Stern's Diatomic Sequence 0,1,1,2,1,3,2,3,1,4,...

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1 Introduction.

Stern's diatomic sequence is a simply defined sequence with an amazing set of properties. Our goal is to present many of these properties—those that have most impressed the author. The diatomic sequence has a long history; perhaps first appearing in print in 1858 [28], it has been the subject of several papers [22, 23, 5, 9, 12, 13, 14]; see also [27, sequence 002487]! It is difficult to present anything truly new on this topic; indeed, most of the properties discussed here have appeared in print several times. Some properties however are apparently "folklore" theorems and some are perhaps new. We try to present either simple proofs or proofs that complement existing proofs. We have tried to give a sufficient set of references for the interested reader to delve further into the topic or to find other proofs. We expect that the interested reader will see clear directions for further research.

Beyond the basic and most well-known properties discussed in Section 2, we shall focus on the combinatorics of the sequence (i.e., what the numbers count) in Section 3, some surprising parallels with the Fibonacci sequence in Section 4, enumeration of the rational numbers in Section 5, some connections with Minkowski's question mark function in Section 6, and a geometric-probabilistic view in Section 7.

2 Stern's diatomic sequence and array.

Consider the following sequence: $a_0 = 0$, $a_1 = 1$, and

$$\begin{cases} a_{2n} & = a_n, \\ a_{2n+1} & = a_n + a_{n+1}. \end{cases} \tag{1}$$

Looking at the first few terms, we get

$$0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, ...$$

This sequence has been well studied and has gone by several names. One of the first was coined by Dijkstra who defined $\text{fusc}(n) := a_n$, apparently in reference to the obfuscation it embodied (see [9]). Another is "Stern's diatomic

sequence" (see [3] for example). The latter name comes from the following array of numbers, called "Stern's diatomic array":

1															1
1								2							1
1				3				2				3			1
1		4		3		5		2		5		3		4	1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

This is similar to Pascal's triangle in that every entry in all but the top row is the sum of certain entries above. Specifically, given the nth row, we get the next row by repeating the nth row but, between each two entries, we put the sum of those entries. Any entry which is at the top of a column is the sum of two entries on the previous row while any other entry just repeats the entry directly above it. All entries not in the first or last column contribute to three below and receive from either one or two above, so that the "valence" (here meaning the number of bonds made with other entries) is 4 or 5. Hence, the term "diatomic": conceivably an alloy with two types of atoms, of chemical valence 4 and 5 (e.g., carbon and gold), could combine to make a kind of crystal described by the diatomic array (see Figure 3 in Section 7). Of course, such a crystal could only exist in hyperbolic space since row size increases exponentially.

We observe that the nth row of the diatomic array contains $2^{n-1}+1$ elements while the sum of the elements in the nth row is $3^{n-1}+1$. The diatomic array can be written symbolically in terms of $\{a_n\}$:

a_1								a_2
a_2				a_3				a_4
a_4		a_5		a_6		a_7		a_8
a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
.	.	.	.	.	.	.	.	.

Each row of the diatomic array is palindromic and we may express this symmetry as a formula: for all n and all $k \leq 2^n$,

$$a_{2^n+k} = a_{2^{n+1}-k}. \tag{2}$$

We also claim, again for all k, n with $k \leq 2^n$,

$$a_{2^n+k} = a_k + a_{2^n-k}. \tag{3}$$

This is left as an exercise in induction (on n).

From now on, we let F_n denote the nth Fibonacci number, which can be defined either recursively by

$$F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1} \ (n \geq 2)$$

or by Binet's formula

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} = \frac{\phi^n - \overline{\phi}^n}{\phi - \overline{\phi}}, \tag{4}$$

where ϕ is the "golden ratio" $(1+\sqrt{5})/2$ and $\overline{\phi}$ is its algebraic conjugate $(1-\sqrt{5})/2$.

Notice that the right column of the diatomic array matches the left. Removal of the right column and squeezing everything to the left yields the "crushed" array:

1									
1	2								
1	3	2	3						
1	4	3	5	2	5	3	4		
1	5	4	7	3	8	5	7	2	...

Theorem 2.1. *The following are properties of the crushed array.*

a) Each column is an arithmetic sequence (i.e., successive differences are constant). The sequence of these differences is

$$0, 1, 1, 2, 1, 3, 2, 3, 1, 4, ...;$$

a sequence with which we are now familiar!

b) The row maxima are 1,2,3,5,8,13,...; part of the Fibonacci sequence!

c) The locations of row maxima are 1,3,5,7,11,13,21,27,43,53,... (e.g., $a_1 = 1$ is the maximum of the first row, $a_5 = a_7 = 3$ is the maximum of the third row), but note that since the diatomic array is palindromic, row maxima appear twice in all but the first two rows. The locations of earliest maxima of the rows are 1,3,5,11,21,43,85,..., a subsequence $J_2, J_3, ...$ of "Jacobsthal's sequence" (J_n) [27, sequence A001045]. It obeys a three-term recurrence relation:

$$J_{n+1} = J_n + 2J_{n-1}.$$

The sequence 2,3,7,13,27,53,..., the locations of the last maxima of the rows, also obeys this three-term recurrence relation.

Proof. Statement (a) can be formulated as $a_{2^n+k} = na_k + c$. This is not hard to see: by (2) and (3),

$$a_{2^n+k} + a_k = a_{2^{n+1}+k},$$

from which it follows that $a_{2^n+k} - na_k$ is independent of n.

Let M_n denote the maximum of the nth row of the diatomic array. Clearly every row maximum must have odd index. Hence $M_{n+1} = a_{2k+1}$ for some k and so M_{n+1} equals a sum of two consecutive terms in the previous row: a_k

and a_{k+1}. One of k and $k+1$ is even and so appears in a previous row, and therefore

$$M_{n+1} \leq M_n + M_{n-1}.$$

Since $M_1 = 1 = F_2$ and $M_2 = 2 = F_3$, by induction, $M_n \leq F_{n+1}$.

