbers, \( y \).

Notice that the features of the logarithm function are flip-flopped in every aspect in comparison to the exponential function.

Sketching the graph, notice that as the input approaches zero from the right, the output of the function decreases in the negative direction, indicating a vertical asymptote at \( x = 0 \).

In symbolic notation we write as \( x \to 0^+ \), \( f(x) \to -\infty \), and as \( x \to \infty \), \( f(x) \to \infty \).
Graphical Features of Logarithms

Graphically, in the function \( f(x) = \log(x) \) and in the function \( g(x) = \ln(x) \):

- The graph has a horizontal intercept at (1, 0)
- The graph has a vertical asymptote at \( x = 0 \)
- The graph is increasing and concave down
- The domain of the function is \( x > 0 \), which can be written as \((0, \infty)\)
- The range of the function is all real numbers, which can be written as \((-\infty, \infty)\)

Video Link: Domain, Range, Intercepts, and Asymptotes of a Logarithmic Function.

When sketching the general logarithm function \( f(x) = \log(x) \), it can be helpful to remember that the graph will pass through the points (1, 0) and (10, 1). Similarly, when sketching the general logarithm function \( g(x) = \ln(x) \), it can be helpful to remember that the graph will pass through the points (1, 0) and (\( e \), 1). In general, when sketching the general logarithm function \( f(x) = \log_b(x) \), it can be helpful to remember that the graph will pass through the points (1, 0) and (\( b \), 1). To get a feeling for how the base affects the shape of the graph, examine the graphs below.

<table>
<thead>
<tr>
<th>Function</th>
<th>Points Passing Through</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_2(x) )</td>
<td>(1,0) and (2,1)</td>
</tr>
<tr>
<td>( \ln(x) = \log_e(x) )</td>
<td>(1,0) and (( e ),1)</td>
</tr>
<tr>
<td>( \log(x) = \log_{10}(x) )</td>
<td>(1,0) and (10,1)</td>
</tr>
</tbody>
</table>

Notice that the larger the base, the slower the graph grows. For example, the common log graph, \( f(x) = \log(x) \), while it grows without bound, it does so very slowly. Every time \( x \) is multiplied by 10, the \( y \)-value increases by 1. For example, to reach an output of 8, the input must be \( 10^8 = 100,000,000 \).

Another important observation made was the domain of the logarithm. Like the reciprocal and square root functions, the logarithm has a restricted domain which must be considered when finding the domain of any composition function involving a log.

Video Link: Graphs of Logarithmic Functions

Example 1 (*Video Link Here)

Find the domain of the function \( f(x) = \log(5 - 2x) \).

Solution:
The logarithm is only defined when the input is positive. So, this function will only be defined when \( 5 - 2x > 0 \). To determine where \( 5 - 2x > 0 \), we solve the inequality:

\[
5 - 2x > 0
\]

Add 2\( x \) on both sides.

\[
5 > 2x
\]

Divide both sides by 2.

\[
2.5 > x
\]

which can also be written as \( x < 2.5 \) or could be written as \((-\infty, 2.5)\).

Therefore, the domain of the function \( f(x) = \log(5 - 2x) \) is \((-\infty, 2.5)\).
Try it Now
1. Find the domain of the function \( f(x) = \log(x - 5) + 2 \).

Example 2
Find the equation of the function that is inverse to each of the following functions.

(a) \( f(x) = 1400e^{-0.3x} \)
(b) \( f(x) = 1250 \log(2x - 7) + 95 \)

Solution:
(a) We will write the equation as \( y = 1400e^{-0.3x} \) and note that an inverse function is a function that will flip-flop the input and output. So, if we can re-write this equation so that the input becomes \( y \) and the output becomes \( x \), then we will have the inverse.

\[
y = 1400e^{-0.3x}
\]

\[
\frac{y}{1400} = e^{-0.3x}
\]

\[
\ln \left( \frac{y}{1400} \right) = \ln(e^{-0.3x})
\]

\[
\ln \left( \frac{y}{1400} \right) = -0.3x \cdot \ln(e)
\]

\[
ln(e)=1
\]

\[
\ln \left( \frac{y}{1400} \right) = -0.3x
\]

\[
\frac{-1}{0.3} \ln \left( \frac{y}{1400} \right) = x
\]

So, the inverse function, \( g(x) \), is \( g(x) = \frac{-1}{0.3} \ln \left( \frac{x}{1400} \right) \).

