Pythagorean Triples

Visualization Using Complex Numbers and Applications

Ryan Gray

First Reader: Irina Shablinsky

Second Reader: Athar Abdul-quader

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Contents

1 Abstract

This paper examines the nature of Pythagorean triples and how complex numbers can be used to generate, as well as, visualize Pythagorean triples. The information in this paper is formatted in a way to take the reader on a journey through Pythagorean triples, exploring all its dimensions in order to gain a deeper understanding of the subject. We start at the beginning by defining what exactly complex numbers are, then move forward into methods of how we can generate Pythagorean triples and primitive Pythagorean triples. By combining both of these sections, we gain a deeper understanding of these triples and use this knowledge to visualize them. Furthermore, this paper presents proofs and applications to help display the level of understanding we intend to gain by formatting the paper as such. We start from the very beginning, then use what is learned along the way to help further the understanding of Pythagorean triples.

2 Introduction

Throughout my years in undergraduate, I spent my time exploring the vast world of mathematics looking for something in my brain to click. I have always been quite fond of math, it has always interested me. However, when I was younger, for example in high school, I would get confused how mathematicians came to conclusions, just in general. Seemingly since numbers and mathematics are arbitrary constructs that humans invented, how exactly did mathematicians come to the conclusions they did? How did they prove something to be true? I craved an answer, I needed to find something that connected everything together. I finally understood once I was enrolled in a Number Theory course.

Everything began to make sense, all of my questions were beginning to be answered. I learned the skills to prove something in math to be definitively true, and it was quite sensational. However, the more questions of mine that were answered, the more that would arise. How did we get to the level of mathematics we are at today? Who were the people that helped us get here? I began to fall down a rabbit hole of sorts. I decided if I was going to gain a deeper understanding of the philosophy of mathematics, I needed to stick to one topic and learn everything there is to know about it. And that is exactly what I did.

It started with Fibonacci and his golden spiral. When I was taking Linear Algebra, I became fascinated by the sequence. I learned it's origins, I learned the methods for generating this sequence. Such a simple rule, $F_{n+1} = F_n + F_{n-1}$. It amazed me that something so simple could produce something so complex and elegant. My obsession with Fibonacci eventually faded, but my obsession with recursive sequences did not. And so I ended up on Euclid, the Father of Geometry. I took my philosophy of sticking to one topic and applied it to another beautiful sequence, Pythagorean triples. And just like the Fibonacci sequence, there is a lot more than meets the eye when it comes to Pythagorean triples. However, if I was going to do what I originally set out to do, then I needed to start where it all began. Pythagoras.

3 History

I find that most people are turned off by math. They think it is all mind numbing equations and proofs. And while I might find myself enjoying said equations and proofs, there is really a whole lot more to math than just, well, math. Mathematics is rich in history. The earliest surviving examples of geometrical and algebraic calculations come from Babylon and Egypt, dating back as far as 1750 BCE.[2] Babylonian mathematics was far more sophisticated than Egyptian mathematics, although there does exist an entire papyrus on Egyptian mathematics that was discovered and salvaged, known as the Rhind papyrus. Within this document, we can find the first historical representation of a variable, h, which translates to the quantity of an unknown number.[2] While this early mathematics is quite important and fascinating, we would not be where we are today without the works of ancient Greek mathematicians from the classical period. One extremely important mathematician comes to mind, Pythagoras.

3.1 Pythagoras

Around 530 BCE, Pythagoras moved from Greece to a Greek seaport in southern Italy, Crotona. There he founded the Pythagorean School of Mathematics.[3] Him and his followers believed that "numbers are the underlying and unchangeable truth of the universe"[2] and that is exactly what they tried to prove with their discoveries. Some of the things he and his followers discovered include three angles of a triangle always add up to the sum of two right angles, or 180 degrees. Another is the discovery of the square root of two being irrational. This discovery troubled Pythagoras and his followers greatly. They were of devout belief that any two lengths were integral multiples of some unit length, and therefore must be expressed as a ratio of two integers. This discovery was so troubling that Pythagoras and his followers tried to suppress this knowledge, and hide it from the general public or other mathematicians. However, it eventually leaked and it is said that the man who leaked the information was drowned at sea for his crimes.[3]

Among his discoveries, the most important is definitely the discovery of the Pythagorean Theorem. The Pythagorean Theorem is one of the earliest theorems known to ancient civilization, and arguable the most famous equation in classical mathematics. The Pythagorean Theorem is as follows, "in any right-angle triangle, the square of the longest side (hypotenuse) is equal to the sum of the squares of the two other sides."[4] Now even though Pythagoras came to this conclusion himself, the Pythagorean theorem was first discovered in ancient Babylon and Egypt. The relationship is shown on a 4000 year old tablet known as Plimpton 322.[5] While Pythagoras was not the first to discover the theorem, he is definitely responsible for the fame the theorem has culminated and also for propelling the future of mathematics and geometry forward.