Jacobsthal's sequence is given by a Binet-type formula:

$$J(n) := J_n = \frac{2^n - (-1)^n}{3},$$

and so $J(n)$ is always odd. Also, $J(n) - 2J(n-1) = \pm 1$, and so

$$a_{J(n+1)} = a_{J(n)+2J(n-1)} = a_{J(n)} + a_{2J(n-1)} = a_{J(n)} + a_{J(n-1)}.$$

Since $a_{J(1)} = a_1 = 1 = F_1$ and $a_{J(2)} = a_2 = 1 = F_2$,

$$a_{J(n)} = F_n.$$

□

This result, and more, goes back to a paper by Lind [23], who gives a formula for the locations of both Fibonacci numbers and Lucas numbers in the diatomic array. Several of these results have been ascribed to Tokita by Sloane [27, sequence A002487].

3 Finding a Counting Interpretation.

An expression of n as a finite sum $n = e_0 + e_1 \cdot 2 + e_2 \cdot 4 + e_3 \cdot 8 + e_4 \cdot 16 + \cdots + e_k \cdot 2^k$ where $e_i \in \{0, 1, 2\}$ is called a *hyperbinary representation* of n. There are, in general, many hyperbinary representations of an integer n.

Theorem 3.1. *For all n, the number of hyperbinary representations of n equals a_{n+1}.*

Proof. We shall prove this using generating functions (see [15] or [31] for a general treatment). Consider the generating function for the diatomic sequence:

$$A(x) := \sum_{n=0}^{\infty} a_{n+1} x^n.$$

By (b) and (c) of Theorem 2.1, if a_k occurs in row $n-1$, then $a_k \leq F_n \leq \phi^n$ and $k \geq 2^{n-2}$, and so $a_k^{1/k} \leq \phi^{n/2^{n-2}}$. It follows that $A(x)$ has a positive radius of convergence and, for x in the interval of convergence,

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} a_{2n+1} x^{2n} + \sum_{n=0}^{\infty} a_{2n+2} x^{2n+1} \text{ [split into sums where } n \text{ is even or odd]} \\ &= \sum_{n=0}^{\infty} (a_n + a_{n+1}) x^{2n} + \sum_{n=0}^{\infty} a_{n+1} x^{2n+1} \text{ [using (1)]} \\ &= (1 + x + x^2) A(x^2) \text{ [using } a_0 = 0 \text{ and cleaning up]}. \end{aligned}$$

Continuity of $A(x)$ at 0 combined with the fact that $A(0) = 1$ gives:

$$\begin{aligned} A(x) &= (1 + x + x^2)(1 + x^2 + x^4)(1 + x^4 + x^8)(1 + x^8 + x^{16}) \cdots \\ &= (x^{0\cdot 1} + x^{1\cdot 1} + x^{2\cdot 1})(x^{0\cdot 2} + x^{1\cdot 2} + x^{2\cdot 2})(x^{0\cdot 4} + x^{1\cdot 4} + x^{2\cdot 4}) \cdots \end{aligned}$$

A generic term in the product looks like

$$x^{e_0 + e_1 \cdot 2 + e_2 \cdot 4 + e_3 \cdot 8 + e_4 \cdot 16 + \cdots},$$

where each $e_i \in \{0, 1, 2\}$, and so, combining like terms, the coefficient of x^n in $A(x)$ is on the one hand the number of ways to write n as a sum of the form $e_0 + e_1 \cdot 2 + e_2 \cdot 4 + e_3 \cdot 8 + e_4 \cdot 16 + \cdots$ where $e_i \in \{0, 1, 2\}$, and on the other hand a_{n+1}.

□

Theorem 3.1 goes back to Carlitz [6], who also showed that a_n is the number of odd values of r for which the Stirling number (of the second kind) $S(n, r)$ is also odd. Bicknell-Johnson [3] showed that a_{n+1} counts the number of partitions of n as a sum of distinct even-subscripted Fibonacci numbers. Finch [11] notes that a_{n+1} is the number of "alternating bit sets" in n (the number of ways to choose a subsequence of the form 1010....10 from the sequence of 1's and 0's which represents n in binary). Finally, there is a relation with the "Towers of Hanoi" problem (see [19]).

4 Parallels with the Fibonacci sequence.

We have seen that the maximum of the nth row of the diatomic array is the $(n + 1)$st Fibonacci number. The numbers a_n are, in several surprising ways, analogues of the Fibonacci numbers.

Consider Pascal's triangle:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & \boxed{1} & & & \\ & & 1 & & \boxed{3} & & 3 & & 1 & & \\ & \boxed{1} & & 4 & & 6 & & 4 & & 1 & \\ & & . & . & . & . & . & . & & & \end{array}$$

which is composed of binomial coefficients $\binom{n}{k} := n!/(k!(n-k)!)$. The Fibonacci numbers appear as diagonal sums across Pascal's triangle. For example, the sum along the boxed numbers is $5 = F_5$ and, in general,

$$\sum_i \binom{n-i}{i} = \sum_{2i+j=n} \binom{i+j}{i} = F_{n+1}. \tag{5}$$

In general, we let $a \bmod b$ denote the remainder of a upon division by b or, equivalently, $a \bmod b := a - b\lfloor a/b \rfloor$.

Consider now "Pascal's triangle mod 2" where each entry $\binom{n}{k}$ is replaced by

$$\binom{n}{k} \bmod 2 := \begin{cases} 1 \text{ if } \binom{n}{k} \text{ is odd,} \\ 0 \text{ if } \binom{n}{k} \text{ is even:} \end{cases}$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 0 & & \boxed{1} & & & \\
 & & 1 & & \boxed{1} & & 1 & & 1 & & \\
 & \boxed{1} & & 0 & & 0 & & 0 & & 1 & \\
 & & . & . & . & . & . & . & & & \\
\end{array}$$

The diatomic sequence appears as diagonal sums over Pascal's triangle mod 2. For example, the boxed entries sum to $3 = a_5$ and, in general, we get the following result.

Theorem 4.1.

$$\sum_{2i+j=n} \left[\binom{i+j}{i} \bmod 2 \right] = a_{n+1}. \tag{6}$$

We can prove this theorem using Kummer's theorem of 1852 (see [21] or [15, exercise 5.36]):

Lemma 4.2. *Let p be a prime number. The largest k such that p^k divides $\binom{i+j}{i}$ is the number of carries when adding i and j base p.*

For example, in base 3, $13 = 111_3$ and $5 = 12_3$, and adding them together requires 2 carries. By the lemma, 9 should be the largest power of 3 which divides $\binom{13+5}{5} = 8568$, which is easily verified.