(b) We will write the equation as \( y = 1250 \log(2x - 7) + 95 \) and note that an inverse function is a function that will flip-flop the input and output. So, if we can re-write this equation so that the input becomes \( y \) and the output becomes \( x \), then we will have the inverse.

\[
y = 1250 \log(2x - 7) + 95
\]

\[
y - 95 = 1250 \log(2x - 7)
\]

\[
\frac{(y-95)}{1250} = \log(2x - 7)
\]

\[
10^{\frac{(y-95)}{1250}} = 2x - 7
\]

\[
10^{\frac{(y-95)}{1250}} + 7 = 2x
\]

\[
\frac{10^{\frac{(y-95)}{1250}} + 7}{2} = x
\]
So, the inverse function, \( g(x) \), is \( g(x) = \frac{10^{(x-95)/12}}{2} + 7 \).

Try it Now
2. Find the equation of the function that is inverse to each of the following functions
   (a) \( f(x) = 45 \ln(x + 8) \)
   (b) \( f(x) = 500(1.045)^x \)

Example 3
Identify the domain of each function. Then use Excel to graph the function.
(a) \( f(x) = 1200 \ln(x + 50) \)
(b) \( f(x) = 575 \log(300x - 25) \)

Solution:
(a) The function \( f(x) = 1200 \ln(x + 50) \) will be defined wherever the input to the logarithm is greater than 0. So, we need to find where \( x + 50 > 0 \). We subtract 50 on both sides, and get a domain of \( x > -50 \). To create the graph in Excel, we can only use input values that are greater than -50.

(b) The function \( f(x) = 575 \log(300x - 25) \) will be defined wherever the input to the logarithm is greater than 0. So, we need to find where \( 300x - 25 > 0 \). We add 25 and then divide by 300 on both sides to get a domain of \( x > \frac{25}{300} \). As a simplified fraction that is \( x > \frac{1}{12} \). To create the graph in Excel, we can only use input values that are greater than 1/12, which is about 0.083.
Important Topics of this Section

Understand features of the graph of the logarithmic functions $f(x) = \log(x)$ and $f(x) = \ln(x)$
Find inverse function for exponential or logarithmic functions
Identify domain of function is logarithmic form
Use Excel to graph functions in logarithmic form

Video Link: Match Graphs with Exponential and Logarithmic Functions

Try it Now Answers

1. Domain: $\{x \mid x > 5\}$

2. (a) The inverse is $g(x) = e^{x/45} - 8$
(b) The inverse is $g(x) = \frac{1}{1.045} \cdot \ln\left(\frac{x}{500}\right)$
Section 5.5 Exercises

For each function, find the domain and use Excel to create a graph of the function.

1. \( f(x) = \log(x - 5) \)  
2. \( f(x) = \log(x + 2) \)

3. \( f(x) = \ln(3 - x) \)  
4. \( f(x) = \ln(5 - x) \)

5. \( f(x) = \log(3x + 1) \)  
6. \( f(x) = \log(2x + 5) \)

7. \( f(x) = 3\log(-x) + 2 \)  
8. \( f(x) = 2\log(-x) + 1 \)

Identify an equation for the inverse function.

9. \( f(x) = 61(0.87)^{0.02x} \)
10. \( f(x) = 1350e^{2x - 13} \)
11. \( f(x) = \ln(15 - 95x) \)
12. \( f(x) = 45\log(800x) \)

13. The function \( A(t) = 15,000e^{0.03t} \) gives the amount of money accumulated in a savings account, \( A(t) \), if the person has left the money invested for \( t \) years.
   (a) Describe the rate at which the money in the account is increasing.
   (b) How much money was initially invested in the account?
   (c) Identify the amount of time that has to pass in order for the money in the account to double.
   (d) Find an equation of the inverse function, \( t(A) \), which allows us to input the accumulated value \( A \) and output the number of years it would take to accumulate that much money.
   (e) Use the function \( t(A) \) to calculate the value \( t(15,000) \) and \( t(30,000) \). Explain how the answers to these two calculations verify your answer to part (c).

14. The function \( t(p) = \frac{27 + \log(p) - \log(45,000)}{\log(2)} \) gives the number of years, \( t \), for a certain population to reach \( p \) total people. In this case, \( t \) represents the number of years since 1950.
   (a) Find the number of years until the population is 90,000 people.
   (b) Find the number of years until the population is 45,000 people.
   (c) What do your answers to parts (a) and (b) tell us about the doubling time of this population?
   (d) Find the inverse function \( p(t) \).
   (e) Use the function \( p(t) \) to calculate the value of \( p(0) \). Check that it matches the answer found in your previous calculations.