The Pythagorean theorem is the cornerstone for finding Pythagorean triples, they simply would not exist without it. However, there is arguably one person more important than Pythagoras when it comes to finding Pythagorean triples, and that person is Euclid. While Pythagoras discovered the foundation for getting Pythagorean triples, Euclid developed a method for generating them.

3.2 Euclid

There is not much known about the famous mathematician, Euclid. There are very few records about the man. It is estimated that Euclid lived from around 330-275 BCE and that he did his work in Alexandria of Egypt. However, besides this, there is very little to go off of. Although very little is known about Euclid, he still holds the title as the "Father of Geometry" and there is very good reason for this. Euclid is responsible for writing arguably the most important and successful mathematical textbooks of all time, the *Elements*. Euclid's *Elements* are a culmination of the mathematical revolution that had been taking place in Greece. They are a comprehensive compilation of the known mathematics of his time done by mathematicians before his time such as Pythagoras, Hippocrates, Theudius, Theaetetus and Eudoxus.[6] There are 13 books in total and are foundational to what mathematics is today. Euclid reworked the mathematical concepts of his predecessors into a consistent whole, later to become known as Euclidean geometry, which is still as valid today. In total, the series of textbooks include 465 theorems and proofs, described in a clear, logical and elegant style, and using only a compass and a straight edge.[6] Euclid fills the other half to our whole when it comes to Pythagorean triples. In Euclid's tenth book is where we find our formula for generating Pythagorean triples.

4 Real and Complex Numbers

There is a very deep connection between Pythagorean triples and complex numbers, or more specifically, Gaussian Integers. Gaussian Integers are the set of numbers of the form $x+yi$ in which x and y are integers and $i = \sqrt{-1}$. Before looking into the connection between Pythagorean triples and Gaussian Integers, it is important that we understand from where complex numbers are derived.

4.1 Deriving Complex Numbers

Real numbers are really just a subset of complex numbers. In order to show this we are going to need a change in perspective. Think of complex numbers as ordered pairs of real numbers (a, b), (c, d) where $a, b, c, d \in \mathsf{R}$.[7] Now if we are going to be thinking of ordered pairs as real numbers, then we need to be able to do operations on them, such as addition and multiplication. We can define addition of two ordered pairs of real numbers as,

$$(a, b) + (c, d) = (a + c, b + d)$$

When we add our ordered pairs of real numbers, our result is also an ordered pair of real numbers. Next we define multiplication of two ordered pairs of real numbers as,

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

The product of our ordered pair of real numbers is also an ordered pair of real numbers. This is important, because it will help us arrive at our intended conclusion. Now we can look at some interesting cases.

$$(a, b) + (0, 0) = (a + 0, b + 0) = (a, b)$$

$$(a, b) \cdot (0, 0) = (a \cdot 0 - b \cdot 0, a \cdot 0 + b \cdot 0) = (0, 0)$$

Here it is shown that the ordered pair $(0, 0)$ acts very much like the 0 in the real numbers when added or multiplied. So one might come to the conclusion that the ordered pair $(1, 1)$ would provide similar results. Let's test this,

$$(a, b) \cdot (1, 1) = (a - b, a + b)$$

The ordered pair $(1, 1)$ does not provide similar results to the ordered pair $(0, 0)$ when multiplied to some ordered pair of real numbers. Instead we can see,

$$(a, b) \cdot (1, 0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a, b)$$

Here it is shown that the ordered pair $(1, 0)$ plays the same role for ordered pairs as the number 1 plays for real numbers. And so it is safe to assume that ordered pairs that take the form $(a, 0)$

behave the same as their real number correspondent. SO what if we were to flip the ordered pair, for example $(0, 1)$? By our definition of multiplication then,

$$(0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0)$$

In other words, $(0, 1)^2 = -1$. The ordered pair $(0, 1)$ does not share the same properties as the real numbers. However, given our new understanding of real number ordered pairs, we can now begin to transition from real numbers to complex numbers. This is quite simple, because ordered pairs are really just complex numbers. We can show this by letting z be our complex number where,

$$z = (x, y)$$

Using what we know from above, we can rewrite this as,

$$z = (x, 0) + (0, y)$$

$$z = x(1, 0) + y(0, 1)$$

Since we know that $(1, 0)$ is just our real number 1, we can write,

$$z = x + y(0, 1)$$

Since $(0, 1)^2 = -1$, it is safe to say that the ordered pair $(0, 1)$ is our imaginary number i and the final form of z looks like,

$$z = x + yi$$

With our new understanding of complex numbers, we will begin to explore the connection between them and Pythagorean triples.