Proof of Theorem 4.1. Lemma 4.2 implies that $\binom{i+j}{i} \bmod 2$ is 1 if and only if adding i and j base 2 requires no carries or, saying that another way, i and j share no 1's in the same locations in their base-2 expansions. The left side of (6) then counts the number of ways to write n as a sum $2i + j$ where i and j share no 1's in the same locations in their base-2 expansions, which is the number of hyperbinary expansions of n. □

Most proofs of Fibonacci identities (of which there are hundreds!—see the books by Koshy [20] and Vajda [30] for example) use or can use one of two basic tools: the Binet formula and the recurrence formula. We find analogues of these for the diatomic sequence.

Proposition 4.3. *Stern's diatomic sequence satisfies*

$$a_{n+1} = a_n + a_{n-1} - 2(a_{n-1} \bmod a_n). \tag{7}$$

Proof. We prove this by a version of induction: we suppose, for a fixed n, that (7) holds and show that (7) holds for $2n+1$ and for $2n$.

Note that $a_{2n} \bmod a_{2n+1} = a_{2n} = a_n$ and so

$$\begin{aligned} a_{2n+2} = a_{n+1} &= (a_{n+1} + a_n) + a_n - 2(a_{2n} \bmod a_{2n+1}) \\ &= a_{2n+1} + a_{2n} - 2(a_{2n} \bmod a_{2n+1}). \end{aligned}$$

Therefore (7) holds for $2n+1$. Also, $a_{2n-1} \bmod a_{2n} = (a_n + a_{n-1}) \bmod a_n = a_{n-1} \bmod a_n$. Thus

$$\begin{aligned} a_{2n+1} = a_n + a_{n+1} &= a_n + (a_n + a_{n-1} - 2(a_{n-1} \bmod a_n) \\ &= a_{2n} + a_{2n-1} - 2(a_{2n-1} \bmod a_{2n}) \end{aligned}$$

and so (7) holds for $2n$. □

We remark that just as (5) can be proven using the recurrence formula for Fibonacci numbers, one can prove Theorem 4.1 by using Proposition 4.3.

Binet's formula (4), a workhorse in the theory of Fibonacci and Lucas numbers, can be rewritten as

$$F_{n+1} = \sum_{k=0}^{n} \phi^k \overline{\phi}^{n-k}. \tag{8}$$

This version has an analogue for Stern's sequence.

Proposition 4.4. *Let $s_2(k)$ denote the number of 1's in the binary expansion of k and let $\sigma, \overline{\sigma} := e^{\pm \pi i/3}$ be the primitive sixth roots of unity. Then*

$$a_{n+1} = \sum_{k=0}^{n} \sigma^{s_2(k)} \overline{\sigma}^{s_2(n-k)}. \tag{9}$$

Proof. We show that the generating functions of both sides agree. Note that $s_2(k)$, the number of 1's in the binary expansion of k, obviously satisfies and can be in fact defined by

$$\begin{cases} s_2(2k) & = s_2(k) \\ s_2(2k+1) & = s_2(k) + 1 \\ s_0 & = 0. \end{cases}$$

Let $\sigma := e^{i\pi/3}$ and, for real x, let

$$B(x) := \sum_{k=0}^{\infty} \sigma^{s_2(k)} x^k.$$

Then $B(0) = 1$, and

$$\begin{aligned} B(x) &= \sum_{k=0}^{\infty} \sigma^{s_2(2k)} x^{2k} + \sum_{k=0}^{\infty} \sigma^{s_2(2k+1)} x^{2k+1} \\ &= \sum_{k=0}^{\infty} \sigma^{s_2(k)} x^{2k} + x \sum_{k=0}^{\infty} \sigma^{s_2(k)+1} x^{2k} = (1 + \sigma x) B(x^2). \end{aligned}$$

Since $(1+x+x^2) = (1+\sigma x)(1+\overline{\sigma}x)$, we have

$$|B(0)|^2 = 1, |B(x)|^2 = (1+x+x^2)|B(x^2)|^2$$

and so $|B(x)|^2$ has the same product formula as $A(x)$. Hence they agree and

$$\begin{aligned}\sum_{n=0}^{\infty} a_{n+1}x^n &= \left(\sum_{k=0}^{\infty} \sigma^{s_2(k)}x^k\right) \cdot \left(\sum_{k=0}^{\infty} \overline{\sigma}^{s_2(k)}x^k\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \sigma^{s_2(k)}\overline{\sigma}^{s_2(n-k)}\right)x^n.\end{aligned}$$

□

The sequence $\{s_2(k)\}$ has been well studied; see [27, sequence A000120] and references therein. The sequence $\{2^{s_2(k)}\}$, also known as Gould's sequence, also is well studied (see [27, sequence A001316])—it is the number of 1's in the kth row of Pascal's triangle modulo 2. General sums across Pascal's triangle modulo 2, specifically

$$x_n := \sum_{ai+bj=n} \left[\binom{i+j}{i} \bmod 2\right],$$

are discussed in [24].

Proposition 4.4 leads to several alternative ways to write a_n. For example, since $\overline{\sigma} = \sigma^{-1}$ and the left side of (9) is real, one can write

$$a_{n+1} = \sum_{k=0}^{n} \Re(\sigma^{s_2(k)-s_2(n-k)}) = \sum_{k=0}^{n} \cos(\pi(s_2(k)-s_2(n-k))/3).$$

Finally, since

$$\Re(\sigma^n) = \frac{1}{2}(-1)^n(3\cdot 1_{(3)}(n)-1),$$

where $1_{(3)}(n)$ is 1 if n is a multiple of 3 and 0 otherwise, we find

$$a_{n+1} = \frac{1}{2}\sum_{k=0}^{n}(-1)^{s_2(k)-s_2(n-k)}[3\cdot 1_{(3)}(s_2(k)-s_2(n-k))-1].$$