5 Pythagorean Triples

A Pythagorean triple is a set of three integers a, b, and c that satisfies the Pythagorean theorem, $a^2 + b^2 = c^2$. This is the most blanket way of defining a Pythagorean triple, however there are actually multiple ways to define them. For example, any point on the coordinate plane where the distance from the origin is a natural number is a Pythagorean triple. Pythagorean triples can also be defined as some Gaussian integer squared. For example, $(x+yi)^2$; this will be further elaborated on in a later section. Our main focus, however, is not on regular Pythagorean triples, and neither was Euclid's. Instead, we are going to focus on a subcategory of Pythagorean triples known as primitive Pythagorean triples.

5.1 Primitive Pythagorean Triples

Primitive Pythagorean triples can be described as a Pythagorean triple in which the natural numbers (a, b, c) are pairwise relatively prime. In order for (a, b, c) to be pairwise relatively prime, (a, b), (b, c), and (a, c) must all have a greatest common divisor of one. If given a Pythagorean triple that is not primitive, there is always an equivalent primitive Pythagorean triple. A non-primitive Pythagorean triple is just a primitive Pythagorean triple that has been scaled by some integer n. This can be shown using the triple $(6, 8, 10)$. In order to find its equivalent primitive Pythagorean triple, all that must be done is to divide through the equation by the $gcd(a, b)$. The $gcd(6, 8) = 2$, and so,

$$\frac{6^2 + 8^2 = 10^2}{2} \rightarrow 3^2 + 4^2 = 5^2$$

The triple we end up with is $(3, 4, 5)$ which is a primitive Pythagorean triple. This specific triple is actually the smallest possible primitive Pythagorean triple.

In *Euclid's Elements Book X Proposition 29*, there is a theorem provided that describes the nature of primitive Pythagorean triples and how they can be generated. In his book, Euclid states the following theorem. "To find two rational straight lines commensurable in square only such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater."[9] In modern notation it can be written as follows:

5.1.1 Theorem 1

Let $a, b, c \in \mathsf{N}$. Then set (a, b, c) is a primitive Pythagorean triple if and only if there exists a pair of distinct relatively prime natural numbers, p and q, that are of opposite parity, $p > q$, and satisfy these three equations: [10]

$$\begin{aligned} a &= 2pq \\ b &= p^2 - q^2 \\ c &= p^2 + q^2 \end{aligned} \tag{1}$$

In order to prove Euclid's algorithm for generating primitive Pythagorean triples to be true, there are multiple things we must first prove true. First we must show that (a, b, c) is indeed a Pythagorean triple. Next we must prove that b and c are relatively prime. After we must prove that c is definitively odd. Finally we must find from where p and q are derived. Only after completing all these steps can we confidently say that we have proven Euclid's algorithm for generating primitive Pythagorean triples. We will now move onto our first proof in our series of proofs.

5.1.2 Proof 1 (part 1)

Suppose that p and q are natural numbers of opposite parity, where $p > q$, and satisfies the above equations, shown in section 1. First we must show that (a, b, c) is indeed a Pythagorean Triple. We will do this by doing a direct proof. We must then show,

$$a^2 + b^2 = (2pq)^2 + (p^2 - q^2)^2$$

In order to show this to be true, we will complete the arithmetic processes of the equation.

$$(2pq)^2 + (p^2 - q^2)^2 = 4p^2q^2 + p^4 - 2p^2q^2 + q^4$$

Now we will begin to simplify the equation.

$$4p^2q^2 + p^4 - 2p^2q^2 + q^4 = p^4 + 2p^2q^2 + q^4$$

$$p^4 + 2p^2q^2 + q^4 = (p^2 + q^2)^2 = c^2$$

As you can see, we achieved c^2 meaning this proof holds true. Since (a, b, c) is indeed a Pythagorean triple, we can move on to the next proof. Now we will show that a, b, c are relatively prime. This can be done using a proof by contradiction.

5.1.3 Proof 1 (part 2)

Claim: b and c are relatively prime.

Proof of Claim by Contradiction

Assume that b and c are not relatively prime, i.e. $gcd(b, c) > 1$.

For this to be true, there must be a prime n such that $n \mid b$ and $n \mid c$. Therefore, we can state that $n \mid b + c$. When we substitute b and c for their equations shown above, 1, we see,

$$n \mid p^2 - q^2 + p^2 + q^2$$

$$= 2p^2$$

It follows from the equation above that b and c are both odd since p and q are of opposite parity. Therefore, n must also be odd. Since this is the case, the Fundamental Property of Primes then guarantees that $n \mid p$. The Fundamental Property of Primes states that if a prime $p \mid ab$, then $p \mid a$ or $p \mid b$. Similarly, if we were then to do $n \mid c - b$, it shows,

$$n \mid p^2 + q^2 - p^2 - q^2$$

$$= 2q^2$$

In this case, the Fundamental Property of Prime also guarantees that $n \mid q$. This is a contradiction because p and q are relatively prime. Therefore, b and c must be relatively prime, thus proving the claim. Since we know that a is even, and b and c are both odd, we can conclude that a is also relatively prime to both b and c. Now we can begin to show that c is definitively odd.

5.1.4 Proof 1 (part 3)

Suppose that (a, b, c) is a primitive Pythagorean triple. We need to find relatively prime $p, q \in \mathsf{N}$ of opposite parity, where $p > q$, such that the equations (1) shown above holds. Since (a, b, c) is relatively prime, one of these numbers must be even. We will show that c is definitively odd using a proof by contradiction.

Claim: c is odd.

Proof of Claim by Contradiction

Assume that c is even.

If c is even, since (a, b, c) is relatively prime, a and b must be of the same parity. Therefore, they must both be odd. Since that is the case, suppose $a = 2k + 1$. Then $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1$, which yields that $a^2 \equiv 1 (\text{mod } 4)$. Similarly, we can show that $b^2 \equiv 1 (\text{mod } 4)$. Since c is even, then $c \equiv 0 (\text{mod } 4)$ or $c \equiv 2 (\text{mod } 4)$, then further implying that $c^2 \equiv 0 (\text{mod } 4)$. Since $a^2 + b^2 \equiv c^2 (\text{mod } 4)$, we then have that $1 + 1 \equiv 0 (\text{mod } 4)$ which is a contradiction. This then proves our claim that c is definitively odd. And because c is odd, we can also conclude that one of a or b is even and the other is odd.

5.1.5 Proof 1 (part 4)

Now, all that is left to do is find exactly from where p and q are derived. This can be done by doing a direct proof. In order to do so, let: $a^2 + b^2 = c^2$, $(a, b, c) \in \mathsf{Z}$. With a little algebraic gymnastics, we should achieve our goal.

$$a^2 = c^2 - b^2$$

$$a^2 = (c + b)(c - b)$$

$$1 = (\frac{c}{a} + \frac{b}{a})(\frac{c}{a} - \frac{b}{a})$$

Since the terms on the right of the equation are reciprocals, and because the sum or difference of two rational numbers is rational, we can then let,

$$(\frac{c}{a} + \frac{b}{a}) = \frac{p}{q} \; ; \; (\frac{c}{a} - \frac{b}{a}) = \frac{q}{p}$$

Now we can solve for $\frac{c}{a}$ and $\frac{b}{a}$,

$$\frac{c}{a} = \frac{1}{2}(\frac{p}{q} + \frac{q}{p}) = \frac{p^2 + q^2}{2pq}$$

$$\frac{b}{a} = \frac{1}{2}(\frac{p}{q} - \frac{q}{p}) = \frac{p^2 - q^2}{2pq}$$

If we plug in the values of $\frac{p}{q}$ and $\frac{q}{p}$, we see that these equations hold true.

$$\frac{c}{a} = \frac{1}{2}((\frac{c}{a} + \frac{b}{a}) + (\frac{c}{a} - \frac{b}{a}))$$

$$= \frac{1}{2}(\frac{2c}{a}) = \frac{c}{a}$$

As you can see, this holds true for $\frac{c}{a}$ and will also hold true for $\frac{b}{a}$

$$\frac{b}{a} = \frac{1}{2}((\frac{c}{a} + \frac{b}{a}) - (\frac{c}{a} - \frac{b}{a}))$$

$$= \frac{1}{2}(\frac{c}{a} + \frac{b}{a} - \frac{c}{a} + \frac{b}{a})$$

$$= \frac{1}{2}(\frac{2b}{a}) = \frac{b}{a}$$

Now we can plug in our new values for $\frac{c}{a}$ and $\frac{b}{a}$ into our original equation, which should result in Euclid's algorithm for generating primitive Pythagorean triples.

$$1 = (\frac{c}{a} + \frac{b}{a})(\frac{c}{a} - \frac{b}{a}) = (\frac{p^2 + q^2}{2pq} + \frac{p^2 - q^2}{2pq})(\frac{p^2 + q^2}{2pq} - \frac{p^2 - q^2}{2pq})$$

$$(2pq^2) = (p^2 + q^2 + p^2 - q^2)(p^2 + q^2 - p^2 - q^2) = (p^2 + q^2)^2 - (p^2 - q^2)^2$$

$$(2pq)^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2$$

With the completion of these series of proofs, we can confidently say that we proved Euclid's algorithm for generating primitive Pythagorean triples to hold true. We will now move onto how we can visualize these primitive Pythagorean triples using complex numbers.