Note that by Binet's formula (4),

$$\lim_{n\to\infty} \frac{F_n}{\phi^n} = \frac{1}{\sqrt{5}}.$$

Finch [11, p. 149] asked if it was true that

$$\limsup_{n\to\infty} \frac{a_n}{\phi^{\log_2(n)}} = 1 \tag{10}$$

(though there is no particular evidence that this should be true and numerical evidence suggests that it is not). To put this into context, Calkin and Wilf, in an unpublished note (referred to in [11]), show that the left side of (10) satisfies $.9588 < L < 1.1709$. By Theorem 4.1, a_n is a diagonal sum across Pascal's triangle mod 2. Summing along rows instead gives, as we saw before, $2^{s_2(n)}$. It is easy to see (we leave it as an exercise) that

$$\limsup_{n\to\infty} \frac{2^{s_2(n)}}{2^{\log_2(n)}} = 1.$$

Harborth [17] shows a similar result: if $S(n)$ is the number of odd numbers in the first n rows of Pascal's triangle (i.e., $S(n) = \sum_{k\le n} 2^{s_2(k)}$), then

$$\limsup_{n\to\infty} \frac{S(n)}{3^{\log_2(n)}} = 1.$$

5 Enumerating the rationals.

Everyone knows that the rational numbers are countable (that is, are in one-to-one correspondence with $\mathbb{Z}^+$, the set of positive integers) but few actually know of an explicit such correspondence. The diatomic sequence gives one. Specifically,

$$\left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, ..., \frac{a_{n+1}}{a_n}, ...\right\}$$

is exactly the set of positive rational numbers (and no number appears twice). We state this as a theorem; our proof (which we present for completeness) is new but can't beat the one by Calkin and Wilf as presented in their charming paper [5].

Theorem 5.1. *The map*

$$n \longmapsto \frac{a_n}{a_{n+1}}$$

is a bijection from $\mathbb{Z}^+$ to the set of positive rational numbers.

Proof. We shall prove the theorem by showing that every relatively prime pair $[a, b]$ appears exactly once in the list

$$\mathcal{L} := [1,1], [1,2], [2,1], [1,3], [3,2], [2,3], [3,1], ..., [a_n, a_{n+1}], ...$$

Define the "Slow Euclidean Algorithm" (SEA) on pairs of positive integers: given a pair $[a, b]$, subtract the smaller from the larger; repeat; stop when equal. For example,

$$[4,7] \mapsto [4,3] \mapsto [1,3] \mapsto [1,2] \mapsto [1,1].$$

Clearly, this algorithm always terminates and, since the greatest common divisor is preserved at each step, it must terminate with $[g, g]$, where $g = \gcd(a, b)$.

Let $\mathcal{L}_n := [a_n, a_{n+1}]$. By (1), it's easy to see that for $n > 1$,

$$\text{SEA} : \mathcal{L}_{2n}, \mathcal{L}_{2n+1} \longmapsto \mathcal{L}_n$$

and, moreover, if $\text{SEA} : [a, b] \mapsto \mathcal{L}_n$ then either $[a, b] = \mathcal{L}_{2n}$ or $[a, b] = \mathcal{L}_{2n+1}$. Since $\mathcal{L}_1 = [1, 1]$, every $\mathcal{L}_n$ is a relatively prime pair.

If there is relatively prime pair $[a, b]$ not in $\mathcal{L}$, then all of its successors under the SEA, including $[1, 1]$, are not in $\mathcal{L}$—a contradiction. Hence, every relatively prime pair appears in $\mathcal{L}$.

The pair $[1, 1]$ appears only once in $\mathcal{L}$. In general, no relatively prime pair appears more than once in $\mathcal{L}$ for, otherwise, there exists a smallest $n > 1$ such that $\mathcal{L}_n = \mathcal{L}_m$ for some $m > n$. Applying one step of the SEA to both $\mathcal{L}_n$ and $\mathcal{L}_m$ forces $\lfloor n/2 \rfloor = \lfloor m/2 \rfloor$ and therefore $m = n + 1$. Thus $a_n = a_{n+1} = a_{n+2}$, a contradiction. □

It is clear that the map $n \mapsto a_n/a_{n+1}$ can be efficiently computed (i.e., computed in polynomial time in the number of digits of n) but, unfortunately, the same is not true of the inverse, since the first k where $(a_k, a_{k+1}) = (1, n)$ is when $k = 2^{n-1}$. So we can't even write this down in polynomial time (n has $\log n$ bits but the first position of $1/n$ has roughly n bits.) For a solution that does run in polynomial time in both directions, see [26].

Equation (7) leads to Theorem 5.2, a fact noted by several authors (e.g., Newman and Somos, in [27, sequence A002487]). Let $\{x\}$ denote the fractional part of x.

Theorem 5.2. *The iterates of* $1 + 1/x - 2\{1/x\}$ *(equivalently, the iterates of* $1 + 2\lfloor 1/x \rfloor - 1/x$*), starting at 1, span the entire set of positive rational numbers.*

Proof. Let $r_n := a_{n+1}/a_n$. Then we may rewrite (7) as

$$r_{n+1} = 1 + \frac{1}{r_n} - 2\left\{\frac{1}{r_n}\right\}$$

and the result follows. □

We conclude with two interesting formulas.

Theorem 5.3. *For* $t > 1$*,*

$$\sum_{m \text{ odd}} \frac{1}{t^{a_m} - 1} = \frac{t}{(t-1)^2}.$$

Also,

$$\sum_{n=1}^{\infty} \frac{2}{a_{2n} a_{2n+1}^2 a_{2n+2}} = 1.$$

Proof. For a rational number r, let $\delta(r)$ be the denominator of r when expressed in lowest terms or, equivalently,

$$\delta(r) := \min\{k \in \mathbb{N} : kr \in \mathbb{Z}\}.$$

Let $A = \{(n,j) : n, j \in \mathbb{Z}^+, j > n\}$ and $B = \{(r,k) : r \in \mathbb{Q}, 0 < r < 1, k \in \mathbb{Z}^+\}$, and let $f : A \to B$ be the function $f(n,j) = (n/j, \gcd(n,j))$. It is easy to verify that f is a bijection and if $f(n,j) = (r,k)$, then $\delta(r) = j/k$ and so $k\delta(r) = j$. Therefore

$$\sum_{0<r<1} \frac{1}{t^{\delta(r)} - 1} = \sum_{0<r<1} \sum_{k=1}^{\infty} t^{-k\delta(r)} = \sum_{n=1}^{\infty} \sum_{j>n} t^{-j}$$
$$= \sum_{n=1}^{\infty} t^{-n} \sum_{j=1}^{\infty} t^{-j} = \frac{1}{(t-1)^2}.$$

Note that for every positive integer j, $a_j < a_{j+1}$ if and only if j is even. Hence every rational number strictly between 0 and 1 is of the form $a_{2m}/a_{2m+1} = a_m/a_{2m+1}$ for a unique $m \geq 1$, and we see that

$$\sum_{m=1}^{\infty} \frac{1}{t^{a_{2m+1}} - 1} = \frac{1}{(t-1)^2}.$$

Since $a_1 = 1$, adding $1/(t-1)$ to both sides gives the result.