5.2 Visualizing Pythagorean Triples

We can visualize Pythagorean triples using Gaussian integers. As shown earlier, complex numbers are ordered pairs of real numbers, and Pythagorean triples are any point on the coordinate plane where the distance from the origin is a natural number. We can easily combine these two ideas simply by squaring any Gaussian integer. Let z be our Gaussian integer,

$$z = (u + vi)$$

$$z^2 = (u + vi)^2$$

$$z^2 = (u + vi)(u + vi)$$

$$z^2 = u^2 + 2uv + i^2v^2$$

$$z^2 = 2uv + (u^2 - v^2)$$

As you might see, by squaring a complex number, we actually ended up deriving Euclid's Algorithm for generating Pythagorean triples, barring the other conditions given in the theorem. We now know that when we take any point on the complex plane, by squaring that number, we end up with an ordered pair that is the distance of some whole number from the origin.

Euclid's algorithm for generating Pythagorean triples is very useful for generating large amounts of these sets. In order to achieve a better visualization, I made a program in Wolfram Mathematica using this algorithm to generate thousands of these triples and then plotted them on a scatter plot. This can be seen below,

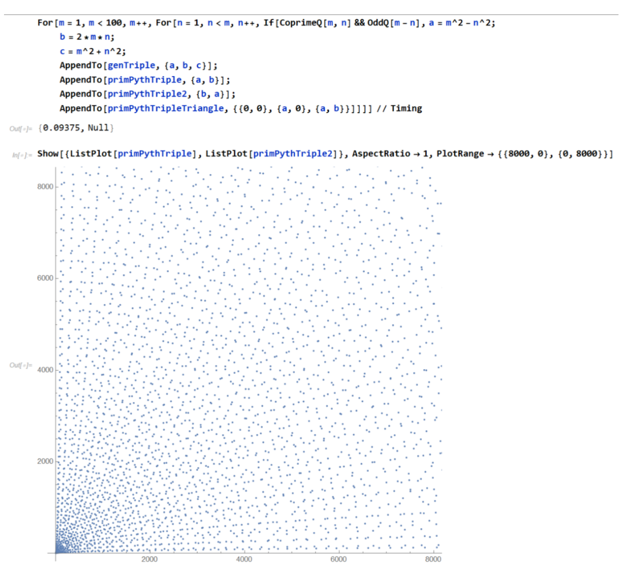

Figure 1: Primitive Pythagorean triples plotted on a scatter plot

When I generated these triples, I ended up with a really beautiful, symmetric pattern. To me, it looks like fireworks. It always fascinates me that these complex sequences of numbers, such as the Fibonacci sequence, can result in such an elegant pattern. Each point on the graph represents a primitive Pythagorean triple and is some whole number distance away from the origin. I also made a program generating every Pythagorean triple, not just primitive Pythagorean triples. This can be seen below,

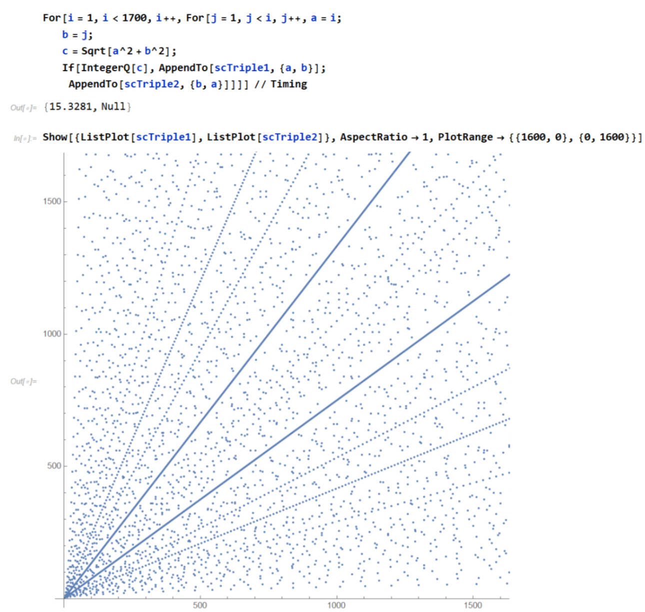

Figure 2: All Pythagorean triples plotted on a scatter plot

There is clearly a very large difference between Figure 1 and Figure 2. The most evident difference between the two are the straight lines that appear in Figure 2. These lines appear because we are generating triples with common factors. For example, the triple $(3, 4, 5)$ when multiplied by 2, we get $(6, 8, 10)$ which is also a triple. We can multiply any primitive Pythagorean triple by some scalar number and we would end up with a new infinite amount of triples.

6 Applications of Pythagorean Triples

When studying the sets of primitive Pythagorean triples that I generated, I noticed some interesting properties. Exactly one of a, b is divisible by 3, exactly one of a, b is divisible by 4, and exactly one of a, b, c is divisible by 5 [8]. Here is a list of some of the sets to help visualize this,

(3, 4, 5)	(5, 12, 13)	(8, 15, 17)	(7, 24, 25)
(20, 21, 29)	(12, 35, 37)	(9, 40, 41)	(28, 45, 53)
(11, 60, 61)	(16, 63, 65)	(33, 56, 65)	(48, 55, 73)
(13, 84, 85)	(36, 77, 85)	(39, 80, 89)	(65, 72, 97)

(20, 99, 101)	(60, 91, 109)	(15, 112, 113)	(44, 117, 125)
(88, 105, 137)	(17, 144, 145)	(24, 143, 145)	(51, 140, 149)
(85, 132, 157)	(119, 120, 169)	(52, 165, 173)	(19, 180, 181)
(57, 176, 185)	(104, 153, 185)	(95, 168, 193)	(28, 195, 197)
(84, 187, 205)	(133, 156, 205)	(21, 220, 221)	(140, 171, 221)
(60, 221, 229)	(105, 208, 233)	(120, 209, 241)	(32, 255, 257)
(23, 264, 265)	(96, 247, 265)	(69, 260, 269)	(115, 252, 277)
(160, 231, 281)	(161, 240, 289)	(68, 285, 293)	

Figure 3: List of some primitive Pythagorean triples

You can very quickly verify these properties to be true by testing it out on any of these primitive Pythagorean triples. However, there was a property that I did not discover just by simply looking over the sets of triples, but rather through my research. This would be extended Pythagorean triples.

6.1 Extended Pythagorean Triples

Something interesting happens when you take multiple Pythagorean triples and put them together. What you end up with is an extended version of the Pythagorean theorem, known as an extended Pythagorean triple. An extended Pythagorean triple can be define as (a, b, c, d, e) where (a, b, c) and (c, d, e) are Pythagorean triples. For example, $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$, therefore $3^2 + 4^2 + 12^2 = 13^2$. An extended This can be broken down into the following general equations, $a^2 + b^2 = c^2$ and $c^2 + d^2 = e^2$, therefore $a^2 + b^2 + d^2 = e^2$ [1].

Table 1: Extended Pythagorean triples

a	b	c	d	e
3	4	5	12	13
5	12	13	84	85
7	24	25	312	313
9	40	41	840	841

Given some examples such as, $5^2 + 6^2 + 30^2 = 31^2$, there is a noticeable pattern when constructing these extended triples which would suggest an infinite set of extended Pythagorean triples. These extended triples can be written as such, $(n, n + 1, n(n + 1), n(n + 1) + 1)$ and we are going to prove this form to hold true for every extended Pythagorean triple using a direct proof.

6.1.1 Theorem 2

For every $n \in \mathsf{N}$, $(n, n+1, n(n+1), n(n+1)+1)$ is an extended Pythagorean triple.

6.1.2 Proof 2

If the above set is in fact an extended Pythagorean triple, then it should hold true that,

$$n^2 + (n+1)^2 + (n(n+1))^2 = (n(n+1)+1)^2$$

In order to show this to be true, we will complete the arithmetic processes of the equation.

$$n^2 + (n+1)^2 + (n^2+n) = (n^2+n+1)^2$$

Now we will begin to simplify the equation.

$$n^2 + n^2 + 2n + 1 + n^4 + 2n^3 + n^2 = n^4 + 2n^3 + 3n^2 + 2n + 1$$

$$n^4 + 2n^3 + 3n^2 + 2n + 1 = n^4 + 2n^3 + 3n^2 + 2n + 1$$

Since this equation seems to hold true, it is safe to say that $(n, n+1, n(n+1), n(n+1)+1)$ is in fact an extended Pythagorean triple.