Let $a \perp b$ mean that a and b are relatively prime. To get the second formula, we rewrite the left-hand side and then use the substitution $m = n + k$:

$$\sum_{n=1}^{\infty} \frac{2}{a_{2n} a_{2n+1}^2 a_{2n+2}} = 2 \sum_{n,k \geq 1, n \perp k} \frac{1}{nk(n+k)^2}$$
$$= \sum_{m=2}^{\infty} \sum_{k<m, k \perp m} \frac{2}{(m-k)km^2}$$
$$= \sum_{m=2}^{\infty} \frac{2}{m^3} \sum_{k<m, k \perp m} \left[\frac{1}{k} + \frac{1}{m-k} \right]$$
$$= \sum_{m=2}^{\infty} \frac{4}{m^3} \sum_{k<m, k \perp m} \frac{1}{k}.$$

As noted by Baney et al. [2, Theorem 7], the mth summand on the right is the probability that m is the lowest possible denominator of any fraction between the two coordinates of a point P chosen uniformly at random from the unit square. Hence the total sum is 1. □

6 Minkowski's ? Function

Recall the notation for continued fractions (see, for example, Hardy and Wright [18]): for positive integers $c_1, c_2, c_3, \ldots$,

$$[0; c_1, c_2, c_3, \ldots] := 1/(c_1 + 1/(c_2 + 1/(c_3 + \cdots))).$$

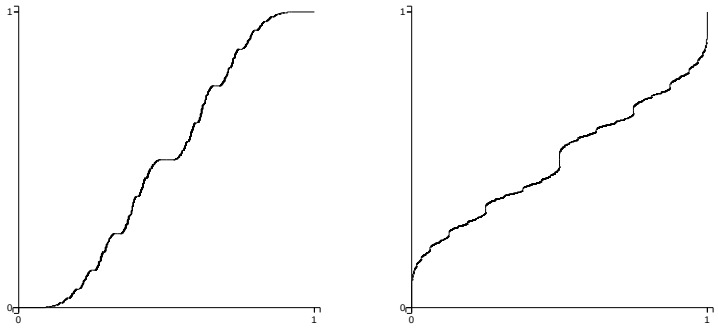

Figure 1: The graphs of $y = ?(x)$ and its inverse $y = f(x)$.

Every irrational positive number between 0 and 1 can be written uniquely in that form (rational numbers have nonunique *finite* expansions).

In 1904, Minkowski introduced a singular function (continuous with derivative existing and 0 almost everywhere) $? : [0,1] \to [0,1]$. It's value at x is defined in terms of the continued fraction expansion of x: if $x = [0, c_1, c_2, ...]$ then

$$?(x) := \sum_k \frac{(-1)^{k-1}}{2^{c_1+\cdots+c_k-1}}. \tag{11}$$

One of the question mark function's most interesting properties is that it maps quadratic surds to rational numbers (since the sequence $c_1, c_2, ...$ is eventually periodic precisely when x is a quadratic surd). Several nice references exist; see especially the thesis of Conley [8]. The results of this section are surely known; I have seen hints in internet references to a talk by Reznick and a note by Conway but am unable to give a specific reference.

Let $\mathcal{D}$ denote the set $\{k/2^n : k, n \in \mathbb{Z}^+, k \le 2^n\}$ of positive dyadic rationals in the unit interval and consider the function $f : \mathcal{D} \to \mathbb{Q}$ defined by

$$f(k/2^n) := \frac{a_k}{a_{2^n+k}}.$$

This function is well-defined since

$$f(2k/2^{n+1}) = \frac{a_{2k}}{a_{2^{n+1}+2k}} = \frac{a_k}{a_{2^n+k}} = f(k/2^n).$$

Theorem 6.1. *The function f extends to a strictly increasing continuous function from the unit interval to itself.*

Proof. Consider the matrices

$$M(m,n) := \begin{pmatrix} a_{m+1} & a_{n+1} \\ a_m & a_n \end{pmatrix}$$

and define

$$A_0 := M(0,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, A_1 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Using (1) it is easy to verify that for $i = 0$ or 1,

$$A_i M(m,n) = M(2m+i, 2n+i).$$

Therefore,

$$\text{if } m = \sum_{k=0}^{n} i_k 2^k \text{ then } A_{i_0} A_{i_1} \cdots A_{i_n} M(0,1) = M(m, 2^{n+1}+m). \tag{12}$$

Thus, for any positive integers n and $k < 2^n$, $M(k, 2^n + k)$ is a product of matrices with determinant 1 and so, for all n and $k < 2^n$,

$$a_{k+1} a_{2^n+k} - a_k a_{2^n+k+1} = 1. \tag{13}$$

Next, note that every odd entry (i.e., entry of odd index) in the nth row of the diatomic array is the sum of two consecutive entries in the previous row and so is at least one more than an odd entry of the previous row. It follows that for all n and $k < 2^n$,

$$a_{2^n+k} a_{2^n+k+1} \geq n+1. \tag{14}$$

Using (13) and (14), we see that

$$\frac{a_{k+1}}{a_{2^n+k+1}} - \frac{a_k}{a_{2^n+k}} = \frac{a_{k+1}a_{2^n+k} - a_k a_{2^n+k+1}}{a_{2^n+k+1}a_{2^n+k}} = \frac{1}{a_{2^n+k+1}a_{2^n+k}}$$

and so

$$0 < \frac{a_{k+1}}{a_{2^n+k+1}} - \frac{a_k}{a_{2^n+k}} \leq \frac{1}{n+1}$$

and the result follows. □

For more information about using the binary representation and n and matrix multiplication to compute a_n, the reader should see Allouche and Shallit [1, Example 7, p. 187].

The function $f(x)$ has been called Conway's box function [8, p. 82]. Since it is strictly increasing, it has a continuous inverse. The inverse function is Minkowski's question mark function.

Theorem 6.2. $f^{-1}(x) = ?(x)$.

Proof. Note that, by (11), $?(x)$ is uniquely determined by $?(1) = 1$ and

$$? \left(\frac{1}{n+x} \right) = \frac{2 - ?(x)}{2^n}.$$

It is enough to show that $f^{-1}(x)$ also satisfies these equations or, equivalently, $f(1) = 1$ and

$$f\left(\frac{2-x}{2^n} \right) = \frac{1}{n + f(x)}. \tag{15}$$

It is then enough to show that (15) holds for dyadic rationals $x := k/2^m$.

By (2) and (3),

$$a_{2^{m+1}+2^{m+1}-k} = a_{2^{m+2}-k} = a_{2^{m+1}+k} = a_k + a_{2^{m+1}-k}$$

and

$$a_{2^{m+n+1}+2^{m+1}-k} = a_{2^{m+n}+2^{m+1}-k} + a_{2^{m+1}-k}.$$

Therefore, by induction,

$$a_{2^{m+n}+2^{m+1}-k} = a_k + na_{2^{m+1}-k} = a_k + na_{2^m+k}.$$

Then

$$\begin{aligned} f\left(\frac{2-k/2^m}{2^n}\right) = f\left(\frac{2^{m+1}-k}{2^{m+n}}\right) &= \frac{a_{2^{m+1}-k}}{a_{2^{m+n}+2^{m+1}-k}} \\ &= \frac{a_{2^m+k}}{a_k + na_{2^m+k}} = \frac{1}{n + f(k/2^m)}. \end{aligned}$$

□

We note another symmetry peculiar to Stern's diatomic sequence (pointed out to the author by Bruce Reznick). Given odd m, let m' be the number with reverse binary expansion:

$$\text{if } m = \sum_{k=0}^{n} i_k 2^k \text{ then } m' = \sum_{k=0}^{n} i_{n-k} 2^k.$$

For example, if $m = 35 = 100011_2$, then $m' = 110001_2 = 49$.

Theorem 6.3. *For all odd m, $a_m = a_{m'}$.*

Proof. Using equation (12) again, it's easy to verify by induction that if $m = \sum i_k 2^k$, then

$$a_m = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) A_{i_0} \cdots A_{i_n} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

For any 2-by-2 matrix M, we define M^* by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* := \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right).$$

It's easy to verify the following for any matrices M, N and $i = 0, 1$:

$$(MN)^* = N^* M^*,$$

$$A_i^* = A_i,$$

$$\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) M^* \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) M \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

Hence

$$\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) A_{i_0} \cdots A_{i_n} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) A_{i_n} \cdots A_{i_0} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$$

and the result follows. □

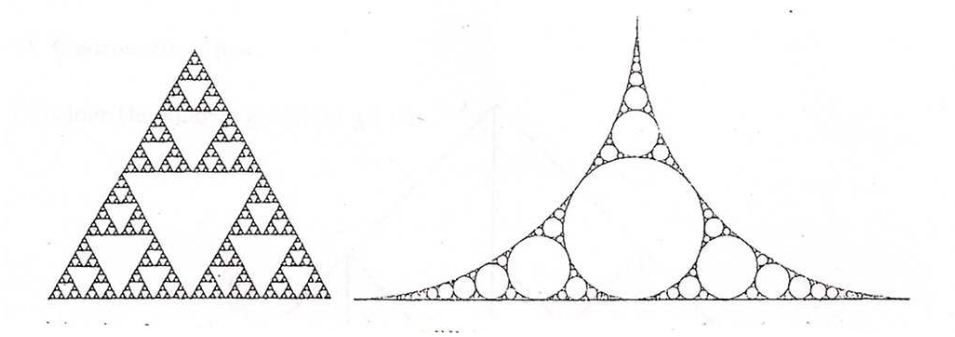

Figure 2: Sierpinski gasket (SG) and Apollonian circle packing (CP).

There is a geometric way of viewing Minkowski's ? function; consider Figure 2. We consider each of the two objects to be lying in the complex plane with bottom edge coinciding with the unit interval. Clearly they are homeomorphic and there is a unique homeomorphism from SG to CP which takes the unit interval to itself and fixes its endpoints. The restriction of this homeomorphism to the unit interval can be shown to be $f(x)$. (If $\oplus$ denotes "mediant addition" $a/b \oplus c/d = (a+c)/(b+d)$ and if $2^n r, 2^n s$ are consecutive integers for some n then $f((r+s)/2) = f(r) \oplus f(s)$.)

The two figures in Figure 2 are called *fractals* because their Hausdorff dimension is fractional. Roughly speaking, since SG is made up of three copies of itself, with each copy having length and width half as big, the dimension of SG is the solution of $2^d = 3$ (namely $d = \ln 3/\ln 2$). However, Hausdorff dimension is not a topological invariant and, indeed, the dimension of CP is unknown (see Graham et al. [16]).

We conclude this section with a connection to a well-known conjecture of number theory (and perhaps give our little sequence—or at least $f(x)$—some additional *gravitas*). The triangles in SG are evenly distributed in the sense that if one takes any horizontal line above the x-axis intersecting SG, it will intersect finitely many triangles in the complement of SG which touch the x-axis. The bottom vertices of these triangles are evenly distributed, being $\{k/2^n : k \text{ odd }\}$ for some n. On the other hand, a horizontal line at a distance $t > 0$ above the x-axis intersects finitely many circles in the complement of CP tangent to the x-axis, but the tangent points of those circles with the x-axis, call that set $\{x_1, x_2, ..., x_N\}$, are not evenly distributed. A measure of that discrepency can be given by

$$\Delta_t := \sum_{j=1}^{N} \left| x_i - \frac{i}{N} \right|.$$

As t decreases towards 0, N grows, and although Δ_t may also grow, it is conjectured that it grows no faster than $1/t^{1/4}$. More precisely, it is conjectured

that for all $\epsilon > 0$,

$$\lim_{t \to 0^+} t^{\frac{1}{4}+\epsilon} \Delta_t = 0.$$

This is known to be equivalent to the Riemann hypothesis(!). If $1/4$ is replaced by $1/2$, then the result is equivalent to the prime number theorem and is known to be true. It is not known to be true for any other $c < 1/2$. (The points $\{x_1, x_2, ..., x_N\}$ form a Farey series—see, for example, [18]—and the connection between Farey series and the Riemann hypothesis, due to Franel and Landau, is nicely explained by Edwards in [10].)