While extended Pythagorean triples are certainly fascinating, I find myself more interested in the connection of Pythagorean triples to other sectors in number theory. My favorite recursive sequence is probably the Fibonacci sequence and so it is very exciting to me that there is a very interesting connection between the Fibonacci sequence and Pythagorean triples.

6.2 Fibonacci And Pythagorean Triples

The Fibonacci sequence is my favorite complex sequence for one reason, it is deceivingly simple. It is incredibly easy to generate Fibonacci numbers, starting the sequence with $0, 1$, but the numbers grow as such an exponential rate that it's almost impossible to calculate them by hand after a certain point. Along with this, it produces an incredibly elegant and beautiful structure known as the golden spiral.

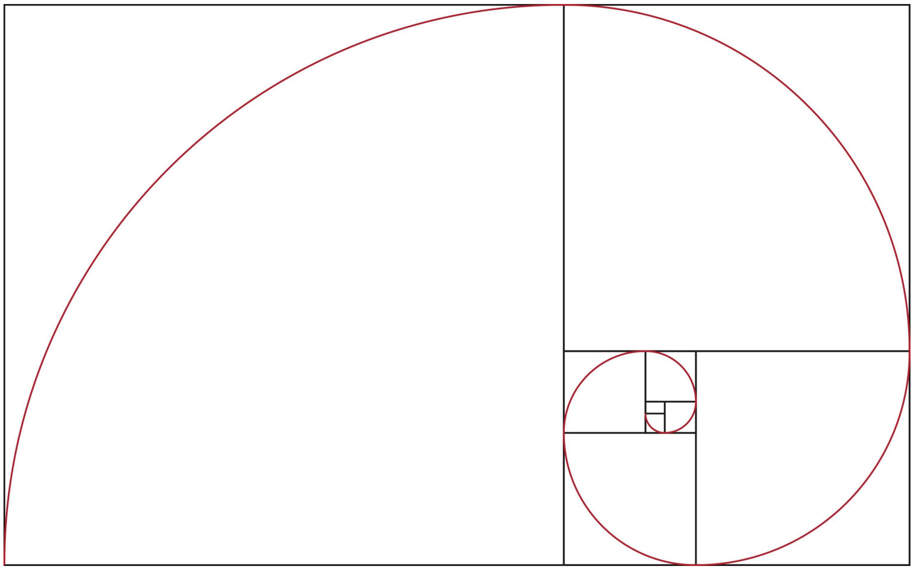

Figure 4: The Golden Spiral

The Fibonacci sequence is as follows: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144...$

The equation for generating Fibonacci numbers is very simple. The n^{th} Fibonacci number can be denoted by F_n and the equation is as follows,

$$F_{n+1} = F_n + F_{n-1}$$

Following a pattern of simplicity, the connection between the Fibonacci sequence and Pythagorean triples is also quite simple.

6.2.1 Theorem 3

Starting with F_5, every second Fibonacci number is the length of the hypotenuse of a right angled triangle, or in other words the longest length (c) in a Pythagorean triple.

6.2.2 Proof 3 (part 1)

In order to prove our theorem, we must first prove two other mathematical statements to be true.

$$F_{2n} = F_n(F_{n+1} + F_{n-1})$$

$$F_{2n+1} = F_{n+1}^2 + F_n^2$$

We will rely on the results of these proofs to eventually complete our proof of Theorem 3. We are going to be using a proof by induction to prove these two mathematical statements.

Base Case: Let $n = 1$. Then:

$$F_{2n} = F_2 = 1 = 1(1 + 0) = F_1(F_2 + F_0)$$

$$F_{2n+1} = F_3 = 2 = 1^2 + 1^2 = F_2^2 + F_1^2$$

These statements hold true for $n = 1$ so the next step is to make our assumption.

Assume: $F_{2n} = F_n(F_{n+1} + F_{n-1})$ and $F_{2n+1} = F_{n+1}^2 + F_n^2$ hold true for some $n \geq 1$.

With our assumption, we can now move onto our inductive proofs.

Prove: Using our assumption, we will now begin to prove these for $n + 1$.

$$F_{2(n+1)} = F_{2(n+2)} = F_{2n+1} + F_{2n}$$

$$= F_{n+1}^2 + F_n^2 + F_n(F_{n+1} + F_{n-1})$$

$$= F_{n+1}(F_n + F_{n+1}) + F_n(F_n + F_{n-1})$$

$$= F_{n+1}F_{n+2} + F_nF_{n+1}$$

$$= F_{n+1}(F_{n+2} + F_n)$$

This statement holds true for $n + 1$ and so we can move onto proving the second statement.

$$F_{2(n+1)+1} = F_{2n+3} = F_{2n+2} + F_{2n+1}$$

$$= F_{n+1}(F_{n+2} + F_n) + F_{n+1}^2 + F_n^2$$

$$= F_{n+1}(F_{n+1} + 2F_n) + F_{n+1}^2 + F_n^2$$

$$= (F_{n+1}^2 + 2F_nF_{n+1} + F_n^2) + F_{n+1}^2$$

$$= (F_{n+1} + F_n)^2 + F_{n+1}^2$$

$$= F_{n+2}^2 + F_{n+1}^2$$

This statement holds true for $n+1$. Therefore, we can begin our proof to show our original theorem is true.