7 A Geometric View

Let $S_{m,n}$ be an infinite graph which is the "boundary" of a tiling of a quarter-plane by $2n$-gons so that every vertex has m edges emanating downwards. For example, Figures 3 and 4 show $S_{3,2}$ and $S_{2,3}$ respectively. $S_{2,2}$ is just one quarter of an infinite square grid. $S_{2,2}$ is a geometric representation of Pascal's triangle; assign to each vertex the corresponding number in Pascal's triangle and note that the number assigned to each vertex is the sum of all numbers in neighbors above. By induction, the number assigned to each vertex v is the number of downward paths starting at the top vertex and ending at v.

Figure 3 shows an infinite graph closely related to Stern's sequence. We call

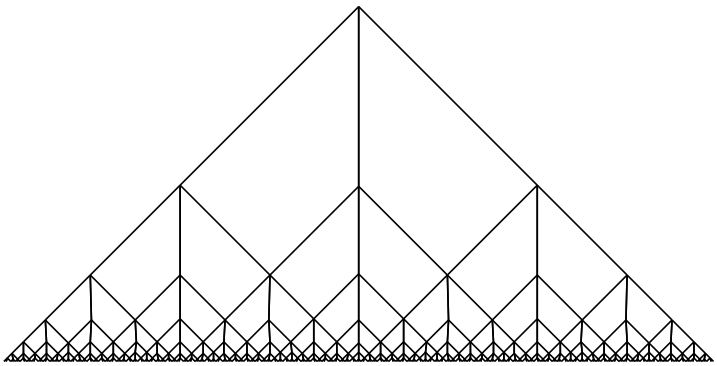

Figure 3: The hyperbolic graph $S_{3,2}$.

this a "hyperbolic" graph since, when it is embedded in the hyperbolic upper half-plane, each quadrilateral has exactly the same size.

For example, if we assign to each vertex v of $S_{3,2}$ the number of downward paths starting at the top and ending at v, we get:

$$\begin{array}{cccccccccccccc}
 & & & & & & & 1 & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1
\end{array}$$

and it's not hard to see that Stern's sequence features in it: the $(n-1)$st row is

$$a_1, a_2, ..., a_{2^n-1}, a_{2^n}, a_{2^n-1}, ..., a_2, a_1.$$

Now consider a random walk starting at the top vertex and always going downwards, choosing one of its three edges at random. If we identify the bottom of $S_{3,2}$ with the unit interval, then the random walk has a limiting distribution μ on the unit interval. For example, let's compute the chance that the random walk is eventually in the interval $[1/4, 3/8]$ (we call this quantity $\mu([1/4, 3/8])$). The only way the random walk can end up in this interval is if, at the second step, it is at the second or third vertex of the third row. There are $a_2 = 1$ and $a_3 = 2$ ways to end up at these vertices respectively. However, given it is at the second vertex or third vertex, only half of the time will it end up in $[1/4, 3/8]$, and so the desired probability is $\frac{1}{2}(a_2 + a_3)/3^2 = a_5/18 = 1/6$. In general,

$$\mu\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) = \frac{a_{2j+1}}{2 \cdot 3^{n-1}},$$

where $j = k$ or $2^n - 1 - k$ according to whether k is less than 2^{n-1} or not.

Let M be the (infinite) matrix defined by $M(i, j) = 1$ if $j \in \{2i-1, 2i, 2i+1\}$, 0 otherwise $(i, j = 1, 2, 3, ...)$. This is the adjacency matrix of a graph with vertices labelled by the positive integers. The powers of M have a limit:

$$M^\infty(i, j) := \lim_{n\to\infty} M^n(i, j)$$

exists for all i, j and

$$M^\infty(1, n) = a_n.$$

Figure 4 shows the graph $S_{2,3}$. Note that every quadrilateral in this particular representation of the graph is either a square or a "golden rectangle" (i.e., a rectangle with length-to-width ratio ϕ)! The row sizes are 1,2,4,7,12,20,...., the partial sums of Fibonacci numbers (see [27, sequence A000145] for more on this particularly well-studied sequence). Counting downward paths as we did earlier for $S_{3,2}$ gives rise to an analogue (b_n) of Stern's sequence:

$$1, 1, 1, 2, 1, 2, 2, 1, 3, 2, 2, 3, 1, 3, 3, 2, 4, 2, 3, 3, 1, ...$$

[27, sequence A000119] called the "Fibonacci diatomic sequence" (see [4]).

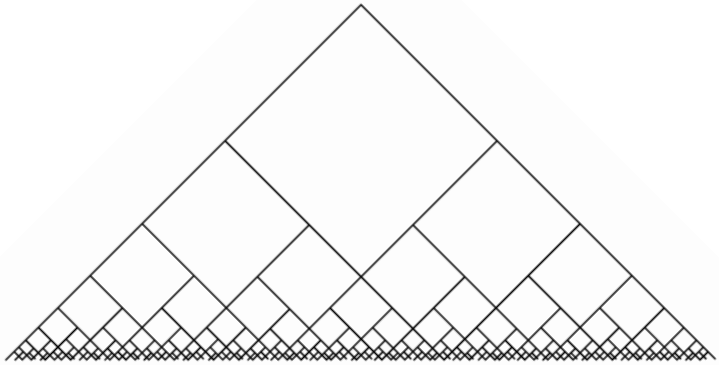

Figure 4: The hyperbolic graph $S_{2,3}$.

This sequence also arises from an analogue of Stern's diatomic array called the "Fibonacci diatomic array" [4]:

1																1
1								2								1
1						2				2						1
1			3			2				2			3			1
1		3		3		2		4		2		3		3		1
1	4	3		3	5	2	4		4	2	5	3		3	4	1
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

This array is contained in $S_{2,3}$ in the same way that Stern's diatomic array is contained in $S_{3,2}$.

If $N(i,j) = 1$ if $j \in \{\lfloor \phi i \rfloor, \lceil \phi i \rceil\}$ and 0 otherwise, then N is the adjacency matrix of a graph and

$$b_n := \lim_{n\to\infty} N^n(1,n).$$

The Fibonacci diatomic sequence (b_n) has a counting interpretation: b_{n+1} is the number of ways that n can be written as a sum of distinct Fibonacci numbers.

8 Future Directions.

Thus ends a tour of this splendid sequence. The references below—and their references—contain more and the reader is encouraged to delve further. Here are some directions of future research.

- Are there any applications for the Binet-type formula (9)?
- The Fibonacci sequence satisfies $F_n \asymp \phi^n$ (here $x_n \asymp y_n$ means x_n/y_n is bounded away from 0 and ∞; equivalently, $x_n = \Theta(y_n)$) while $a_n \asymp \phi^{\log_2 n}$ even though the conjectured equation (10) might not be true. More generally, other diagonal sums across Pascal's triangle satisfy

$$\sum_{ai+bj=n} \binom{i+j}{i} \asymp \gamma^n$$

where γ is the unique positive solution to $\gamma^a + \gamma^b = \gamma^{a+b}$ (see [24]). For the same a, b, and γ, is

$$\sum_{ai+bj=n} \left[\binom{i+j}{i} \bmod 2 \right] \asymp \gamma^{\log_2 n}$$

true? What if we sum mod 3 instead of mod 2?
- Stern's sequence is characterized in Section 5 by having every relatively prime pair of positive integers appear exactly once (as two consecutive terms). There are other sequences with this property, but is Stern's sequence minimal or universal in some sense? Are there sequences so that every relatively prime pair appears exactly twice? What if the condition of being relatively prime is removed? Is there a sequence where every relatively prime *triple* appears exactly once?
- Is it possible to rephrase the Riemann hypothesis in terms of a conjecture about a_k?
- The Fibonacci diatomic array and its associated graph $S_{3,2}$ gives rise to the "Fibonacci diatomic sequence". Which properties listed above for Stern's diatomic sequence have analogues for the Fibonacci diatomic sequence? How about for $S_{m,n}$ or other regular or not-so-regular hyperbolic tilings?

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References

[1] J-P. Allouche and J. Shallit, The ring of k-regular sequences, *Theoret. Comput. Sci.* **98** (1992) 163–197.

[2] D. Baney, S. Beslin, and V. DeAngelis, Farey tree and distribution of small denominators, in *Topology Proc.* **22** (1997) 23-37.

[3] M. Bicknell-Johnson, Stern diatomic array applied to Fibonacci representations, *Fibonacci Quart.* **41** (2003) 169-180.

[4] ———, The Fibonacci diatomic array applied to Fibonacci representations, in *Applications of Fibonacci Numbers*, vol. 9, Kluwer Academic, Dordrecht, 2004, 29–38.

[5] N. Calkin and H. S. Wilf, Recounting the rationals, this MONTHLY **107** (2000) 360-367.

[6] L. Carlitz, A problem in partitions related to the Stirling numbers, *Bull. Amer. Math. Soc.* **70** (1964) 275-278.

[7] R. M. Conley, *A Survey of the Minkowski ?(x) Function*, M.S. thesis, West Virginia University, Morgantown, WV, 2003.

[8] J. H. Conway, *On Numbers and Games*, Academic Press, New York, 1976.

[9] E. W. Dijkstra, *Selected Writings on Computing: A Personal Perspective*, Springer-Verlag, New York, 1982.

[10] H. M. Edwards, *Riemann's Zeta Function*, Pure and Applied Mathematics, vol. 58, Academic Press, New York, 1974.

[11] S. R. Finch, *Mathematical Constants*, Encylopedia of Mathematics and its Applications, vol. 94, Cambridge University Press, Cambridge, 2003.

[12] C. Giuli and R. Giuli, A primer on Stern's diatomic sequence I, *Fibonacci Quart.* **17** (1979) 103-108.

[13] ———, A primer on Stern's diatomic sequence II, *Fibonacci Quart.* **17** (1979) 246-248.

[14] ———, A primer on Stern's diatomic sequence III, *Fibonacci Quart.* **17** (1979) 318-320.

[15] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, Reading, MA, 1994.

[16] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. R.Wilks, and C. H. Yan, Apollonian circle packings: Number theory, *J. Number Theory* **100** (2003) 1-45.

[17] H. Harborth, Number of odd binomial coefficients, *Proc. Amer. Math Soc.* **62** (1977) 19-22.

[18] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Oxford University Press, New York, 1979.

[19] A. M. Hinz, The Tower of Hanoi, in *Algebras and Combinatorics: An International Congress, ICAC '97*, Springer, Singapore, 1999, 277-289.

[20] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Pure and Applied Mathematics, Wiley Interscience, New York, 2001.

[21] E. E. Kummer, Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen, *J. Reine Angew. Math.* **44** (1852) 93-146.

[22] D. H. Lehmer, On Stern's diatomic sequence, this MONTHLY **36** (1929) 59-67.

[23] D. A. Lind, An extension of Stern's diatomic series, *Duke Math. J.* **36** (1969) 55-60.

[24] S. Northshield, Sums across Pascal's triangle modulo 2 (to appear).

[25] J. Paradis, P. Viader, and L. Bibiloni, The derivative of Minkowski's ?(x) Function, *J. Math. Anal. Appl.* **253** (2001) 107-125.

[26] J. Shallit, Formal languages and number theory, in *Unusual Applications of Number Theory, Proc. DIMACS Workshop, Rutgers, 2000*, American Mathematical Society, Providence, RI, 2004, 169-181.

[27] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, available at `http://www.research.att.com/~njas/sequences/`.

[28] M. A. Stern, Über eine zahlentheoretische Funktion, *J. Reine Agnew. Math.* **55** (1858) 193-220.

[29] I. Urbiha, Some properties of a function studied by de Rham, Carlitz and Dijkstra and its relation to the (Eisenstein-)Stern's diatomic sequence, *Math. Commun.* **6** (2001) 181-198.

[30] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Ellis Horwood, Chichester, UK, 1989.

[31] H. Wilf, *generatingfunctionology*, Academic Press, Boston, 1990.

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