6.2.3 Proof 3 (part 2)

Since we are only interested in odd terms of the Fibonacci sequence greater than F_5, we can say $n \geq 2$ and look at F_{2n+1}.

With the knowledge from our mathematical statements that we just proved, we know,

$$F_{2n+1}^2 = (F_{n+1} + F_n)^2$$

Since we are looking to show that every second Fibonacci number from F_5 is the length of the hypotenuse of a right angled triangle, we can say we are looking to show something resembling the form of the Pythagorean theorem, or,

$$F_{2n+1}^2 = a^2 + b^2$$

This is actually quite simple to show thanks to the our earlier proof.

$$F_{2n+1}^2 = (F_{n+1} + F_n)^2$$

$$= F_{n+1}^4 + 2F_{n+1}^2 F_n^2 + F_n^4$$

$$= F_{n+1}^4 + 4F_{n+1}^2 F_n + 1^2 - 2F_{n+1}^2 F_n^2 + F_n^4$$

$$= 4F_{n+1}^2 F_n^2 + (F_{n+1}^4 - 2F_{n+1}^2 F_n^2 + F_n^4)$$

$$= (2F_{n+1}F_n)^2 + (F_{n+1}^2 - F_n^2)^2$$

So for any $n \geq 2$ we can define,

$$a = 2F_{n+1}F_n$$

$$b = F_{n+1}^2 - F_n^2$$

$$c = F_{2n+1}$$

Since this holds true for the Pythagorean theorem, $a^2 + b^2 = c^2$, we have therefore arrived at our intended result, thus proving true that every other Fibonacci number from F_5 is in fact the length of the hypotenuse of a right angled triangle.

7 Reflection about the Research Process

7.1 Wolfram Mathematica

As seen in section 4.2, I used Wolfram Mathematica in order to create the program which generated the Pythagorean triples as well graph them. I wanted to use Mathematica for a portion of my Senior

Project because I am quite literally obsessed with the program. It is such an extremely powerful tool, and when utilized properly, can proved results that would take a combination of other programs to achieve. The learning curve for Mathematica was quite steep, I began learning the program when I was enrolled in the Number Theory course. I struggled a lot at first, but eventually I started to grow a deep understanding of how to use the program efficiently and started making really efficient code. The library of built in functions is extremely useful, making Mathematica extremely versatile depending on the type of project one might be pursuing. Overall, I am glad I decided to use Mathematica. I believe it helped me better realize what sectors in mathematics I am truly interested in.

7.2 LaTeX

I originally began writing my Senior Thesis Paper in a word document last semester. However, this semester I began taking the course Discrete Mathematics and in the class we use the language LaTeX exclusively for our work. After learning the language and becoming familiar with it, I decided that I will rewrite the portion of my paper that I had done in the word document, in LaTeX, as well as complete the rest of the paper in LaTeX. Like Mathematica, LaTeX is an extremely powerful tool for mathematicians. It made writing mathematics simpler than I ever could have imagined. I struggled writing equations in the word document, so when I was exposed to LaTeX for the first time, you might be able to imagine how excited I was to learn the language. There was definitely a bit of a learning curve, and quite honestly I am still continuing to learn the language and have been learning throughout the writing process of this paper. I believe it was the right decision to make the switch to LaTeX because I know that knowing LaTeX will only benefit me in the future and I feel that I have grown as a mathematician.

8 Conclusion

The intended purpose of this paper was not for it to be purely academic. I struggle a lot with outlining this paper because I was unsure of the direction I wanted to take it. Ultimately, I decided on a more personal approach, really getting to the root of my fascination of mathematics. Displaying the multiple layers involved when researching a topic like Pythagorean triples. If I chose to make this purely academic, I very likely would have just focused on the proofs. However, I felt it was necessary to highlight certain aspects of the subject that one typically would not find in an academic paper. Some things you would not normally see are the history of the mathematicians that made it possible to write this paper, or my own personal fascination in the subject. I believe

I succeeded in not only achieving my goal in providing a broad encapsulation of the subject, but also showing my own deeper understanding of Pythagorean triples.

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